SOME PROPERTIES OF BIORTHOGONAL POLYNOMIALS

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In studying Padé approximations of functions with the aid of generalized moment representations [1] one is obliged to construct and investigate various systems of biorthogonal polynomials. In the present paper we establish some general results concerning biorthogonal polynomials.

The starting point for obtaining these results is the following proposition, applied here in a somewhat more general form than in the primary source.

THEOREM 1 (Dzyadyk [1]). Let $\{s_k\}_{k=0}^{\infty}$ be a numerical sequence such that all its Hankel determinants are different from zero:

$$H_N = \det S_N = \det \| s_{i+j} \|_{i,j=0}^{N} \neq 0, \quad N = \overline{0, \infty},$$

and let us assume that in a Banach space X we are given the sequence $\{y_i\}_{i=0}^{\infty}$, and assume also that in the space X* conjugate to it we are given a sequence of functionals $\{l_j\}_{j=0}^{\infty}$, for which, moreover, we have the equations

$$l_j(y_i) = s_{i+j}, \quad i, j = \overline{0, \infty}.$$

Then if for some N = $\overline{0, \infty}$ we construct the generalized polynomials

$$Y_0 = \varepsilon_0 y_0, \quad Y_M = \varepsilon_M \begin{vmatrix} s_0 & s_1 & s_2 & \cdots & s_M \\ s_1 & s_2 & s_3 & \cdots & s_{M+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ s_{M-1} & s_M & s_{M+1} & \cdots & s_{2M-1} \\ y_0 & y_1 & y_2 & \cdots & y_M \end{vmatrix}, \quad M = \overline{1, \infty}, \tag{1}$$

$$L_0 = \varepsilon_0 l_0, \quad L_N = \varepsilon_N \begin{vmatrix} s_0 & s_1 & s_2 & \cdots & s_N \\ s_1 & s_2 & s_3 & \cdots & s_{N+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ s_{N-1} & s_N & s_{N+1} & \cdots & s_{2N-1} \\ l_0 & l_1 & l_2 & \cdots & l_N \end{vmatrix}, \quad N = \overline{1, \infty}, \tag{1'}$$

where $\varepsilon_N := \dfrac{1}{\sqrt{H_N H_{N-1}}}$, $N = \overline{0, \infty}$, $H_{-1} := 1$, we shall then have the biorthogonality relations

$$L_N(Y_M) = \delta_{M,N}, \quad M, N = \overline{0, \infty}.$$

For these biorthogonal systems of polynomials we construct three-term recursion relations under additional restrictions which prove to be sufficiently natural in problems of rational approximation.

THEOREM 2. Subject to the conditions of Theorem 1, let there exist a linear bounded operator A:X → X such that

$$A y_i = y_{i+1}, \quad i = \overline{0, \infty}.$$

Then the following recursion relations hold for the biorthogonal polynomials (1) and (1'):

$$A Y_M = \alpha_M Y_{M+1} + \gamma_M Y_M + \alpha_{M-1} Y_{M-1}, \quad M \geqslant 0,$$

$$A^* L_N = \alpha_N L_{N+1} + \gamma_N L_N + \alpha_{N-1} L_{N-1}, \quad N \geqslant 0,$$

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where $A^*:X^* \to X^*$ is an operator conjugate to A, $\alpha_M = \sqrt{H_{M-1}H_{M+1}}/H_M$, $M \geqslant 0$, $\alpha_{-1} := 0$, $\gamma_M = H_M/H_M + \tilde{H}_{M-1}/H_{M-1}$,

$$\tilde{H}_M := \begin{vmatrix} s_0 & s_1 & \dots & s_M \\ s_1 & s_2 & \dots & s_{M+1} \\ \cdot\ \cdot & \cdot\ \cdot & \dots & \cdot\ \cdot \\ \cdot\ \cdot & \cdot\ \cdot & \dots & \cdot\ \cdot \\ s_{M-1} & s_M & \dots & s_{2M-1} \\ s_{M+1} & s_{M+2} & \dots & s_{2M+1} \end{vmatrix}.$$

<u>Proof.</u> We write Y_M in the form

$$Y_M = \sum_{i=0}^{M} c_i^{(M)} y_i. \tag{2}$$

Applying operator A to relation (2), we obtain

$$AY_M = \sum_{i=0}^{M} c_i^{(M)} y_{i+1} = \sum_{i=0}^{M+1} d_i^{(M+1)} Y_i. \tag{3}$$

Similarly,

$$A^* L_N = \sum_{j=0}^{N+1} \tilde{d}_j^{(N+1)} L_j. \tag{4}$$

To determine the coefficients $d_i^{(M+1)}$, $i = \overline{0, M+1}$, $\tilde{d}_j^{(N+1)}$, $j = \overline{0, N+1}$, we apply to relation (3) the functionals L_N, $N = \overline{0, \infty}$. Obviously,

$$L_N(AY_M) = \begin{cases} 0 & \text{for} \quad N \geqslant M+2; \\ d_N^{(M+1)} & \text{for} \quad N \leqslant M+1. \end{cases}$$

