Ukrainian Mathematical Journal, Vol. 45, No. 10, 1993

ON SEQUENCES THAT DO NOT INCREASE THE NUMBER OF REAL ROOTS OF POLYNOMIALS

A. G. Bakan and A. P. Golub

UDC 519.64

A complete description is given for the sequences $\{\lambda_k\}_{k=0}^{\infty}$ such that, for an arbitrary real polynomial $f(t) = \sum_{k=0}^{n} a_k t^k$, an arbitrary $A \in (0, +\infty)$, and a fixed $C \in (0, +\infty)$, the number of roots of the polynomial $(Tf)(t) = \sum_{k=0}^{n} a_k \lambda_k t^k$ on $[0, C]$ does not exceed the number of roots of $f(t)$ on $[0, A]$.

The following problem was formulated in [1, p. 382]:

Karlin's problem. Describe the sequences of factors $\{\lambda_k\}_{k=0}^{\infty}$ that do not increase the number of real zeros of polynomials, i.e., the sequences such that, for any real polynomial $f(x) = \sum_{k=0}^{n} a_k x^k$,

$$\mathbb{Z}_{\mathbb{R}}\Big(\sum_{k=0}^{n} a_k \lambda_k x^k \Big) \le \mathbb{Z}_{\mathbb{R}}\Big(\sum_{k=0}^{n} a_k x^k \Big), \tag{1}$$

where $\mathbb{Z}_{\mathbb{R}}(f)$ is the number of real zeros of f taking account of their multiplicities.

In [6], it was proved that the solution of this problem presented in [2–5] is not correct and, thus, Karlin's problem remains open.

In this paper, we describe the sequences of factors $\{\lambda_k\}_{k=0}^{\infty}$ such that, for an arbitrary real polynomial $f(t) = \sum_{k=0}^{n} a_k t^k$, an arbitrary $A \in (0, \infty)$, and a fixed $C \in (0, \infty)$, the number of roots of the polynomial $\sum_{k=0}^{n} a_k \lambda_k x^k$ on $[0, C]$ does not exceed the number of roots of $f(t)$ on $[0, A]$.

Denote by τ the class of all sequences that do not increase the number of real zeros, i.e., sequences satisfying property (1); the transformations determined by sequences of this type are denoted by T, i.e., if $f(x) = \sum_{k=0}^{n} a_k x^k$, then

$$(Tf)(x) = \sum_{k=0}^{n} a_k \lambda_k x^k. \tag{2}$$

Let us prove some auxiliary results.

Lemma 1. *If* $\{\lambda_k\}_{k=0}^{\infty} \in \tau$, *then there exists a nondecreasing function* $\mu(t)$ *on* $[0, +\infty)$ *and numbers* $\delta_1 = \pm 1$ *and* $\delta_2 = \pm 1$ *such that*

$$\lambda_k = \delta_1 \int_0^{\infty} (\delta_2 t)^k \, d\mu(t), \quad k = \overline{0, \infty}. \tag{3}$$

Institute of Mathematics, Ukrainian Academy of Sciences, Kiev. Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 45, No. 10, pp. 1323–1331, October, 1993. Original article submitted March 20, 1992.

0041–5995/93/4510–1481 $12.50

Proof. We take an arbitrary algebraic polynomial

$$P(t) = \sum_{k=0}^{n} \xi_k t^k$$

and construct the polynomial

$$f(x) = [P(t)]^2 + \varepsilon = \sum_{k,j=0}^{n} \xi_k \xi_j t^{k+j} + \varepsilon,$$

where $\varepsilon > 0$. It is obvious that the polynomial $f(t)$ is strictly positive on the entire real axis and, therefore, does not have real roots. By applying to $f(t)$ the linear transformation T determined by relation (2), we get

$$(Tf)(t) = \sum_{k,j=0}^{n} \xi_k \xi_j t^{k+j} \lambda_{k+j} + \varepsilon \lambda_0.$$

