GENERALIZED MOMENT REPRESENTATIONS AND PADÉ-CHEBYSHEV APPROXIMATIONS

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An approach to the application of Dzyadyk's generalized moment representations in problems of construction and investigation of the Padé-Chebyshev approximants is developed. With its help, certain properties of the Padé-Chebyshev approximants of a class of functions that is a natural analog of the class of Markov functions are studied. In particular, it is proved that the poles of the Padé-Chebyshev approximants of these functions lie outside their domain of analyticity.

In the study of the Padé approximations on the first plan, as a rule, we introduce functions of the form

$$f(z) = \int_{\Gamma} \frac{d\mu(\zeta)}{1-\zeta z}, \tag{1}$$

where $d\mu(\zeta)$ is a measure on the compactum $\Gamma \subset \mathbb{C}$. This is connected with the classical ideas of Chebyshev, concerning the moments problem for the numerical sequence $\{s_k\}_{k=0}^{\infty}$ (see, e.g., [1]):

$$s_k = \int_{\Gamma} \zeta^k d\mu(\zeta), \quad k = \overline{0, \infty}. \tag{2}$$

In the case of a nonnegative measure $d\mu(\zeta)$ on a real set, many problems of rational approximation of functions of form (1) are solved in terms of polynomials that are orthogonal with respect to the measure $d\mu(\zeta)$. For the extension of this class of functions, some investigators (see, e.g., [2, 3]) have studied the properties of sequences of orthogonal polynomials, corresponding to variable-sign and complex-valued measures. Dzyadyk suggested in 1981 another method, included in the generalization of moments problem (2).

Definition 1 [4]. The set of equations

$$s_{i+j} = \int_{\Gamma} a_i(t)\, b_j(t)\, d\mu(t), \quad i, j = \overline{0, \infty}, \tag{3}$$

in which Γ is a Borel set (most often, a segment of the real axis), $d\mu(t)$ is a measure on Γ, and $\{a_i(t)\}_{i=0}^{\infty}$ and $\{b_j(t)\}_{j=0}^{\infty}$ are sequences of measurable functions on Γ, for which all the integrals in (3) exist, is called a generalized moment representation of the sequence of complex numbers $\{s_k\}_{k=0}^{\infty}$.

In some works, Dzyadyk and the author have indicated methods and examples of applications of the generalized moment representations to the problems of Padé approximations, many-point Padé approximations, joint Padé approximations, etc. Dzyadyk and Chyp have obtained with the help of generalized moment representations integral representations for a series of special functions [5, Chap. VI]. These applications are based on the transformation of Eq. (3) to the form

$$f(z)\, Q_N(z) - P_{N-1}(z) = z^N \int_{\Gamma} A(z, t)\, B_N(t)\, d\mu(t), \tag{4}$$

where

$$f(z) = \sum_{k=0}^{\infty} s_k z^k, \quad A(z, t) = \sum_{i=0}^{\infty} a_i(t)\, z^i, \quad B_N(t) = \sum_{j=0}^{N} c_j^{(N)} b_j(t),$$

$$Q_N(z) = \sum_{j=0}^{N} c_j^{(N)} z^{N-j}, \quad P_{N-1}(z) = \sum_{j=1}^{N} c_j^{(N)} z^{N-j} \sum_{k=0}^{j-1} s_k z^k.$$

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The present article is devoted to the application of generalized moment representations to the Padé–Chebyshev approximation problem, which is realized in a manner somewhat different from that given above.

Definition 2 [6]. Let a function $f(x) \in C[-1, 1]$ be expanded in a uniformly convergent Fourier–Chebyshev series of the form

$$f(x) = \sum_{k=0}^{\infty} s_k T_k(x), \tag{5}$$

where $T_k(x) = \cos k \arccos x$ are Chebyshev polynomials of first kind. The rational polynomial $[M/N]_f^{\tau}(x) = P_M(x)/Q_N(x) \in R[M/N] := \{r(x) : r(x) = p(x)/q(x), \deg p(x) = M, \deg q(x) = N\}$ such that

$$f(x) Q_N(x) - P_M(x) = \sum_{k=M+N+1}^{\infty} \tau_k T_k(x) \tag{6}$$

is called the Padé–Chebyshev approximant of f(x) of order [M/N].

The following theorem establishes a connection between the generalized moment representations and the Padé–Chebyshev approximations.

THEOREM 1. Let the function f(x) be expanded in a uniformly convergent Fourier–Chebyshev series of form (5) and a sequence $\{s_k\}_{k=0}^{\infty}$ be such that a generalized moment representation of the form

$$s_{i+j} = \int_{\Gamma} a_i(t) b_j(t) d\mu(t), \quad i = \overline{0, \infty} \tag{7}$$

holds. Moreover, let the determinant

$$\Delta[M/N] = \det \| s_{M+1+i+j} + s_{M+1+i-j} \|_{i,j=0}^{N} \neq 0 \tag{8}$$

for certain integers $M \geqslant N \geqslant 0$.

