PADE APPROXIMATION OF arcsin z

A. P. Golub

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In this article we obtain a representation of arcsin z in the form of a Markov–Stieltjes integral which allows us to resolve the question of the convergence of its Pade approximation on the basis of a classical result. It is to be noted that arcsin z is the only basic elementary function for which this question remains unresolved. We also obtain bounds for the Hankel determinants.

1. Integral Representation of arcsin z. The expansion

$$\arcsin z = \sum_{k=0}^{\infty} a_k z^{2k+1} \tag{1}$$

for $|z| \leqslant 1$, where $a_k = (2k-1)!!/(2k)!!(2k+1)$, is well known. We establish that $\{a_k\}_{k=0}^{\infty}$ is a sequence of moments for some measure $\mu(t)dt$ on $[0, 1]$. In fact, it is easy to express $(2k-1)!!/(2k)!!$ in terms of Euler's beta function:

$$\frac{(2k-1)!!}{(2k)!!} = \frac{\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)\cdot\ldots\cdot\frac{1}{2}\cdot 2^k}{k!\cdot 2^k} = \frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(k+1)} =$$

$$= \frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\pi\Gamma(k+1)} = \frac{1}{\pi}\int_0^1 x^{k-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\,dx = \frac{1}{\pi}\int_0^1 x^k \frac{dx}{\sqrt{x(1-x)}}.$$

We have further

$$a_k = \frac{(2k-1)!!}{(2k)!!(2k+1)} = \frac{1}{\pi}\int_0^1 \frac{x^k}{2k+1}\frac{dx}{\sqrt{x(1-x)}} = \frac{1}{2\pi}\int_0^1 \frac{1}{x\sqrt{1-x}}\int_0^x t^{k-\frac{1}{2}}\,dt\,dx.$$

Interchanging the order of integration, we get

$$a_k = \frac{1}{2\pi}\lim_{\varepsilon\to+0}\int_\varepsilon^1 \frac{1}{x\sqrt{1-x}}\int_0^\varepsilon t^{k-\frac{1}{2}}\,dt + \frac{1}{2\pi}\lim_{\varepsilon\to+0}\int_\varepsilon^1 t^{k-\frac{1}{2}}\int_t^1 \frac{1}{x\sqrt{1-x}}\,dx\,dt.$$

Since the first term admits the estimate

$$\left|\int_\varepsilon^1 \frac{1}{x\sqrt{1-x}}\,dx\int_0^\varepsilon t^{k-\frac{1}{2}}\,dt\right| \leqslant \varepsilon^{k+\frac{1}{2}}\frac{1}{\varepsilon^r}\mathrm{B}\left(r, \frac{1}{2}\right) \to 0$$

for $0 < r < k + 1/2$ and $\varepsilon \to 0$, it follows that

$$a_k = \int_0^1 t^k \mu(t)\,dt, \tag{2}$$

where

$$\mu(t) = \frac{1}{2\pi\sqrt{t}}\int_t^1 \frac{1}{x\sqrt{1-x}}\,dx = \frac{1}{2\pi\sqrt{t}}\ln\left(\frac{1+\sqrt{1-t}}{1-\sqrt{1-t}}\right).$$

If we take (1) and (2) into account, we get a representation of arcsin z in the form of a Markov–Stieltjes integral:

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$$\arcsin z = z\int_0^1 \frac{1}{1-z^2t}\mu(t)\,dt. \qquad (3)$$

Since the functions on both sides of (3) are analytic in $D = \mathbf{C}\setminus((-\infty, -1]\cup[+1, +\infty))$, it follows that it will hold not only for $|z| \leqslant 1$ but for all $z \in D$.

2. Pade Approximant of arcsin z. We consider the function

$$\varphi(z) = \frac{\arcsin\sqrt{z}}{\sqrt{z}} = \int_0^1 \frac{1}{1-zt}\mu(t)\,dt$$
$$\varphi(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{for} \quad |z| \leqslant 1. \qquad (4)$$

A rational polynomial $\pi_{M,N}(z) = P_M(z)/Q_N(z)$, where $P_M(z)$ and $Q_N(z)$ are polynomials of degrees not greater than M and N, respectively, for which the relation $\varphi(z) - \pi_{M,N}(z) = O(z^{M+N+1})$ holds for $z \to 0$ is called a Pade approximant of $\varphi(z)$ of order [M, N] at $z = 0$ (see, e.g., [1, p. 5]).

It is well known that the Pade approximant $\pi_{N+J,N}(z)$, $J \geqslant -1$ of the function $\varphi(z)$ represented in (4), where $\mu(t)$ is a function which is nonnegative, integrable on [0, 1], and different from zero on a set of positive measure, can be expressed in the form

$$\pi_{N+J,N}(\varphi;\, z) = \sum_{k=0}^{J} a_k z^k + \frac{1}{Q_{J,N}\left(\frac{1}{z}\right)}\int_0^1 \frac{Q_{J,N}\left(\frac{1}{z}\right) - Q_{J,N}(t)}{1-zt} t^{J+1}\mu(t)\,dt,$$

where $\{Q_{J,N}\}_{N=0}^{\infty}$ is a sequence of polynomials which are orthonormal on [0, 1] with respect to the measure $t^{J+1}\mu(t)dt$ (see, e.g., [2, p. 267]). It has been shown [2, p. 268] that the sequence $\pi_{N+J,N}(\varphi;\, z)$ converges uniformly to $\varphi(z)$ as $N \to \infty$ on every compact subset of $\mathbf{C}\setminus[+1, +\infty)$. By virtue of this, the Pade approximant of order $[2(N + j) + 1, 2N]$, $J \geqslant -1$, of arcsin z converges uniformly to arcsin z as $N \to \infty$ on every compact set contained in D.