One can easily find that

$$(Tf)(0) = \xi_0^2 \lambda_0 + \varepsilon \lambda_0.$$

If $\{\lambda_k\}_{k=0}^{\infty} \in \tau$, then $(Tf)(t)$ also has no real roots and, thus, it preserves its sign on the entire real axis. For example,

$$\operatorname{sign}(Tf)(1) = \operatorname{sign}\Big[\sum_{k,j=0}^{n} \xi_k \xi_j \lambda_{k+j} + \varepsilon \lambda_0 \Big] = \operatorname{sign}(Tf)(0) = \operatorname{sign} \lambda_0 =: \delta_1 .$$

Since $\varepsilon > 0$ is arbitrary, we conclude that the sequence $\{ \delta_1 \lambda_k \}_{k=0}^{\infty}$ is positive [7, p. 10]. Given an arbitrary polynomial

$$P(t) = \sum_{k=0}^{n} \xi_k t^k,$$

we construct the following one:

$$f(t) = t\Big([P(t)]^2 + \varepsilon \Big) = \sum_{k,j=0}^{n} \xi_k \xi_j t^{k+j+1} + \varepsilon t, \quad \varepsilon > 0.$$

Let us apply the transformation T to $f(t)$,

$$(Tf)(t) = \sum_{k,j=0}^{n} \xi_k \xi_j t^{k+j+1} \lambda_{k+j+1} + \varepsilon t \lambda_1 .$$

Since $f(t)$ has exactly one root $t = 0$ and $(Tf)(0) = 0$, we conclude that $Q(t) := \frac{1}{t}(Tf)(t)$ cannot have any real roots. Obviously,

$$Q(0) = \xi_0^2\lambda_1 + \varepsilon\lambda_1.$$

Therefore,

$$\operatorname{sign} Q(1) = \operatorname{sign}\Big[\sum_{k,j=0}^{n} \xi_k\xi_j\lambda_{k+j+1} + \varepsilon\lambda_1\Big] = \operatorname{sign} Q(0) = \operatorname{sign}\lambda_1 =: \delta_2/\delta_1.$$

Consequently, in view of the arbitrariness of $\varepsilon > 0$, the sequence $\{(\delta_2/\delta_1)\lambda_{k+1}\}_{k=0}^{\infty}$ is also positive. This implies that representation (3) holds [7, p. 93].

Without loss of generality, we assume in what follows that $\delta_1 = \delta_2 = 1$. The subclass of the class τ, determined by these conditions, is denoted by τ^+.

Assume that the sequence $\{\lambda_k\}_{k=0}^{\infty} \in \tau^+$ possesses the following additional property: There exists a positive constant C such that

$$\mathbb{Z}_{[0, CA]}(Tf) \le \mathbb{Z}_{[0, A]}(f) \tag{4}$$

for any real polynomial $f(t)$ and any $A \in (0, +\infty)$, where, by analogy with the notation introduced above, $\mathbb{Z}_{[0,A]}(f)$ denotes the number of zeros of the polynomial $f(t)$ on the interval $[0, A]$ taking account of their multiplicities. Let τ_C^+ denote the subclass of the class τ^+ for which this condition is satisfied.

Lemma 2. *If a sequence* $\{\lambda_k\}_{k=0}^{\infty}$ *belongs to* τ_C^+, *then it can be represented in the form*

$$\lambda_k = \int_0^C t^k d\mu(t), \quad k = \overline{0,\infty}, \tag{5}$$

where $\mu(t)$ *is a nondecreasing function on the segment* $[0, C]$.