Then the Padé–Chebyshev approximant of f(z) of order [M/N] can be represented in the form

$$[M/N]_f^{\tau}(x) = P_M(x)/Q_N(x), \tag{9}$$

where

$$Q_N(x) = \sum_{j=0}^{N} c_j^{(N)} T_j(x), \tag{10}$$

$$P_M(x) = \frac{1}{2} s_0 \sum_{j=0}^{N} c_j^{(N)} T_j(x) - \frac{1}{2} \sum_{j=0}^{N} c_j^{(N)} s_j + \frac{1}{2} \sum_{i=0}^{M} T_i(x) \times \sum_{j=0}^{N} c_j^{(N)} [s_{i+j} + s_{|i-j|}], \tag{11}$$

and the coefficients $c_j^{(N)}$, $j = \overline{0, N}$, not all equal to zero, are determined from the conditions of biorthogonality for the polynomial

$$B_N(t) = \sum_{j=0}^{N} c_j^{(N)} [b_{M+1+j}(t) + b_{M+1-j}(t)],$$
$$\int_{\Gamma} a_i(t) B_N(t) d\mu(t) = 0, \quad i = \overline{0, N-1}. \tag{12}$$

The approximation error has the integral representation

$$f(z) Q_N(z) - P_M(z) = \frac{1}{2} \int_{\Gamma} \sum_{k=0}^{\infty} T_{k+M+1}(z) a_k(t) B_N(t) d\mu(t). \tag{13}$$

Proof. By virtue of (5) and (10), we have

$$f(x) Q_N(x) = \sum_{k=0}^{\infty} s_k T_k(x) \sum_{j=0}^{N} c_j^{(N)} T_j(x) = \frac{1}{2} \sum_{j=0}^{N} c_j^{(N)} \sum_{k=0}^{\infty} s_k [T_{k+j}(x) + T_{|k-j|}(x)] =$$
$$= \frac{1}{2} \sum_{k=0}^{\infty} T_k(x) \sum_{j=0}^{N} c_j^{(N)} [s_{k+j} + s_{|k-j|}] - \frac{1}{2} \sum_{j=0}^{N} c_j^{(N)} s_j T_0(x) + \frac{1}{2} s_0 \sum_{j=0}^{N} c_j^{(N)} T_j(x) =$$

$$= P_M(x) + \frac{1}{2}\sum_{k=M+1}^{\infty} T_k(x)\sum_{i=0}^{N} c_i^{(N)}[s_{k+i} + s_{k-i}] = P_M(x) + \frac{1}{2}\int_{\Gamma}\sum_{k=0}^{\infty} T_{k+M+1}(x)\, a_k(t)\, B_N(t)\, d\mu(t).$$

All assertions of the theorem follow from the last equation.

As an example, let us consider the class of the functions f(x) that can be represented in the form

$$f(x) = \int_{\alpha}^{\beta} \frac{1 - xt}{1 - 2xt + t^2}\, d\mu(t), \tag{14}$$

where $d\mu(t)$ is a positive measure on the segment $[\alpha, \beta] \subset [-1, 1]$. Since $\frac{1 - xt}{1 - 2xt + t^2}$ is the generating function of Chebyshev polynomials of first kind, the coefficients in the expansion

$$f(x) = \sum_{k=0}^{\infty} s_k T_k(x) \tag{15}$$

have the form

$$s_k = \int_{\alpha}^{\beta} t^k d\mu(t), \quad k = \overline{0, \infty}. \tag{16}$$

Equations (16) can be rewritten in the form

$$s_{i+j} = \int_{\alpha}^{\beta} t^i t^j d\mu(t), \quad i, j = \overline{0, \infty}. \tag{17}$$

In order to construct the Padé–Chebyshev approximant of f(x) of order $[M/N]$, $M \geqslant N \geqslant 0$, according to Theorem 1, it is necessary to construct the biorthogonal polynomial

$$B_N(t) = t^{M+1}\sum_{i=0}^{N} c_i^{(N)} (t^i + t^{-i}), \tag{18}$$

satisfying the conditions

$$\int_{\alpha}^{\beta} t^i B_N(t)\, d\mu(t) = 0, \quad i = \overline{0, N-1}. \tag{19}$$

It is easily seen that the polynomial $B_N(t)/t^{M+1}$ is an algebraic polynomial of degree N in the variable t + 1/t. Let us denote it by $U_N(x)$. Thus,

$$\frac{B_N(t)}{t^{M+1}} = U_N\left(t + \frac{1}{t}\right). \tag{20}$$

So, biorthogonalization (18), (19) reduces in obvious manner to the biorthogonalization of the systems of functions $\{t^k\}_{k=0}^{N}$ and $\{t^{M+1}(t + 1/t)^i\}_{i=0}^{N}$ with respect to the measure $d\mu(t)$ on the segment $[\alpha, \beta]$.

