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GENERALIZED MOMENT REPRESENTATIONS AND PADÉ APPROXIMANTS ASSOCIATED WITH BILINEAR TRANSFORMATIONS

Using the method of generalized moment representations [1] with operator of bilinear transformation of independent variable Padé approximants of orders $[N-1/N]$, $N \geq 1$, are constructed for some special functions.

1^0. **Introduction.** V.K. Dzyadyk [1] in 1981 had proposed the method of generalized moment representations allowing to construct and to investigate rational Padé approximants for a number of elementary and special functions.

Definition 1. *We shall call by generalized moment representation of the numerical sequence $\{s_k\}_{k=0}^{\infty}$ on the product of linear spaces $\mathscr{X}$ and $\mathscr{Y}$ the two-parameter collection of equalities*

$$s_{k+j} = \langle x_k, y_j \rangle, \quad k, j = \overline{0, \infty}, \tag{1}$$

where $x_k \in \mathscr{X}$, $k = \overline{0, \infty}$, $y_j \in \mathscr{Y}$, $j = \overline{0, \infty}$, and $\langle ., . \rangle$ - bilinear form defined on $\mathscr{X} \times \mathscr{Y}$.

In the case when linear operator $A : \mathscr{X} \to \mathscr{Y}$ exists such that

$$Ax_k = x_{k+1}, \quad k = \overline{0, \infty},$$

and in the space $\mathscr{Y}$ linear operator $A^* : \mathscr{Y} \to \mathscr{Y}$ exists such that

$$\langle Ax, y \rangle = \langle x, A^* y \rangle, \quad \forall x \in \mathscr{X}, \quad \forall y \in \mathscr{Y},$$

(we shall call operator A^* as conjugate to operator A with respect to bilinear form $\langle ., . \rangle$), the representation (1) as it was shown in [2] is equivalent to the representation

$$s_k = \langle A^k x_0, y_0 \rangle, \quad k = \overline{0, \infty}, \tag{2}$$

In this paper the representation of the form (2) will be considered with operator A defined by bilinear transformation of independent variable.

Let us introduce some necessary definitions. We shall denote by $\mathscr{R}[M/N]$ a class of rational functions with nominators of degree $\leq M$ and denominators of degree $\leq N$

$$\mathscr{R}[M/N] = \left\{ r(z) = \frac{p(z)}{q(z)}, \quad \deg p(z) \leq M, \quad \deg q(z) \leq N \right\}.$$

Definition 2 [3, Part 1, Chap.1, Par.B]. *We shall call by Padé approximant of the order* $[M/N]$, $M, N = \overline{0, \infty}$, *for power series*

$$f(z) = \sum_{k=0}^{\infty} s_k z^k$$

the rational function

$$[M/N]_f(z) = \frac{P_M(z)}{Q_N(z)} \in \mathscr{R}[M/N]$$

such that

$$f(z) - [M/N]_f(z) = O\left(z^{M+N+1}\right)$$

in the neighbourhood of $z = 0$.

2^0. **Compositions of bilinear transformations.** Let us consider for some $\gamma \in (0, +\infty) \setminus \{1\}$ bilinear transformation

$$\sigma(t) = \frac{t}{(1-\gamma)t + \gamma}.$$

It is easily seen that transformation σ maps the segment $[0, 1]$ onto itself, and in addition $\sigma(0) = 0$ as well as $\sigma(1) = 1$. Let us define in the space $\mathscr{X} = C[0, 1]$ of continuous on $[0, 1]$ functions linear bounded operator

$$(A\varphi) = \varphi(\sigma(t)) = \varphi\left(\frac{t}{(1-\gamma)t + \gamma}\right).$$

It is simple to calculate its degrees

$$\left(A^k\varphi\right) = \varphi\left(\frac{t}{(1-\gamma^k)t + \gamma^k}\right).$$

Let us assume for some $\delta \in (0, +\infty) \setminus \{1\}$

$$x_0(t) = \frac{t}{(1-\delta)t + \delta},$$

and construct a system of functions

$$x_k(t) = \left(A^k x_0\right)(t) = \frac{t}{(1-\delta\gamma^k)t + \delta\gamma^k}, \quad k = \overline{0,\infty}. \qquad (3)$$