Proof. One can easily see that it suffices to consider the case where $C = 1$. Consider the algebraic polynomial

$$f(x) = (1-t)^m t^n + \varepsilon = \sum_{k=0}^{m} \binom{m}{k} (-1)^k t^{k+n} A^{m-k} + \varepsilon,$$

where m and n are nonnegative integers and $\varepsilon > 0$. This polynomial is strictly positive on $[0, A]$ and, hence, it have no zeros on this segment. Let us apply the transformation T determined by the sequence $\{\lambda_k\}_{k=0}^{\infty} \in \tau_1^+$ to this polynomial,

$$(Tf)(t) = \sum_{k=0}^{m} \binom{m}{k} (-1)^k t^{k+n} \lambda_{k+n} A^{m-k} + \varepsilon\lambda_0.$$

Taking into account the assumptions concerning the sequence $\{\lambda_k\}_{k=0}^{\infty}$, we conclude that $(Tf)(t)$ is strictly positive on $[0, A]$ and, in particular,

$$(Tf)(A) = A^{m+n}\sum_{k=0}^{m} \binom{m}{k}(-1)^k\lambda_{k+n} + \varepsilon\lambda_0 > 0.$$

Letting ε tend to zero, we obtain the Hausdorff condition [7, p. 97], which implies the possibility of representation (5). Note that in our proof we only have used the validity of condition (4) for a single fixed number $A > 0$.

Representation (5) allows one to define the transformation T not only on algebraic polynomials but also on arbitrary functions continuous on the semiaxis $[0, +\infty)$. Consider the polynomial in powers of a logarithm

$$g(t) = A^{m+n}\sum_{k=0}^{m} c_k (\log t)^k. \tag{6}$$

Since the function $g(t)$ can take arbitrarily large values at the point $t = 0$, it is impossible to define the transformation T directly on it. Therefore, we define the transformation T that corresponds to the sequence $\{\lambda_k\}_{k=0}^{\infty} \in \tau_1^+$ on the function

$$g_\rho(t) = t^\rho g(t), \quad \rho > 0, \tag{7}$$

by the relation

$$(Tg_\rho)(x) = \int_0^1 g_\rho(xt)d\mu(t) = x^\rho \int_0^1 t^\rho g(xt)d\mu(t). \tag{8}$$

Let us prove the following lemma:

Lemma 3. *Assume that the sequences* $\{\lambda_k\}_{k=0}^{\infty}$ *belong to* τ_1^+. *Then, for an arbitrary polynomial*

$$g(t) = \sum_{k=0}^{m} c_k(\log t)^k$$

with real coefficients c_k, $k = \overline{0, m}$, $\sum_{k=0}^{m} c_k^2 > 0$, *for any* $\rho > 0$ *and* $A \in (0, +\infty)$, *the following inequality holds:*

$$\mathbb{Z}_{(0,A]}(Tg_\rho) \le \mathbb{Z}_{(0,A]}(g_\rho). \tag{9}$$

Proof. Assume that the polynomial $g(t)$ has r roots (taking account of their multiplicities) $t_1, t_2, \ldots, t_r$ on $(0, A]$. Consider an auxiliary function

$$\varphi(t) = \frac{g_\rho(t)}{(t-t_1)(t-t_2)\ldots(t-t_r)}.$$

The function $\varphi(t)$ is continuous on $[0, A]$ and preserves its sign on $(0, A]$. Without loss of generality, we assume that it is strictly positive on $(0, A]$. According to the Weierstrass theorem, the function $\sqrt{\varphi(t)}$ can be approximated on $[0, A]$ by an algebraic polynomial $P(t)$ so that

$$\| \sqrt{\varphi(t)} - P(t) \|_{C[0,A]} < \varepsilon$$

for an arbitrary given $\varepsilon > 0$.

