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SOME PROPERTIES OF BIORTHOGONAL POLYNOMIALS AND THEIR APPLICATION TO PADÉ APPROXIMATIONS

A. P. Golub

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Transformations of biorthogonal polynomials under certain transformations of biorthogonalizable sequences are studied. The obtained result is used to construct Padé approximants of orders $[N-1/N]$, $N \in \mathbb{N}$, for the functions

$$\tilde{f}(z) = \sum_{m=0}^{M} \alpha_m \frac{f(z) - T_{m-1}[f;z]}{z^m},$$

where $f(z)$ is a function with known Padé approximants of the indicated orders, $T_j[f;z]$ are Taylor polynomials of degree j for the function $f(z)$, and α_m, $m = \overline{1, M}$, are constants.

1. Introduction

One of the methods for construction and investigation of Padé approximations is the method of generalized moment representations suggested by Dzyadyk in 1981 [1].

Definition 1. *For a numerical sequence* $\{s_k\}_{k=0}^{\infty}$ *or for a function representable as a power series*

$$f(z) = \sum_{k=0}^{\infty} s_k z^k,$$

a two-parametric set of equalities

$$s_{k+j} = l_j(x_k), \quad k, j = \overline{0, \infty}, \tag{1}$$

with $x_k \in X$, $k = \overline{0, \infty}$, *and* $l_j \in X^*$, $j = \overline{0, \infty}$, *is called its generalized moment representation in a Banach space* X.

The application of the method of generalized moment representations to the problem of finding Padé approximants encounters serious difficulty connected with the necessity to construct and investigate biorthogonal polynomials.

Definition 2. *Sequences of generalized polynomials*

$$L_M = \sum_{j=0}^{M} c_j^{(M)} l_j, \quad M = \overline{0, \infty},$$

and

$$X_N = \sum_{k=0}^{N} c_k^{(N)} x_k, \quad N = \overline{0, \infty},$$

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in the systems of functions that appear in equalities (1) are called biorthogonal if

$$L_M(X_N) = 0 \quad \textit{for } M \neq N. \tag{2}$$

Some properties of biorthogonal polynomials were studied in [2–4].

However, the difficulty indicated above can be overcome in many cases where the role of biorthogonal polynomials is played by ordinary orthogonal polynomials, whose properties are well studied. Let us illustrate this by examples.

Example 1 [5]. For the sequence $s_k = 1/(k+1)!$ [or the function $f(z) = (\exp z - 1)/z$], the following generalized moment representation is true:

$$s_{k+j} = \int_0^1 a_k(t) b_j(t) d\mu(t); \tag{3}$$

here, $a_k(t) = t^k/k!$, $k = \overline{0, \infty}$, $b_j(t) = (1-t)^j/j!$, $j = \overline{0, \infty}$, and $d\mu(t) = dt$. Since $\deg a_k(t) \equiv k$ and $\deg b_j(t) \equiv j$, the biorthogonalization of these sequences yields the classical Legendre polynomials shifted by the interval [0, 1] (see [6, p. 116]).

Example 2 [5]. For the sequence $s_k = (\kappa + \nu + 1)_k/(\nu+1)_k$, $k = \overline{0, \infty}$, where $(\alpha)_k := \alpha(\alpha+1)\ldots(\alpha+k-1)$, $k = \overline{1, \infty}$, $(\alpha)_0 := 1$ [or for the function $f(z) = {}_2F_1(\kappa+\nu+1, 1; \nu+2; z)/(\nu+1)$ with $\nu > -1$, $\kappa + \nu + 1 \notin \mathbb{Z}^-$, and $1 - \kappa \notin \mathbb{Z}^-$], a generalized moment representation of the form (3) is also true with

$$a_k(t) = \frac{(\kappa+\nu+1)_k}{(\nu+1)_k} t^k, \quad k = \overline{0, \infty},$$

$$b_j(t) = \frac{(1-\kappa)_j}{j!} t^j + \frac{(\kappa)_j}{j!} \sum_{m=0}^{j-1} \frac{(1-\kappa)_m}{m!} t^m, \quad j = \overline{0, \infty},$$

$$d\mu(t) = t^\nu dt.$$

In this case, the biorthogonalization leads to the shifted classical Jacobi polynomials, which are orthogonal on [0, 1] with the weight $\omega(t) = t^\nu$ (see [6, p. 268]).