Since both the systems of functions are Chebyshev on [−1, 1], nondegenerate biorthogonalization is possible in the present situation. Moreover, in this connection the polynomial $U_N(t + 1/t)$ has exactly N simple zeros in (α, β) [7]. Let us now consider

$$U_N(2z) = \sum_{i=0}^{N} c_i^{(N)} [(z - \sqrt{z^2 - 1})^i + (z + \sqrt{z^2 - 1})^i] = 2\sum_{j=0}^{N} c_j^{(N)} T_j(z) = 2Q_N(z). \tag{21}$$

Thus, the denominator for the Padé–Chebyshev approximant has the form

$$Q_N(z) = \frac{1}{2} U_N(2z). \tag{22}$$

Since $U_N(t + 1/t)$ has N zeros in (α, β), we conclude that all the zeros of the denominator $Q_N(z)$ lie in the interval $(\beta + 1/\beta, \alpha + 1/\alpha)$ if $0 < \alpha < \beta \leqslant 1$, on the ray $(\beta + 1/\beta, +\infty)$ if $0 = \alpha < \beta \leqslant 1$, on the ray $(-\infty, \alpha + 1/\alpha)$ if $-1 \leqslant \alpha < \beta = 0$, and on the union of rays $(-\infty, \alpha + 1/\alpha) \cup (\beta + 1/\beta, +\infty)$ if $-1 \leqslant \alpha < 0 < \beta < 1$.

Summing up all the above arguments, we formulate the following theorem.

THEOREM 2. Let a function f(x) be represented in the form

$$f(x) = \int_{\alpha}^{\beta} \frac{1 - xt}{1 - 2xt + t^2} d\mu(t), \tag{23}$$

where $\mu(t)$ is a function that is nondecreasing and has infinite number of growth points on $[\alpha, \beta] \subset [-1, 1]$. Then the Padé-Chebyshev approximant of f(x) of order $[M/N]$, $M \geqslant N \geqslant 0$, is a function that is analytic in the real domain of analyticity of f(x) and can be represented in the form

$$[M/N]_f^T(x) = P_M(x)/Q_N(x), \tag{24}$$

where

$$Q_N(x) = \frac{1}{2} U_N(2x), \tag{25}$$

$$P_M(x) = \frac{1}{2} \int_{\alpha}^{\beta} \left[\frac{(1 - xt)(U_N(2x) - U_N(t + 1/t))}{1 - 2xt + t^2} - \sum_{k=0}^{M} t^k T_k(x) U_N(t + 1/t) \right] d\mu(t), \tag{26}$$

and the algebraic polynomials $U_N(t)$ are defined by the biorthogonality relations

$$\int_{\alpha}^{\beta} t^{M+1+i} U_N(t + 1/t)\, d\mu(t) = 0, \quad i = 0, 1, \ldots, N-1. \tag{27}$$

Proof. Using Eqs. (23) and (25), we write

$$f(x) Q_N(x) = Q_N(x) \int_{\alpha}^{\beta} \frac{1 - xt}{1 - 2xt + t^2} d\mu(t) = Q_N(x) \left[\int_{\alpha}^{\beta} \left(\frac{1 - xt}{1 - 2xt + t^2} - \right.\right.$$

$$\left.\left. - \sum_{k=0}^{M} t^k T_k(x) \right) d\mu(x) + \sum_{k=0}^{M} s_k T_k(x) \right] = \frac{1}{2} \int_{\alpha}^{\beta} \left(\frac{1 - xt}{1 - 2xt + t^2} - \right.$$

$$\left. - \sum_{k=0}^{M} t^k T_k(x) \right) U_N(2x)\, d\mu(t) + Q_N(x) \sum_{k=0}^{M} s_k T_k(x) = \frac{1}{2} \times$$

$$\times \int_{\alpha}^{\beta} \left(\frac{1 - xt}{1 - 2xt + t^2} - \sum_{k=0}^{M} t^k T_k(x) \right) [U_N(2x) - U_N(t + 1/t)]\, d\mu(t) +$$

$$+ \frac{1}{2} \int_{\alpha}^{\beta} \left(\frac{1 - xt}{1 - 2xt + t^2} - \sum_{k=0}^{M} t^k T_k(x) \right) U_N(t + 1/t)\, d\mu(t) + Q_N(x) \times$$

$$\times \sum_{k=0}^{M} s_k T_k(x) = \frac{1}{2} \int_{\alpha}^{\beta} \left[\frac{(1 - xt)[U_N(2x) - U_N(t + 1/t)]}{1 - 2x + t^2} - \sum_{k=0}^{M} t^k T_k(x) U_N(t + 1/t) \right] d\mu(t) + R_{M+N+1}(x).$$

Hence the theorem follows.

