property is satisfied for the measures $\nu_\beta^{V_\Lambda}$ and is preserved for ν_β^V thanks to the convergence of the finite-dimensional distributions, which follows from the convergence of the characteristic functionals.

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CONVERGENCE OF DENOMINATORS OF JOINT PADÉ APPROXIMATIONS OF A SET OF CONFLUENT HYPERGEOMETRIC FUNCTIONS

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This paper continues the investigation begun in [1] concerning the convergence of joint Padé approximations of a set of confluent hypergeometric functions $\{{}_1F_1(1;\nu_k+1;z)\}_{k=1}^n$. We recall the main definitions and results.

Definition 1 (see [2]). Let $F=\{f_k(z)\}_{k=1}^n$ be a set of functions, analytic in the neighborhood of z = 0, and $\vec{r}=(r_1,\ldots,r_n)$ a vector whose coordinates are nonnegative integers whose sum is some number $N=N(\vec{r})\in\mathbb{N}^1$. Joint Padé approximations of the set $\{f_k(z)\}_{k=1}^n$, of order $([N/N];\vec{r})$, are rational polynomials $\pi_{N,N}^{(k)}\{F;\vec{r};z\}$, $k=\overline{1,n}$, of degree [N/N] with a common denominator, such that the following asymptotic relations are true:

$$f_k(z)-\pi_{N,N}^{(k)}\{F;\vec{r};z\}=O(z^{N+r_k+1}),\quad z\to 0;\qquad k=\overline{1,n}. \tag{1}$$

The following theorem was proved in [1].

THEOREM 1. Joint Padé approximations of a set of confluent hypergeometric functions $F=\{f_k(z)\}_{k=1}^n$

$$f_k(z)={}_1F_1(1;\nu_k+1;z),\quad k=\overline{1,n},\quad \nu_k-\nu_m\notin\mathbb{Z}\ \text{for}\ k\neq m;$$
$$\nu_k>-1:\qquad k=\overline{1,n}$$

of order $([N/N];\vec{r})$ are uniformly convergent to the functions $f_k(z)$ on any compact set K in the complex plane as $N\to\infty$.

The arguments used to prove this theorem imply that if $B_N(t)$ are polynomials that satisfy the biorthogonality conditions

$$\int_0^1 B_N(t)\,t^{i+\nu_k}dt=0,\quad i=\overline{0,r_k-1};\qquad k=\overline{1,n},$$

in the interval [0, 1], having zeros so located that their arithmetic means satisfy relation

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$$\alpha_N = \frac{t_1^{(N)} + t_2^{(N)} + \ldots + t_N^{(N)}}{N} \to \varkappa \quad \text{as} \quad N \to \infty,$$

then for the denominators of the corresponding joint approximations one has

$$\frac{1}{N!} Q_N(z) \to \exp\{(\varkappa - 1) z\}, \text{ as } N \to \infty \tag{2}$$

uniformly on compact sets.

We wish to investigate the behavior of the numbers α_N as $N \to \infty$. We may assume without loss of generality that the leading coefficient of the polynomial $B_N(t)$ is unity:

$$B_N(t) = t^N + \lambda_N t^{N-1} + P_{N-2}(t), \tag{3}$$

where $P_{N-2}(t)$ is an algebraic polynomial of degree $\leqslant N-2$. Obviously, $\alpha_N = -\lambda_N/N$. We construct a generalized polynomial

$$A_{N-1}(t) = \sum_{k=1}^{n} \sum_{j=0}^{r_k - 1} c_j^{(N,k)} t^{j+\nu_k} = \sum_{j=0}^{N-1} c_j^{(N)} t^{k_j} \tag{4}$$

defined by the conditions

$$\int_0^1 t^j A_{N-1}(t)\, dt = 0 \text{ for } j < N-1.$$

If $\nu_k - \nu_m \notin \mathbb{Z}$ for $k \neq m$, such a polynomial indeed exists and in the interval $(0, 1)$, it has exactly $N - 1$ simple roots. Multiplying (3) by $A_{N-1}(t)$ and integrating over $[0, 1]$ with respect to the measure dt, we obtain

$$\int_0^1 (t^N + \lambda_N t^{N-1}) A_{N-1}(t)\, dt = 0,$$

whence

$$\lambda_N = -\frac{\int_0^1 t^N A_{N-1}(t)\, dt}{\int_0^1 t^{N-1} A_{N-1}(t)\, dt}. \tag{5}$$