Example 3 [7]. Consider the sequence $s_k = 1/(k+1)_q!$, $k = \overline{0, \infty}$, where $k_q = (1-q^k)/(1-q)$, $k = \overline{1, \infty}$, and

$$k_q! := \prod_{i=1}^{k} i_q, \quad k = \overline{1, \infty}, \quad 0_q! := 1.$$

The elements of this sequence are the coefficients of the power expansion of the function called a q-analog of the exponential [8]; this function is a special case of the basic hypergeometric series [9, pp. 195–196], $0 < q < 1$. For the sequence considered, the following generalized moment representation is true:

$$s_{k+j} = \int_0^1 a_k(t) b_j(t) d_q t; \tag{4}$$

here, $a_k(t) = t^k / k_q!$, $k = \overline{0, \infty}$, and

$$b_j(t) = \prod_{n=1}^{j} (1 - tq^n) / j_q!, \quad j = \overline{0, \infty}.$$

The integral on the right-hand side of (4) is defined by the equality

$$\int_0^1 \varphi(t) d_q t := (1-q) \sum_{n=0}^{\infty} \varphi(q^n) q^n$$

and called the Jackson q-integral [10].

In this case, by biorthogonalizing the sequences $\{a_k(t)\}_{k=0}^{\infty}$ and $\{b_j(t)\}_{j=0}^{\infty}$, we obtain orthogonal polynomials in a discrete variable, which are a generalization of the classical Legendre polynomials (see, e.g., [11]).

Below, we show how the situations described above and similar ones can be used for constructing biorthogonal polynomials in more complicated cases.

2. Principal Result

First, we need a modification of representation (1) (see [12]). In a Banach space X, we consider a linear bounded operator $A: X \to X$ such that $A x_k = x_{k+1}$, $k = \overline{0, \infty}$. It is easy to see that its adjoint operator A^*: $X^* \to X^*$ acts so that $A^* l_j = l_{j+1}$, $j = \overline{0, \infty}$. In this case, we can rewrite (1) as

$$s_{k+j} = A^{*j} l_0(A^k x_0), \quad k, j = \overline{0, \infty},$$

or, equivalently,

$$s_k = l_0(A^k x_0), \quad k = \overline{0, \infty}.$$

Under the assumption that the biorthogonalization is known for some fixed $x_0 \in X$ and $l_0 \in X^*$, we construct biorthogonal polynomials for the case where the functional l_0 is replaced by a functional $\tilde{l}_0$ representable in the form

$$\tilde{l}_0 = \prod_{m=1}^{M} (1 + \beta_m A^*) l_0 = \sum_{m=0}^{M} \alpha_m A^{*m} l_0. \tag{5}$$

The following assertion is true:

Theorem 1. *Assume that the sequence* $\{X_k\}_{k=0}^{\infty}$ *of generalized polynomials*

$$X_k = \sum_{i=0}^{k} c_i^{(k)} A^i x_0 = P_k(A) x_0, \quad k = \overline{0, \infty}, \tag{6}$$

possesses the biorthogonal properties in the sense that

$$l_j(X_k) = \delta_{k,j}, \quad j = \overline{0, k}. \tag{6'}$$

Then, for any $N = \overline{0, \infty}$, *a nontrivial polynomial* $\tilde{X}_N$ *of the form*

$$\tilde{X}_N = \sum_{i=0}^{N} \tilde{c}_i^{(N)} A^i x_0$$

with the biorthogonal properties

$$\tilde{l}_j(\tilde{X}_N) = 0, \quad j = \overline{0, N-1}, \quad \tilde{l}_j = A^{*j} \tilde{l}_0, \tag{7}$$

where $\tilde{l}_0$ *is given by (5), can be represented as follows:*

$$\tilde{X}_N = \prod_{m=1}^{M} (1 + \beta_m A)^{-1} \sum_{k=N}^{M+N} \gamma_k X_k.$$