Consider the algebraic polynomial

$$Q(t) = ([P(t)]^2 + \varepsilon^2)(t - t_1) \dots (t - t_r).$$

Obviously, $Q(t)$ has exactly as many zeros on $(0, A]$ as $g(t)$. Let us estimate on $[0, A]$ the difference

$$|(Tg_\rho)(x) - (TQ)(x)| = \left| \int_0^1 [x^\rho t^\rho g(xt) - Q(xt)]\, d\mu(t) \right|$$

$$= \left| \int_0^1 [\varphi(xt)(xt - t_1) \dots (xt - t_r) - ([P(xt)]^2 + \varepsilon^2)\right.$$

$$\times \left.(xt - t_1) \dots (xt - t_r)]\, d\mu(t) \right| \le \int_0^1 |xt - t_1| \dots |xt - t_r| \big| \varphi(xt)$$

$$- [P(xt)]^2 \big| d\mu(t) + \varepsilon^2 \int_0^1 |xt - t_1| \dots |xt - t_r| d\mu(t)$$

$$\le A^r \int_0^1 | \sqrt{\varphi(xt)} - P(xt) | | \sqrt{\varphi(xt)} + P(xt) | d\mu(t) + A^r \varepsilon^2$$

$$\le A^r \lambda_0 \| \sqrt{\varphi(t)} - P(t) \|_{C[0,A]} \| \sqrt{\varphi(t)} + P(t) \|_{C[0,A]} + A^r \varepsilon^2$$

$$\le A^r \varepsilon \left(2 \| \sqrt{\varphi(t)} \|_{C[0,A]} + \varepsilon \right) \lambda_0 + A^r \varepsilon^2. \tag{10}$$

Thus, the value

$$\| (Tg_\rho)(x) - (TQ)(x) \|_{C[0,A]}$$

can be made as small as desired by the proper choice of the polynomial $P(t)$. We complete the proof by contradiction. Let $(Tg_\rho)(x)$ have $q > r$ zeros on $(0, A]$. Without loss of generality, we can assume that these roots are distinct; otherwise, we can make these roots distinct by changing insignificantly the coefficients of the polynomial $g(t)$ so that the number of its zeros remains unchanged. In exactly the same way, we can arrange that none of these roots would coincide with the point $x = A$. We order the roots of $(Tg_\rho)(x)$ so that $0 < x_1 < x_2 < \dots < x_q < A$ and denote $x_0 := 0$ and $x_{q+1} := A$. Let us introduce the value

$$\kappa := \min_{j = \overline{0, q}} \sup_{x \in [x_j, x_{j+1}]} |(Tg_\rho)(x)| > 0.$$

Taking the previous reasoning into account, we get

$$\| (Tg_\rho)(x) - (TQ)(x) \|_{C[0,A]} < \kappa.$$

It is now easy to show that the algebraic polynomial $(TQ)(x)$ has at least as many zeros on $(0, A]$ as $(Tg_\rho)(x)$. Thus,

$$\mathbb{Z}_{(0,A]}(TQ) \geq \mathbb{Z}_{(0,A]}(Tg_\rho) = q > r = \mathbb{Z}_{(0,A]}(g_\rho) = \mathbb{Z}_{(0,A]}(Q).$$

This contradicts our assumption that $\{\lambda_k\}_{k=0}^{\infty} \in \tau_1^+$. Lemma 3 is proved.

We can now prove the principal result of this paper.

Theorem. *In order that a sequence* $\{\lambda_k\}_{k=0}^{\infty}$ *belong to the class* τ_C^+, $0 < C < +\infty$, *it is necessary and sufficient that the following conditions be satisfied:*

(i) The sequence $\{\lambda_k\}_{k=0}^{\infty}$ *can be represented in the form*

$$\lambda_k = \int_0^C t^k d\mu(t), \quad k = \overline{0,\infty},$$

where $\mu(t)$ *is a nondecreasing function on* $[0, C]$,

(ii) The function

$$\Phi(z) = \int_0^C t^z d\mu(t)$$

is analytic in $\mathbb{C}\setminus(-\infty, 0)$ *and can be represented in the form*

$$\Phi(z) = \frac{\lambda_0 e^{\delta z}}{\prod_{i=1}^{\infty}(1+a_i z)}, \tag{11}$$

where $a_i \geq 0$, $i = \overline{0,\infty}$, $\sum_{i=0}^{\infty} a_i < \infty$, $\delta \leq \log C$.