3. Estimates for the Hankel Determinant of arcsin z. A study of the behavior of the Hankel determinant of a given function has a great deal of significance in problems concerning its Pade approximants. The determinant

$$H_{k,n} = \begin{vmatrix} a_k & a_{k+1} & \dots & a_n \\ a_{k+1} & a_{k+2} & \dots & a_{n+1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_n & a_{n+1} & \dots & a_{2n-k} \end{vmatrix}, \quad k \leqslant n$$

is called the Hankel determinant of $\varphi(z)$.

In the case where $\mu(t)$ is nonnegative, integrable on [0, 1], and different from zero on a set of positive measure, the Hankel determinant of a function of the form (4) is always positive (see [1, p. 210]). We consider the sequence $\{Q_{k,n}(t)\}_{n=0}^{\infty}$ of polynomials which are orthonormalized on [0, 1] with respect to the measure $t^{k+1}\mu(t)dt$, $k = 1, 0, 1, \dots$. It is clear that

$$Q_{k,n}(t) = \alpha_{k,n}\begin{vmatrix} a_{k+1} & a_{k+2} & \dots & a_{n+k+1} \\ a_{k+2} & a_{k+3} & \dots & a_{n+k+2} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n-1} & a_n & \dots & a_{2n-k-1} \\ 1 & t & \dots & t^n \end{vmatrix}.$$

Therefore $\int_0^1 [Q_{k,n}(t)]^2\mu(t)\,t^{k+1}dt = \alpha_{k,n}^2 H_{k+1,n+k+1} H_{k+1,n+k}$. Therefore, the coefficient of the leading term of $Q_{k,n}(t)$ is $\sqrt{\dfrac{H_{k+1,n+k}}{H_{k+1,n+k+1}}}$.

It is well known (see, e.g., [3, p. 39]) that the minimum of the integral $\int_0^1 [A_n(t)]^2\sigma(t)dt$ for all polynomials $A_n(t)$ of degree not greater than n with leading coefficient 1 is achieved if and only if $A_n(t) = (1/\mu_n)P_n(t)$, where $P_n(t)$ is a polynomial which is orthonormalized on [0, 1] with respect to the measure $\sigma(t)dt$, and μ_n is its leading coefficient; in addition, the desired minimum is $1/\mu_n^2$. Thus

$$\frac{H_{k+1,n+k+1}}{H_{k+1,n+k}} = \min_{A_n=t^n+\dots} \int_0^1 [A_n(t)]^2 t^{k+1}\mu(t)\,dt.$$

For the function $\mu(t) = \frac{1}{2\pi\sqrt{t}} \ln\left(\frac{1+\sqrt{1-t}}{1-\sqrt{1-t}}\right)$ the inequalities $\mu(t) \geqslant \frac{1}{\pi}\frac{\sqrt{1-t}}{\sqrt{t}}$, $\mu(t) \leqslant C_\varepsilon \frac{\sqrt{1-t}}{t^{\frac{1}{2}+\varepsilon}}$, $\varepsilon > 0$, hold. Therefore,

$$\frac{H_{k+1,n+k+1}}{H_{k+1,n+k}} \geqslant \frac{1}{\pi} \min_{A_n=t^n+\dots} \int_0^1 [A_n(t)]^2 t^{k+\frac{1}{2}} \sqrt{1-t}\,dt.$$

This last minimum, as is well known, is achieved for shifted Jacobi polynomials, and it can be calculated (see [3, p. 273]). As a final result, we get

$$\frac{H_{k+1,n+k+1}}{H_{k+1,n+k}} \geqslant \frac{1}{\pi} 2^{2k+4}(k+2n+2)\,\mathrm{B}\left(n+\frac{3}{2},\, n+k+\frac{3}{2}\right)\mathrm{B}(n+1,\ n+k+2).$$

For k = −1, 0, we can simplify the right-hand side: $H_{0,n}/H_{0,n-1} \geqslant 1/\pi 2^{4n-3}$, $H_{1,n+1}/H_{1,n} \geqslant 1/\pi 2^{4n-1}$. We can obtain an upper bound similarly:

$$\frac{H_{k+1,n+k+1}}{H_{k+1,n+k}} \leqslant C_\varepsilon 2^{2k+4-2\varepsilon}(k+2n+2-\varepsilon)\,\mathrm{B}\left(n+\frac{3}{2}, n+k+\frac{3}{2}-\varepsilon\right)\mathrm{B}(n+1,\ n+k+2-\varepsilon).$$

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