Integration by parts gives

$$\int_0^1 t^N A_{N-1}(t)\, dt = (-1)^{N-1} \cdot N! \int_0^1 t \int_0^t \int_0^{t_1} \ldots \int_0^{t_{N-2}} A_{N-1}(t_{N-1})\, dt_{N-1} \ldots dt_1 dt = (-1)^{N-1} \cdot N! \int_0^1 t \psi_N(t)\, dt,$$

where

$$\psi_N(t) = \int_0^t \int_0^{t_1} \ldots \int_0^{t_{N-2}} A_{N-1}(t_{N-1})\, dt_{N-1} \ldots dt_2 dt_1 = (-1)^{N-2} \int_0^t \frac{\tau^{N-2}}{(N-2)!} A_{N-1}(\tau)\, d\tau. \tag{6}$$

Similarly,

$$\int_0^1 t^{N-1} A_{N-1}(t)\, dt = (-1)^{N-1} \cdot (N-1)! \int_0^1 \psi_N(t)\, dt.$$

Thus,

$$\lambda_N = -N \frac{\int_0^1 t\psi_N(t)\, dt}{\int_0^1 \psi_N(t)\, dt}, \quad \alpha_N = \frac{\int_0^1 t\psi_N(t)\, dt}{\int_0^1 \psi_N(t)\, dt}.$$

From (3) and (5) we obtain

$$(-1)^{N-2}(N-2)!\,\psi_N(t)=\sum_{j=0}^{N-1}c_j^{(N)}\frac{t^{N-1+k_j}}{N-1+k_j}=t^{N-1+k_0}\sum_{j=0}^{N-1}c_j^{(N)}\frac{t^{k_j-k_0}}{N-1+k_j}.$$

Note that if $\nu_m-\nu_k\notin\mathbb{Z}$ the system of functions $\{t^{\nu_k}\}_{k=1}^{n}$ is an AT-system on $[0, 1+\varepsilon]$ (see [2]), and moreover any polynomial of type (4) has at most $N-1$ roots in $(0, 1+\varepsilon)$ counting multiplicities. Therefore $\psi_N(t)$ also has at most $N-1$ roots in $(0, 1+\varepsilon)$, counting multiplicities. But it is readily seen that $\psi_N(t)$ has a root of multiplicity $N-1$ at the point $t=1$. Consequently, up to a constant factor we can represent $\psi_N(t)$ as

$$\psi_N(t)=t^{N-1+k_0}\begin{vmatrix}1 & 1 & 1 & \dots & 1\\ 0 & k_1-k_0 & k_2-k_0 & \dots & k_{N-1}-k_0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 1 & t^{k_1-k_0} & t^{k_2-k_0} & \dots & t^{k_{N-1}-k_0}\end{vmatrix}.$$

In particular, when $k_j-k_0=j/n$, $j=\overline{0,\infty}$, which corresponds to joint Padé approximations of the functions $\{{}_1F_1(1;\nu_k+1;z)\}_{k=1}^{n}$, $\nu_k=\nu_1+\frac{(k-1)}{N}$, $\nu_1>-1$, $k=\overline{1,n}$ of order $([N/N];\vec{r})$, $\vec{r}=(r_1,\dots,r_n)$
$r_k=\begin{cases}\left[\frac{N}{n}\right]+1, & k=\overline{1,m}\\ \left[\frac{N}{n}\right], & k=\overline{m+1,n}\end{cases}$, where m is the remainder upon division of N by n, we have $\psi_N(t)=t^{N-1+k_0}(t^{1/n}-1)^{N-1}$. Then we obtain

$$\alpha_N=\frac{\int_0^1 t^{N+k_0}(t^{1/n}-1)^{N-1}dt}{\int_0^1 t^{N-1+k_0}(t^{1/n}-1)^{N-1}dt}=\frac{\int_0^1 u^{n(N+k_0)}(u-1)^{N-1}u^{n-1}du}{\int_0^1 u^{n(N+k_0-1)}(u-1)^{N-1}u^{n-1}du}=$$

$$=\frac{\Gamma(n(N+k_0+1))\,\Gamma(N)\,\Gamma(n(N+k_0)+N)}{\Gamma(n(N+k_0+1)+N)\,\Gamma(n(N+k_0))\,\Gamma(N)}=$$

$$=\frac{n(N+k_0)[n(N+k_0)+1]\dots[n(N+k_0)+n-1]}{[n(N+k_0)+N][n(N+k_0)+N+1]\dots[n(N+k_0)+N+n-1]}.$$

Hence $\lim_{N\to\infty}\alpha_N=\left(\frac{n}{n+1}\right)$

We have thus proved the following.