On the other hand,

$$L_N(AY_M) = (A^* L_N)(Y_M) = \begin{cases} 0 & \text{for} \quad M \geqslant N+2; \\ \tilde{d}_M^{(N+1)} & \text{for} \quad M \leqslant N+1. \end{cases}$$

In relation (3) we equate coefficients of y_{M+1}:

$$c_M^{(M)} = d_{M+1}^{(M+1)} c_{M+1}^{(M+1)}.$$

Taking into account that, by virtue of Eq. (1),

$$c_M^{(M)} = \sqrt{\frac{H_{M-1}}{H_M}},$$

we obtain

$$d_{M+1}^{(M+1)} = \frac{c_M^{(M)}}{c_{M+1}^{(M+1)}} = \frac{\sqrt{H_{M-1}H_{M+1}}}{H_M}.$$

Similarly, equating coefficients of ℓ_{N+1} in Eq. (4), we find

$$\tilde{d}_{N+1}^{(N+1)} = \frac{\sqrt{H_{N-1}H_{N+1}}}{H_N}.$$

In ending the proof of the theorem it remains to establish the formula for γ_M, $M = \overline{0, \infty}$. Obviously,

$$\gamma_M = d_M^{(M+1)} = L_M(AY_M) = L_M\left(\sum_{i=0}^{M} c_i^{(M)} y_{i+1}\right) = L_M(c_M^{(M)} y_{M+1} + c_{M-1}^{(M)} y_M) = c_M^{(M)} L_M(y_{M+1}) + c_{M-1}^{(M)} L_M(y_M).$$

From relation (1) we have

$$c_M^{(M)} = \sqrt{\frac{H_{M-1}}{H_M}}, \quad c_{M-1}^{(M)} = \varepsilon_M \begin{vmatrix} s_0 & s_1 & \dots & s_{M-2} & s_M \\ s_1 & s_2 & \dots & s_{M-1} & s_{M+1} \\ \cdot\ \cdot & \cdot\ \cdot & \dots & \cdot\ \cdot & \cdot\ \cdot \\ \cdot\ \cdot & \cdot\ \cdot & \dots & \cdot\ \cdot & \cdot\ \cdot \\ s_{M-1} & s_M & \dots & s_{2M-3} & s_{2M-1} \end{vmatrix}.$$

It is easy to show that

$$L_M(y_{M+1}) = \varepsilon_M \begin{vmatrix} s_0 & s_1 & \dots & s_M \\ s_1 & s_2 & \dots & s_{M+1} \\ \cdot\ \cdot & \cdot\ \cdot & \dots & \cdot\ \cdot \\ \cdot\ \cdot & \cdot\ \cdot & \dots & \cdot\ \cdot \\ s_{M-1} & s_M & \dots & s_{2M-1} \\ s_{M+1} & s_{M+2} & \dots & s_{2M+1} \end{vmatrix} = \varepsilon_M \tilde{H}_M .$$

$$L_M(y_M) = \varepsilon_M H_M.$$

We obtain

$$\gamma_M = \sqrt{\frac{H_{M-1}}{H_M}}\,\frac{1}{\sqrt{H_M H_{M-1}}}\,\tilde{H}_M + \sqrt{\frac{H_M}{H_{M-1}}}\,\frac{1}{\sqrt{H_M H_{M-1}}}\,\tilde{H}_{M-1} = \frac{\tilde{H}_M}{H_M} + \frac{\tilde{H}_{M-1}}{H_{M-1}},$$

which is the required result.

Remark. A similar result was obtained in [2, p. 358] in a somewhat narrower formulation.

In applications of systems of biorthogonal polynomials to rational approximation an important question is that of finding criteria of nondegenerate biorthogonality more effective than the nonvanishing of the Hankel determinant. The following theorem is a partial answer to this question:

THEOREM 3. Let $\{s_k\}_{k=0}^{\infty}$ be a numerical sequence for which generalized moment representations of the form

$$s_{i+j} = \int_0^1 a_i(t)\, b_j(t)\, d\mu(t) = (a_i, b_j)_{L_2([0,1],d\mu(t))}, \quad i, j = \overline{0, \infty},$$

have been constructed, where $\{a_i(t)\}_{i=0}^{\infty}$, $\{b_j(t)\}_{j=0}^{\infty}$ are sequences of continuous functions and $\mu(t)$ is a nondecreasing function having an infinite number of points of increase. Then in order that $\forall M, N = \overline{0, \infty}$ there exist polynomials of the form

$$A_M(t) = \sum_{i=0}^{M} c_i^{(M)} a_i(t), \quad c_M^{(M)} \neq 0,$$
$$B_N(t) = \sum_{j=0}^{N} c_j^{(N)} b_j(t), \quad c_N^{(N)} \neq 0, \tag{5}$$

possessing properties of biorthogonality

$$\int_0^1 A_M(t)\, B_N(t)\, d\mu(t) = \delta_{M,N}, \quad M, N = \overline{0, \infty},$$

it is necessary and sufficient that there exist two sequences T_k, of U_k continuous operators $L_2([0,1], d\mu(t)) \to L_2([0,1], d\mu(t))$ such that

$$(T_k a_i, U_k b_j)_{L_2} = (a_i, b_j)_{L_2} = s_{i+j}, \quad i, j = \overline{0, k},$$

and such that the systems of functions $\{(T_k a_i)(t)\}_{i=0}^{M}$, $\{(U_k b_j)(t)\}_{j=0}^{N}$ are Chebyshevian on [0, 1] for all $M, N = \overline{0, k}$, $k = \overline{0, \infty}$.