Proof. First, we prove the necessity of conditions (i) and (ii). We again restrict ourselves to the case where $C = 1$, since the proof can be easily generalized for arbitrary $C \in (0, +\infty)$ (for this purpose, it suffices to consider the sequence $\{\lambda_k / C^k\}_{k=0}^{\infty}$). It follows from Lemma 3 that

$$\mathbb{Z}_{(0,A]}\Big(\int_0^1 x^\rho t^\rho \sum_{k=0}^{m} c_k (\log xt)^k \, d\mu(t)\Big) \leq \mathbb{Z}_{(0,A]}\Big(t^\rho \sum_{k=0}^{m} c_k (\log t)^k\Big) \tag{12}$$

for all real c_k, $k = \overline{0,m}$, that are not equal to zero simultaneously. Let us make the change $u = -\log t$ in the integral on the left-hand side of (12) and set $w = \log x$. We get

$$\mathbb{Z}_{(-\infty, \log A]}\Big(-\int_0^\infty e^{\rho(w-u)}\sum_{k=0}^m c_k (w-u)^k d\mu(e^{-u})\Big) \le \mathbb{Z}_{(-\infty, \log A]}\Big(e^{\rho w}\sum_{k=0}^m c_k w^k\Big).$$

Since $A \in (0, +\infty)$, this yields

$$\mathbb{Z}_{(-\infty, 0]}\Big(-\int_0^\infty e^{-\rho u}\sum_{k=0}^m c_k (w-u)^k d\mu(e^{-u})\Big) \le \mathbb{Z}_{(-\infty, 0]}\Big(\sum_{k=0}^m c_k w^k\Big). \tag{13}$$

The proof can be completed by an argument analogous to that used in the proof of Theorem 3.2 in [1, p. 342]. Denote

$$f(w) = \sum_{k=0}^m c_k w^k, \tag{14}$$

$$F(w) = -\int_0^\infty e^{-\rho u} f(w-u)\, d\mu(e^{-u}), \tag{15}$$

$$s_k = -\int_0^\infty u^k e^{-\rho u} d\mu(e^{-u})\Big), \quad k = \overline{0,\infty}. \tag{16}$$

Thus, by setting $D = d/dw$ and

$$U_m(D) = \sum_{k=0}^m (-1)^k \frac{s_k D^k}{k!}, \tag{17}$$

we can rewrite equality (15) in the form

$$F(w) = U_m(D) f(w). \tag{18}$$

By applying the Laplace transform to the density $-e^{-\rho u} d\mu(e^{-u})$, $0 \le u < +\infty$, we obtain

$$\Phi(z) = -\int_0^\infty e^{-zu} e^{-\rho u} d\mu(e^{-u}) = \sum_{k=0}^m (-1)^k \frac{s_k z^k}{k!}, \tag{19}$$

Series (19) converges in a disk whose radius is nonzero. Since $\Phi(0) = s_0 = \lambda_0 \ne 0$, the series

$$\frac{1}{\Phi(z)} = \Psi(z) = \sum_{k=0}^\infty \frac{r_k}{k!} z^k \tag{20}$$

also converges in a certain neighborhood V_0 of the point $z = 0$.

Further, since $f(w)$ and $F(w)$ are polynomials, equality (15) can be converted, and we get

$$f(w) = \Psi(D) F(w) = \Big(\sum_{k=0}^m \frac{r_k}{k!} D^k\Big) F(w). \tag{21}$$

It follows from equality (13) that

$$\mathbb{Z}_{(-\infty,0]}(F) \le \mathbb{Z}_{(-\infty,0]}(f).$$

Consequently, if $F(w) = w^m$, then $\mathbb{Z}_{(-\infty,0]}(f) = m$. In this case, equality (21) yields

$$f(w) = \Psi(D)\,w^m = A_m(w) = \sum_{k=0}^{m} \frac{r_k}{k!}\,\frac{m!}{(m-k)!}\,w^{m-k}.$$