For arbitrary system of points

$$0 < t_0 < t_1 < \ldots < t_N < 1, \quad N = \overline{0,\infty}$$

let us consider determinants

$$\Delta_N = \Delta_N(t_0, t_1, \ldots, t_N) = \det \|x_k(t_j)\|_{k,j=0}^N =$$

$$= \det \left\| \frac{t_j}{(1-\delta\gamma^k)t_j + \delta\gamma^k} \right\|_{k,j=0}^N = \prod_{j=0}^N t_j \times \prod_{k=0}^N \frac{1}{1-\delta\gamma^k} \times \det \left\| \frac{1}{t_j + \varkappa_k} \right\|_{k,j=0}^N,$$

where $\varkappa_k = \frac{\delta\gamma^k}{1-\delta\gamma^k}$, $k = \overline{0,N}$, $N = \overline{0,\infty}$. The last determinant is determinant of Cauchy matrix (see [4, Chapter I, §3, example 4]) which is equal

$$\det \left\| \frac{1}{t_j + \varkappa_k} \right\|_{k,j=0}^N = \frac{\prod\limits_{j<k} (t_k - t_j)(\varkappa_k - \varkappa_j)}{\prod\limits_{j,k} (t_j + \varkappa_k)}.$$

Because as easily seen $\varkappa_k \neq \varkappa_j$ for $k \neq j$ then last determinant as well as determinant Δ_n is different from zero, hence, system of functions $\{x_k(t)\}_{k=0}^N$ for any $N = \overline{0,\infty}$ is Tchebycheff on segment $[0,1]$ (see [4, Chapter I, §1, Def.1.1]).

Let us consider on the product of spaces $\mathscr{X} \times \mathscr{X}$ bilinear form

$$\langle x, y \rangle = \int\limits_0^1 x(t)y(t)dt. \qquad (4)$$

Simple calculations give expressions for the degrees of operator A^* conjugate to operator A with respect to bilinear form (4)

$$\left(A^*\psi\right)(t) = \frac{\gamma}{\left(1-(1-\gamma)t\right)^2}\psi\left(\frac{\gamma t}{1-(1-\gamma)t}\right).$$

Let us assume now that $y_0(t) \equiv 1$, and construct system of functions

$$y_j(t) = \left(A^* y_0\right)(t) = \frac{\gamma^j}{\left(1-(1-\gamma^j)t\right)^2}. \tag{5}$$

Let us verify that system of functions (5) is also Tchebycheff. It is easily seen that

$$\frac{d^m}{dt^m} y_j(t) = \frac{(m+1)!\gamma^j\left(1-\gamma^j\right)^m}{\left(1-(1-\gamma^j)t\right)^{m+2}}.$$

Therefore Wronskian of system of functions (5) will have a form

$$W_N = \det\left\|\frac{d^m}{dt^m}y_j(t)\right\|_{j,m=0}^N = \det\left\|\frac{(m+1)!\gamma^j\left(1-\gamma^j\right)^m}{\left(1-(1-\gamma^j)t\right)^{m+2}}\right\|_{j,m=0}^N =$$

$$= \prod_{m=0}^N (m+1)! \times \prod_{j=0}^N \gamma^j \times \prod_{j=0}^N \frac{1}{\left(1-(1-\gamma^j)t\right)^2} \times \det\left\|\frac{1}{\left(\frac{1}{1-\gamma^j}-t\right)^m}\right\|_{j,m=0}^N.$$

The last determinant is Vandermonde determinant (see [4, Chapter I, §1])

$$\det\left\|\frac{1}{\left(\frac{1}{1-\gamma^j}-t\right)^m}\right\|_{j,m=0}^N = \prod_{k<j}\left(\frac{1}{\frac{1}{1-\gamma^k}-t} - \frac{1}{\frac{1}{1-\gamma^j}-t}\right) =$$

$$= \prod_{k<j} \frac{\gamma^j - \gamma^k}{(1 - t(1 - \gamma^k))\,(1 - t(1 - \gamma^j))} \neq 0.$$

It implies that system of functions (5) is Tchebycheff on $[0, 1]$ for any $N = \overline{0, \infty}$ (see [4, Chapter XI, §1, Theorem 1.1]).

3^0. **Generalized moment representations associated with bilinear transformations.** Previous considerations may be summarized in the following results.

Theorem 1. *For the sequence*

$$s_k = \frac{t_0}{(1 - \delta\gamma^k)t_0 + \delta\gamma^k}, \quad k = \overline{0, \infty},$$

where $\gamma \in (0, \infty) \setminus \{1\}$, $\delta \in (0, \infty) \setminus \{1\}$, $t_0 \in (0, 1)$ *the generalized moment representation holds in Banach space* $\mathscr{X} = C[0, 1]$

$$s_{k+j} = y_j(x_k), \quad k, j = \overline{0, \infty}$$

where

$$x_k(t) = \frac{t}{(1 - \delta\gamma^k)t + \delta\gamma^k}, \quad k = \overline{0, \infty},$$

and functionals $y_j(x)$, $j = \overline{0, \infty}$ *are defined by formulae*

$$y_j(x) = x(t_j) = x\left(\frac{t_0}{(1 - \gamma^j)t_0 + \gamma^j}\right), \quad j = \overline{0, \infty}, \tag{6}$$