Here, the coefficients γ_k, $k = \overline{N, M+N}$, *are determined by the homogeneous system of linear algebraic equations*

$$\sum_{k=N}^{M+N} \gamma_k P_k^{(n)}\left(-\frac{1}{\beta_m}\right) = 0, \quad n = \overline{0, r_{m-1}}, \quad m = \overline{1, M^*},$$

where M^* *is the number of different numbers* β_m, $m = \overline{1, M}$, *and* r_m *is the multiplicity of* β_m, $m = \overline{1, M^*}$.

Proof. Properties (7) imply that

$$l_j\left(\prod_{m=1}^{M} (1 + \beta_m A) \tilde{X}_N\right) = 0, \quad j = \overline{0, N-1}.$$

Clearly, the representation

$$\Phi_N := \prod_{m=1}^{M} (1 + \beta_m A) \tilde{X}_N = \sum_{k=0}^{M+N} \gamma_k X_k \tag{8}$$

is possible. Further, we have

$$\tilde{l}_j(\tilde{X}_N) = \tilde{l}_j(\Phi_N) = l_j\left(\sum_{k=0}^{M+N} \gamma_k X_k\right) = \gamma_j + l_j\left(\sum_{k=0}^{j-1} \gamma_k X_k\right) = 0.$$

Therefore, $\gamma_j = 0$, $j = \overline{0, N-1}$. Thus,

$$\Phi_N = \sum_{k=N}^{M+N} \gamma_k X_k.$$

Taking (8) into account, we obtain

$$\tilde{X}_N = \prod_{m=1}^{M} (1 + \beta_m A)^{-1} \sum_{k=N}^{M+N} \gamma_k X_k.$$

By virtue of the expansion

$$(1+\beta A)^{-1} = \sum_{l=0}^{\infty} (-\beta)^l A^l,$$

we have

$$\tilde{X}_N = \sum_{l_1,\ldots,l_M=0}^{\infty} (-\beta_1)^{l_1} \ldots (-\beta_M)^{l_M} A^{l_1+\ldots+l_M} \sum_{k=N}^{M+N} \gamma_k \sum_{i=0}^{k} c_i^{(k)} A^i x_0. \tag{9}$$

Since the elements $x_k = A^k x_0$, $k = \overline{0,\infty}$, are linearly independent [otherwise, the nondegenerate biorthogonalization (6′) is impossible], their coefficients on the both sides of (9) can be equated. In particular, the coefficients of x_i, $i = \overline{N+1,\infty}$, on the right-hand side of (9) must be zero. Assuming that $M \geq 1$ (the case of $M = 0$ is of no interest), we equate the coefficients of x_{N+M+j}, $j = \overline{0,M}$, on the right-hand side of (9) with zero. As a result, we get

$$\sum_{k=N}^{M+N} \gamma_k \sum_{i=0}^{k} c_i^{(k)} \sum_{l_1+\ldots+l_M = N+M+j-i} (-\beta_1)^{l_1} \ldots (-\beta_M)^{l_M} = 0, \quad j = \overline{0,M}. \tag{10}$$

Denote

$$F_p^{(M)}(y_1,\ldots,y_M) := \sum_{l_1+\ldots+l_M=p} y_1^{l_1} \ldots y_M^{l_M}. \tag{11}$$

It is easy to see that

$$F_{p+1}^{(M)}(y_1,\ldots,y_M) = y_1 F_p^{(M)}(y_1,\ldots,y_M) + F_{p+1}^{(M-1)}(y_2,\ldots,y_M). \tag{12}$$