Hence, the polynomial

$$w^m A_m\Big(\frac{1}{w}\Big) = \sum_{k=0}^{m} \binom{m}{k} r_k w^k = A_m^*(w)$$

has only nonpositive zeros. Moreover, $A_m^*\big(\frac{w}{m}\big)$ converges uniformly to $\Psi(w)$ inside any compact subregion $K_0 \subset V_0$. Indeed, for given $\varepsilon > 0$, we choose $N = N(\varepsilon)$ such that

$$\sum_{k=N(\varepsilon)}^{\infty} \frac{|r_k|}{k!}|w|^k \le \varepsilon \quad \forall\, w \in K_0.$$

Then, for any $n > N$, we obtain the following estimate:

$$\Big|A_n^*\Big(\frac{w}{n}\Big) - \Psi(w)\Big| \le \Big|\sum_{k=0}^{N} \frac{r_k}{k!}\,w^k\Big(\frac{n!}{(n-k)!n^k} - 1\Big)\Big| + 2\sum_{k=N}^{\infty} \frac{|r_k|}{k!}|w|^k,$$

which implies the convergence. Thus, the function $\Psi(z)$ is a uniform limit of a sequence of polynomials having only real nonpositive zeros. It is known [1, p. 336] that this function admits the following representation:

$$\Psi(z) = \alpha z^k e^{\delta z} \prod_{i=1}^{\infty} (1 + a_i z),$$

where $\alpha \in \mathbb{R}$, $\delta \ge 0$, $a_i \ge 0$, $i = \overline{0, \infty}$, and $0 < \sum_{i=0}^{\infty} a_i < \infty$, $k \in \mathbb{N} \cup \{0\}$.

Hence,

$$\Phi(z) = \frac{1}{\Psi(z)} = \frac{e^{-\delta z}}{\alpha z^k \prod_{i=1}^{\infty}(1 + a_i z)}.$$

On the other hand,

$$\Phi(z) = -\int_0^{\infty} e^{-u(\rho+z)} d\mu(e^{-u}) = \int_0^{\infty} t^{\rho+z}\, d\mu(t).$$

Since $\rho > 0$ is arbitrary, we have the representation

$$\Phi(z) = \Phi_0(z) = \int_0^1 t^z \, d\mu(t) = \frac{e^{-\delta z}}{\alpha z^k \prod_{i=1}^{\infty}(1+a_i z)}. \tag{22}$$

Since the function $\Phi(z)$ is equal to λ_0 at the point $z=0$, we have $k=0$ and $1/\alpha = \lambda_0$ in (22). Note that the condition $\sum_{i=1}^{\infty} a_i > 0$ may be not satisfied after passing to the limit as $\rho \to 0$. Thus, the necessity of conditions (i) and (ii) is proved. Sufficiency follows from the Laguerre theorem [2, p. 544] and Theorem 2.1 in [1, p. 336].

REFERENCES

1. S. Karlin, *Total Positivity*, Vol. 1, Stanford Univ. Press, Stanford (1968).
2. T. Graven and G. Csordas, "Zero-diminishing linear transformations," *Proc. Amer. Math. Soc.*, 80, No. 4, 544–546 (1980).
3. T. Graven and G. Csordas, "An inequality for the distribution of zeros of polynomials and entire functions," *Pacif. J. Math.*, **95**, No. 2, 263–280 (1981).
4. T. Graven and G. Csordas, "On the number of real roots of polynomials," *Pacif. J. Math.*, **102**, No. 1, 15–28 (1981).
5. T. Graven and G. Csordas, "Locations of zeros. Pt 1: Real polynomials and entire functions," *Pacif. J. Math.*, **27**, No. 2, 244–278 (1983).
6. A. G. Bakan and A. P. Golub, "Some negative results on sequences of factors of the first kind," *Ukr. Mat. Zh.*, **44**, No. 3, 305–309 (1992).
7. N. I. Akhiezer, *The Classical Moment Problem and Some Related Problems in Analysis* [in Russian], Fizmatgiz, Moscow (1961).