Proof. Necessity. Let us assume that the polynomials (5) exist. Then it is readily seen that the determinants $H_N = \det \| s_{i+j} \|_{i,j=0}^{N}$, $N = \overline{0, \infty}$ are different from zero. But in this case, according to Theorem 1.1 of [3], we can construct generalized moment representations of the form

$$s_{i+j} = \int_0^1 \tilde{a}_i(t)\, \tilde{b}_j(t)\, d\mu(t), \quad i, j = \overline{0, \infty},$$

where $\tilde{a}_i(t)$, $\tilde{b}_j(t)$ are algebraic polynomials of degrees exactly i and j, respectively. Therefore, if we construct finite-dimensional linear operators with the properties

$$T_k a_i = \tilde{a}_i, \quad i = \overline{0, k},$$

$$T_k\varphi = 0, \text{ if } \quad \varphi \perp a_i, \; i = \overline{0, k},$$

$$U_k b_j = \tilde{b}_j, \quad j = \overline{0, k},$$

$$U_k\psi = 0, \text{ if } \quad \psi \perp b_j, \quad j = \overline{0, k},$$

they will then satisfy all the conditions of the theorem.

Sufficiency. Assume that the systems of functions $\{(T_k a_i)(t)\}_{i=0}^M$ and $\{(U_k b_j)(t)\}_{j=0}^N$ are Chebyshevian. We show that in this case we can biorthogonalize them, in particular, we can show that $\forall M, N = \overline{0, k}$, $k = \overline{0, \infty}$, there exist polynomials

$$A_M^T(t) = \sum_{i=0}^{M} c_i^{(M)} (T_k a_i)(t),$$
$$B_N^U(t) = \sum_{j=0}^{N} d_j^{(N)} (U_k b_j)(t), \tag{6}$$

having exactly M and N simple zeros on (0, 1), respectively, and such that

$$\int_0^1 A_M^T(t)\, B_N^U(t)\, d\mu(t) = \delta_{M,N}. \tag{7}$$

It is obvious that $\forall M = \overline{0, \infty}$ we can construct a polynomial $\tilde{A}_M^T(t)$, not identically zero, possessing the properties

$$\int_0^1 \tilde{A}_M^T(t)\, (U_k b_j)(t)\, d\mu(t) = 0, \quad j = \overline{0, M-1}. \tag{8}$$

We assume that $\tilde{A}_M^T$ has m < M zeros on (0, 1). Then, using the Chebyshev properties of the system of functions $\{(U_k b_j)(t)\}_{j=0}^m$ [4, p. 21], we can construct a polynomial $R_m(t)$ with respect to this system having simple zeros at the zeros of $\tilde{A}_M^T(t)$. But then the sign of the product $\tilde{A}_M^T(t) R_m(t)$ will be preserved on [0, 1] and, hence, $\int_0^1 \tilde{A}_M^T(t)\, R_m(t)\, d\mu(t) \neq 0$, which contradicts Eq. (8).

Thus, $\tilde{A}_M^T(t)$ has m simple zeros on (0, 1) and hence $c_M^{(M)} \neq 0$ in Eq. (6). Moreover,

$$\int_0^1 \tilde{A}_M^T(t)\, (U_k b_M)(t)\, d\mu(t) \neq 0,$$

since otherwise we would be able to construct a polynomial $R_M(t)$ with respect to the system of functions $\{(U_k b_j)(t)\}_{j=0}^M$, having simple zeros at the zeros of $\tilde{A}_M^T(t)$ and again arrive at a contraction.

Similarly, we can construct a polynomial $\tilde{B}_N^T(t)$, not identically zero, such that

$$\int_0^1 (T_k a_i)(t)\, \tilde{B}_N^T(t)\, d\mu(t) = 0, \quad i = \overline{0, N-1}.$$

Following this, in order to obtain polynomials $A_M^T(t)$, $B_N^U(t)$, satisfying Eq. (7) it is sufficient to carry out a normalization of polynomials $\tilde{A}_M^T(t)$ and $\tilde{B}_N^U(t)$. It follows from Eq. (7) that the determinants $H_N = \det \| s_{i+j} \|_{i,j=0}^N$, $N = \overline{0, k}$, $k = \overline{0, \infty}$, are different from zero; hence it is easy to conclude that polynomials (5) actually exist. This completes proof of the theorem.

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