LITERATURE CITED

1. N. I. Akhiezer, The Classical Moments Problem and Certain Questions of Analysis Connected with Them [in Russian], Nauka, Moscow (1961).
2. H. Stahl, "Orthogonal polynomials of complex-valued meaures and the convergence of Padé approximants," in: Fourier Analysis and Approximation Theory, Vol. II, North-Holland, Amsterdam (1978), pp. 771-788.
3. J. Gilewicz and E. Leopold, "Location of the zeros of polynomials satisfying three term recurrence relations. I. General case with complex coefficients," J. Approx. Theory, 43, No. 1, 1-14 (1985).
4. V. K. Dzyadyk, "On a generalization of the moments problem," Dokl. Akad. Nauk UkrSSR, Ser. A., No. 6, 8-12 (1981).
5. V. K. Dzyadyk, Approximation Methods for the Solution of Differential and Integral Equations [in Russian], Naukova Dumka, Kiev (1988).

6. G. A. Baker, Jr. and P. R. Graves-Morris, Padé Approximations, Parts I and II, Addison-Wesley, Reading, Mass. (1981).
7. A. P. Golub, Generalized Moment Representations and Rational Approximations [in Russian], Kiev (1987). (Preprint No. 87.25, Inst. Mat., Akad. Nauk UkrSSR.)

DIFFERENTIAL INVARIANTS OF A EUCLIDEAN ALGEBRA

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Functional bases of second-order differential invariants of a Euclidean algebra and a conformal algebra are found for a set of scalar functions depending on n variables.

1. INTRODUCTION AND BASIC RESULTS

In the present article we construct explicitly the functional bases of second-order differential invariants of Euclidean algebra AE(n) with basis operators

$$\partial_a = \partial/\partial x_a, \quad J_{ab} = x_a\partial_b - x_b\partial_a \tag{1}$$

for a set of m scalar functions. (Here and in the following, the letters a, b, c, and d used as indices take values from 1 to n, where n is a collection of spatial variables, $n \geqslant 3$.)

Algebra AE(n) (1) is an algebra of invariance of a wide class of multidimensional equations of mathematical physics [1].

Definition 1 [2, 3]. The function

$$F(x, u, \underset{1}{u}, \ldots, \underset{l}{u}), \tag{2}$$

where $x = (x_1, \ldots, x_n)$, $u = (u^1, \ldots, u^m)$, $\underset{l}{u}$ is the set of all partial derivatives of the functions of ℓ-th order, is called a differential invariant of a Lie algebra with basis operators X_i if it is an invariant of the ℓ-the extension of this algbera

$$\overset{l}{X}_i F(x, u, \underset{1}{u}, \ldots, \underset{l}{u}) = \lambda_i (x, u, \underset{1}{u}, \ldots, \underset{l}{u}) F, \tag{3}$$

where if $\lambda_i \equiv 0$, F is called an absolute differential invariant, and if $\lambda_i \not\equiv 0$, F is called a relative differential invariant.

In the following, we shall call absolute invariants simply invariants.

Definition 2. The set of functionally independent invariants of order $r \leqslant l$ of Lie algebra $\{X_i\}$ through which it is possible to express any of its invariants of order $r \leq \ell$ is called a functional basis of order ℓ of algebra $\{X_i\}$.

We shall use the following notation:

$$u_a = \partial u/\partial x_a, \quad u_{ab} = \partial^2 u/\partial x_a \partial x_b,$$
$$S_k(u_{ab}) = u_{a_1a_2}u_{a_2a_3} \ldots u_{a_{k-1}a_k}u_{a_ka_1},$$
$$S_{jk}(u_{ab}, v_{ab}) = u_{a_1a_2} \ldots u_{a_{j-1}a_j}v_{a_ja_{j+1}} \ldots v_{a_ka_1},$$
$$R_k(u_a, u_{ab}) = u_{a_1}u_{a_k}u_{a_1a_2}u_{a_2a_3} \ldots u_{a_{k-1}a_k}.$$

Here and in the following, the repeated indices will signify summation from 1 to n. Everywhere in lists of invariants, k will take values from 1 to n, j from 0 to k.

We shall state the basic results of the article in the form of theorems.

THEOREM 1. A functional basis of second-order differential invariants of Euclidean AE(n) with basis operators (2) for a scalar function $u = u(x_1, \ldots, x_n)$ consists of 2n + 1

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