In view of (11), we can rewrite equalities (10) as follows:

$$\sum_{k=N}^{M+N} \gamma_k \sum_{i=0}^{k} c_i^{(k)} F_{N+M+j-i}^{(M)}(-\beta_1,\ldots,-\beta_M) = 0, \quad j = \overline{0,M}. \tag{13}$$

By multiplying each jth equality in (13) by $-\beta_1$, $j = \overline{0,M-1}$, subtracting it from the $(j+1)$th one, and taking (12) into account, we obtain

$$\sum_{k=N}^{M+N} \gamma_k \sum_{i=0}^{k} c_i^{(k)} F_{N+M+j-i}^{(M-1)}(-\beta_2,\ldots,-\beta_M) = 0, \quad j = \overline{0,M-1}.$$

By continuing this procedure, we arrive at the equality

$$\sum_{k=N}^{M+N} \gamma_k \sum_{i=0}^{k} c_i^{(k)} F_{N+M-i}^{(1)}(-\beta_M) = 0,$$

which, in view of (11), can be rewritten in the form

$$\sum_{k=N}^{M+N} \gamma_k \sum_{i=0}^{k} c_i^{(k)} (-\beta_M)^{-i} = 0$$

or

$$\sum_{k=N}^{M+N} \gamma_k P_k\left(-\frac{1}{\beta_M}\right) = 0.$$

Since conditions (13) are symmetric with respect to $\beta_1, \ldots, \beta_M$, we have

$$\sum_{k=N}^{M+N} \gamma_k P_k\left(-\frac{1}{\beta_m}\right) = 0, \quad m = \overline{1, M}. \tag{14}$$

We now assume that the multiplicity of some numbers β_m, $m = \overline{1, M}$, is greater than one. For example, assume that a number β has multiplicity r. In this case, we consider a perturbed problem with r different values β, $\beta/(1-\beta h), \ldots, \beta/[1-(r-1)\beta h]$ instead of multiple β; moreover, we assume that $h > 0$ is so small that no one of these values coincides with the other numbers β_m. As a result, we obtain conditions of the form (14), namely,

$$\sum_{k=N}^{M+N} \gamma_k P_k\left(-\frac{1}{\beta} + jh\right) = 0, \quad j = \overline{0, r-1}.$$

This implies that the divided differences are also equal to zero and, hence, the corresponding derivatives satisfy the relation

$$\sum_{k=N}^{M+N} \gamma_k P_k^{(j)}\left(-\frac{1}{\beta}\right) = 0, \quad j = \overline{0, r-1},$$

as $h \to 0$. Theorem 1 is proved.

3. Application to Padé Approximations

Recall the following definition [13, p. 31]:

Definition 3. *A rational polynomial* $[M/N]_f(z) = P_M(z)/Q_N(z)$, *where* $P_M(z)$ *and* $Q_N(z)$ *are algebraic polynomials of degrees* M *and* N, *respectively, is called the Padé approximant of order* $[M/N]$, $M, N \in \mathbb{Z}^+$, *for a function* $f(z)$ *analytic in a neighborhood of the point* $z = 0$ *if*

$$f(z) - [M/N]_f(z) = O(z^{M+N+1}), \quad z \to 0.$$

Theorem 1 enables one to construct Padé approximants of orders $[N-1/N]$, $N \in \mathbb{N}$, for functions of the form

$$\tilde{f}(z) = \sum_{m=0}^{M} \alpha_m \frac{f(z) - T_{m-1}[f; z]}{z^m},$$

where $f(z)$ is a function for which the Padé approximants of indicated orders are known, and $T_j[f; z]$ are Taylor polynomials of degree j for the function $f(z)$. In particular, the following assertion is true:

Theorem 2. *Assume that the Padé approximants of orders* $[N+m-1/N+m]$, $m = \overline{1, M}$, *are known for a function* $f(z)$ *and let*

$$\Psi_M(t) = \prod_{m=1}^{M^*} (1 + \beta_m t)^{r_m} = \sum_{m=0}^{M} \alpha_m t^m$$

be a polynomial of degree M. *Then the denominator* $\tilde{Q}_N(z)$ *of the Padé approximant of order* $[N-1/N]$ *for the function*

$$\tilde{f}(z) = \sum_{m=0}^{M} \alpha_m \frac{f(z) - T_{m-1}[f; z]}{z^m}$$

can be represented in the form

$$\tilde{Q}_N(z) = \frac{C}{z^M \Psi_M(1/z)} \det U_M(z), \tag{15}$$

where the matrix $U_M(z) = \| u_{k,j} \|_{k,j=1}^{M}$ *is composed of the elements*

$$u_{k,j} = \frac{d^i}{dw^i}\{w^i Q_{N+j}(w)\}\Big|_{w=-\beta m}, \tag{16}$$

$$k = \overline{1, M-1}, \quad j = \overline{1, M}, \quad m = \max\left\{ l\colon \sum_{p=1}^{l-1} r_p \le k \right\}, \quad i = k - \sum_{p=1}^{m} r_p,$$

$$u_{M,j} = z^j Q_{N+j}(z), \quad j = \overline{1, M}.$$

Here, $Q_{N+j}(z)$, $j = \overline{1, M}$, *are the denominators of the Padé approximants of orders* $[N+j-1/N+j]$ *for the function* $f(z)$, *and* $C = \text{const}$.

Proof. Consider the case where all β_m, $m = \overline{1, M^*}$, are different, i.e., $r_m = 1$, $m = \overline{1, M^*}$, and $M^* = M$. It is known [1] that the denominator of the Padé approximant of order $[N-1/N]$ for the function $\tilde{f}(z)$ can be represented as follows:

$$\tilde{Q}_N(z) = \sum_{k=0}^{N} \tilde{c}_k^{(N)} z^{N-k};$$

here, $\tilde{c}_k^{(N)}$, $k = \overline{0, N}$, are the coefficients of the biorthogonal polynomial

$$\tilde{X}_N = \sum_{i=0}^{N} \tilde{c}_i^{(N)} A^i x_0.$$

It follows from Theorem 1 that these coefficients satisfy the relations

$$\tilde{c}_k^{(N)} = \sum_{m=N}^{M+N} \gamma_m \sum_{i=0}^{k} c_i^{(m)} \sum_{l_1+\ldots+l_m=k-i} (-\beta_1)^{l_1} \ldots (-\beta_M)^{l_M},$$

where γ_m, $m = \overline{N, M+N}$, are determined by the system of linear algebraic equations

$$\sum_{k=N}^{M+N} \gamma_k P_k\left(-\frac{1}{\beta_m}\right) = 0, \quad m = \overline{1, M}. \tag{17}$$

Thus,

$$\begin{aligned}\tilde{Q}_N(z) &= \sum_{k=0}^{N} z^{N-k} \sum_{m=N}^{M+N} \gamma_m \sum_{i=0}^{k} c_i^{(m)} \sum_{l_1+\ldots+l_M=k-1} (-\beta_1)^{l_1} \ldots (-\beta_M)^{l_M} \\ &= \sum_{m=N-}^{M+N} \gamma_m \sum_{i=0}^{N} c_i^{(m)} \sum_{k=i}^{N} z^{N-k} \sum_{l_1+\ldots+l_M=k-i} (-\beta_1)^{l_1} \ldots (-\beta_M)^{l_M} \\ &= \sum_{m=N}^{M+N} \gamma_m \sum_{i=0}^{N} c_i^{(m)} z^{N-i} \sum_{p=0}^{N-i} \sum_{l_1+\ldots+l_M=p} \left(-\frac{\beta_1}{z}\right)^{l_1} \ldots \left(-\frac{\beta_M}{z}\right)^{l_M}.\end{aligned} \tag{18}$$