THEOREM 2. The denominators of joint Padé approximations of a set of confluent hypergeometric functions

$$f_k(z)={}_1F_1(1;\nu_k+1;z),\quad k=\overline{1,n},\quad \nu_k=\nu_1+\frac{(k-1)}{n},\quad k=\overline{1,n},\nu_1>-1,$$

of order $([N/N],\vec{r})$, $\vec{r}=(r_1,\dots,r_n)$, $r_k=\begin{cases}\left[\frac{N}{n}\right]+1, & k=\overline{1,m};\\ \left[\frac{N}{n}\right], & k=\overline{m+1,n}\end{cases}$, where m is the remainder upon division of N by n, converge uniformly as $N\to\infty$ on any compact set in the complex plane:

$$\frac{1}{N!}Q_N(z)\to\exp\left\{\left[\left(\frac{n}{n+1}\right)^n-1\right]z\right\}.$$

Remark 1. Formula (5) yields an elementary derivation of a result due to de Bruin [3], concerning the convergence of the denominators of the off-diagonal Padé approximations for ${}_1F_1(1;c;x)$, $c\in\mathbb{N}^-$. To this end, by [4], one need only substitute the expressions given by Rodrigues' formula for the appropriate Jacobi polynomials into (5), and then use property (2).

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LINDELÖF'S THEOREM IN $\mathbb{C}^n$

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In this article we make more precise Lindelöf's theorem for holomorphic functions in several complex variables.

Let D be a domain in $\mathbb{C}^n$ with C^2-smooth boundary ∂D. For any $\alpha > 0$ and $\varepsilon \geqslant 0$ we denote

$$D_\alpha^\varepsilon(\xi) = \{z \in \mathbb{C}^n : |(z-\xi, \nu_\xi)| < (1+\alpha)\,\delta_\xi(z),\quad |z-\xi|^2 < \alpha(\delta_\xi z))^{1+\varepsilon}\},$$

where $(.,.)$ is the usual Hermitian scalar product in $\mathbb{C}^n$, ν_ξ is the vector unit outer normal to ∂D at the point ξ, $\delta_\xi(z) = \min\{d_\xi(z), \delta(z)\}$. Here $d_\xi(z)$ is Euclidean distance from the point z to the real tangent plane $T_\xi = T_\xi(\partial D)$ to ∂D at the point ξ, and $\delta(z)$ is Euclidean distance from the point z to ∂D.

We denote the set $D_\alpha^0(\xi)$ by $D_\alpha(\xi)$. Clearly $D_\alpha^\varepsilon(\xi) \subset D_\alpha(\xi)$ for all $\varepsilon > 0$.

We will say that a function $f: D \to \mathbb{C}$ has K-limit a at the point $\xi \in \partial D$ if, for any $\alpha > 0$ and for any sequence of points $\{z^m\}$ from $D_\alpha(\xi)$ converging to ξ, $f(z^m) \to a$ as $m \to \infty$ (see [1, p. 83]). A function $f: D \to \mathbb{C}$ has limit a along the normal ν_ξ to ∂D at the point ξ, if $f(\xi - t\nu_\xi) \to a$ as $t \to 0$.

We denote by H(D) the algebra of all functions holomorphic in the domain D.

For any point $z \in D$ sufficiently close to ∂D there is defined a unique point $\xi(z) \in \partial D$ such that $|z - \xi(z)| = \delta(z)$.

Let $z_1, \ldots, z_n$ be coordinates in $\mathbb{C}^n$. For any real function φ of class C^2 in the domain D the Hermitian quadratic form is called its Levi form

$$L_z(\varphi, dz) = \sum_{\mu,\nu=1}^{n} \frac{\partial^2\varphi(z)}{\partial z_\mu \partial \bar{z}_\nu}\, dz_\mu d\bar{z}_\nu.$$

<u>Definition.</u> A function $f \in H(D)$, where D is a domain in $\mathbb{C}^n$ with boundary ∂D of class C^2 belongs to the class N(D) if there is a constant K such that for all $Z \in D$

$$L_z(\log(1+|f|^2), dz) \leqslant K\left\{\frac{|dz_T|^2}{\delta(z)} + \frac{|dz_N|^2}{(\delta(z))^2}\right\}, \tag{1}$$

where dz_T and dz_N are the projections of the vector dz onto the complex tangent plane $T_{\xi(z)}^c = T_{\xi(z)} \cap iT_{\xi(z)}$ to ∂D at the point $\xi(z)$ and the complex normal $N_{\xi(z)}^c$, $\mathbb{C}^n = N_{\xi(z)}^c \oplus T_{\xi(z)}^c$, to ∂D at the point $\xi(z)$, respectively.

Geometrically this condition on the function f means that its spherical product in the normal and complex tangent directions grows no faster than $K/\delta(z)$ and $K/\sqrt{\delta(z)}$, respectively.

In strongly pseudoconvex domains of the space $\mathbb{C}^n$ the right-hand side of inequality (1) is equivalent to the standard invariant metrics of Carathéodory, Kobayashi, and Bergman.

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