Theorem 2. *For the sequence*

$$s_k = \frac{1}{1-\delta\gamma^k} + \frac{(\ln\delta + k\ln\gamma)\,\delta\gamma^k}{(1-\delta\gamma^k)^2}, \quad k=\overline{0,\infty},$$

where $\gamma \in (0,\infty)\setminus\{1\}$, $\delta \in (0,\infty)\setminus\{1\}$, *the generalized moment representation holds on the product of spaces* $C[0,1]\times C[0,1]$

$$s_{k+j} = \int\limits_0^1 \frac{t}{(1-\delta\gamma^k)\,t+\delta\gamma^k} \times \frac{\gamma^j}{(1-(1-\gamma^j)t)^2} dt, \quad k,j=\overline{0,\infty}.$$

4^0. **Applications to Padé approximants.** Using the main result by V.K. Dzyadyk [1] on application of generalized moment representations to the problem of Padé approximation we can receive the following results.

Theorem 3. *Padé approximants for the power series*

$$f(z) = \sum_{k=0}^{\infty} \frac{t_0 z^k}{(1-\delta\gamma^k)t_0+\delta\gamma^k},$$

where $\gamma \in (0,\infty)\setminus\{1\}$, $\delta \in (0,\infty)\setminus\{1\}$, $t_0 \in (0,1)$ *of orders* $[N-1/N]$, $N \geq 1$, *exist and are nondegenerate and may be represented in the form*

$$[N-1/N]_f(z) = \frac{P_{N-1}(z)}{Q_N(z)},$$

where

$$Q_N(z) = \sum_{k=0}^{N} c_k^{(N)} z^{N-k},$$

$$P_{N-1}(z) = \sum_{m=0}^{N-1} z^m \sum_{k=0}^{m} c_{N-k}^{(N)} \frac{t_0}{(1-\delta\gamma^{m-k})\, t_0 + \gamma^{m-k}},$$

and $c_k^{(N)}$, $k = \overline{0, N}$ *- coefficients of biorthogonal polynomial*

$$Y_N = \sum_{j=0}^{N} c_j^{(N)} y_j,$$

defined by the relations

$$Y_N(x_k) = 0, \quad k = \overline{0, N-1},$$

(functions $x_k(t)$, $k = \overline{0, \infty}$, *are defined by formulae (3), and functionals* y_j, $j = \overline{0, \infty}$, *are defined by formulae (6)).*

Theorem 4. *Padé approximants for the power series*

$$f(z) = \sum_{k=0}^{\infty} \left\{ \frac{1}{1-\delta\gamma^k} + \frac{(\ln\delta + k\ln\gamma)\,\delta\gamma^k}{(1-\delta\gamma^k)^2} \right\} z^k,$$

where $\gamma \in (0,\infty)\backslash\{1\}$, $\delta \in (0,\infty)\backslash\{1\}$, *of orders* $[N-1/N]$, $N \geq 1$, *exist and are nondegenerate and may be represented in the form*

$$[N-1/N]_f(z) = \frac{P_{N-1}(z)}{Q_N(z)},$$

where

$$Q_N(z) = \sum_{k=0}^{N} c_k^{(N)} z^{N-k},$$

$$P_{N-1}(z) = \sum_{j=0}^{N-1} z^j \sum_{m=0}^{j} c_{N-m}^{(N)} \left\{ \frac{1}{1-\delta\gamma^{j-m}} + \frac{(\ln\delta + (j-m)\ln\gamma)\,\delta\gamma^{j-m}}{(1-\delta\gamma^{j-m})^2} \right\},$$

and $c_k^{(N)}$, $k=\overline{0,N}$ *are coefficients of generalized polynomial*

$$X_N(t)=\sum_{k=0}^{N} c_k^{(N)} \frac{t}{(1-\delta\gamma^k)\,t+\delta\gamma^k},$$

for which biorthogonality conditions

$$\int\limits_0^1 X_N(t)\frac{\gamma^j}{\left(1-(1-\gamma^j)t\right)^2}dt=0,\quad j=\overline{0,N-1},$$

are satisfied.

References

1. Dzyadyk V.K. A generalization of moment problem//Dokl.Akad.Nauk Ukr.SSR., Ser.A. – 1981. – N 6. – P. 8–12.

2. Golub A.P. Generalized moment representations and rational approximations. – Kiev, 1987. – 50 p. (Preprint/Akad. Nauk Ukrain. SSR. Inst.Mat.; 87.25).

3. Baker G.A.Jr. *Essentials of Padé approximants.* – N.Y.: Academic Press, 1975. – 306 p.

4. Karlin S., Studden W.J. *Tchebycheff systems: with applications in analysis and statistics.* – N.Y.: Interscience Publishers, 1966.