It follows from (17) that

$$\sum_{k=N}^{M+N} \gamma_k \kappa_k = C \begin{vmatrix} Q_{N+M}(-\beta_1) & (-\beta_1) Q_{N+M-1}(-\beta_1) & \ldots & (-\beta_1)^M Q_N(-\beta_1) \\ \cdots & \cdots & \cdots & \cdots \\ Q_{N+M}(-\beta_M) & (-\beta_M) Q_{N+M-1}(-\beta_1) & \cdots & (-\beta_M)^M Q_N(-\beta_M) \\ \kappa_{N+M} & \kappa_{N+M-1} & \cdots & \kappa_N \end{vmatrix}$$

for any numbers κ_k, $k = \overline{N, M+N}$. Therefore, the polynomial determined by (15) and (16) can be rewritten in the form

$$\tilde{Q}(z) = \frac{1}{z^M \Psi_M(1/z)} \sum_{m=N}^{M+N} \gamma_m z^{N+M-m} Q_m(z). \tag{19}$$

Let us prove that equalities (18) and (19) define the same polynomial. For this purpose, it suffices to show that the difference between the right-hand sides of (18) and (19) is equal to zero. We have

$$\begin{aligned}&\sum_{m=N}^{M+N} \gamma_m \sum_{i=0}^{N} c_i^{(m)} z^{N-i} \sum_{p=0}^{N-i} \sum_{l_1+\ldots+l_m=p} \left(-\frac{\beta_1}{z}\right)^{l_1} \ldots \left(-\frac{\beta_M}{z}\right)^{l_M} - \frac{1}{z^M \Psi_M(1/z)} \sum_{m=N}^{M+N} \gamma_m z^{N+M-m} Q_m(z) \\ &= \sum_{m=N}^{M+N} \gamma_m \left\{ \sum_{i=0}^{N} c_i^{(N)} z^{N-i} \sum_{p=0}^{N-i} \sum_{l_1+\ldots+l_m=p} \left(-\frac{\beta_1}{z}\right)^{l_1} \ldots \left(-\frac{\beta_M}{z}\right)^{l_M} - \prod_{m=1}^{M} (z+\beta_m)^{-1} z^{N+M-m} Q_m(z) \right\}.\end{aligned} \tag{20}$$

The expression in braces can be transformed as follows:

$$\sum_{i=0}^{N} c_i^{(m)} z^{N-i} \sum_{p=0}^{N-i} \sum_{l_1+\ldots+l_M=p} \left(-\frac{\beta_1}{z}\right)^{l_1} \ldots \left(-\frac{\beta_M}{z}\right)^{l_M} - \prod_{m=1}^{M}\left(z+\frac{\beta_m}{z}\right)^{-1} z^{N-m} \sum_{i=0}^{m} c_i^{(m)} z^{m-i}$$

$$= -\sum_{i=0}^{N} c_i^{(m)} z^{N-i} \sum_{p=N-i+1} \sum_{l_1+\ldots+l_m=p} \left(-\frac{\beta_1}{z}\right)^{l_1} \ldots \left(-\frac{\beta_M}{z}\right)^{l_M}$$

$$- \sum_{i=N+1}^{m} c_i^{(m)} z^{N-i} \sum_{p=0}^{\infty} \sum_{l_1+\ldots+l_m=p} \left(-\frac{\beta_1}{z}\right)^{l_1} \ldots \left(-\frac{\beta_M}{z}\right)^{l_M}. \qquad (21)$$

The right-hand side of (21) contains only negative powers of z. Therefore, since the initial difference (20) is a polynomial of degree not higher than N, it is equal to zero.

This proves representation (15) in the case where all β_m are different. As in Theorem 1, the obtained result can easily be extended to the case of multiple β_m. Thus, Theorem 2 is proved.

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