Anatoly GOLUB

GENERALIZED MOMENT REPRESENTATIONS AND PADÉ APPROXIMANTS

Abstract. Using the method of generalized moment representations Padé approximants of orders $[N-1/N]$, $N \geq 1$, are constructed for some elementary functions.

1. *Introduction.* In the theory of Padé approximants for functions that are not represented by Markov-Stieltjes integrals there are not unique approach to construction and investigation of diagonal and quasi-diagonal Padé approximants, and appropriate problems are solved only for some individual functions such as $\exp z$, $(1+z)^\alpha$, etc. (majority of known examples are cited in [1]). Proposed by V.K.Dzyadyk method of generalized moment representations [2] admitted to receive practically all known examples from unique positions as well as to widen substantially the number of these examples.

Let us introduce necessary definitions.

Definition 1 ([3]). The rational function

$$[M/N]_f(z) = \frac{P_M(z)}{Q_N(z)},$$

where $P_M(z)$ and $Q_N(z)$ are algebraic polynomials of degrees $\leq M$ and $\leq N$ recpectively, is called to be Padé approximant of order $[M/N]$ for analytic function

$$f(z) = \sum_{k=0}^{\infty} s_k z^k, \tag{1}$$

if $f(z) - [M/N]_f(z) = O(z^{M+N+1})$ for $z \to 0$, i.e. power expansion of rational function $[M/N]_f(z)$ coinsides with expansion (1) up to the term, containing z^{M+N}.

Definition 2 ([2]). The generalized moment representation of the number sequence $\{s_k\}_{k=0}^{\infty}$ in Banach space X is defined as two-parametric set of equalities

$$s_{k+j} = l_j(x_k),\ k,j = \overline{0,\infty}, \tag{2}$$

where $x_k \in X$, $k = \overline{0,\infty}$, $l_j \in X^*$, $j = \overline{0,\infty}$.

In the case when in X there exists linear continuous operator $A : X \to X$ such that

$$Ax_k = x_{k+1},\ k = \overline{0,\infty},$$

the representation (2) is equivalent to the representation:

$$s_k = l_0(A^k x_0),\ k = \overline{0,\infty}. \tag{3}$$

Then the function having power expansion of the form (1) with coefficients represented in the form (3) will have the representation:

$$f(z) = l_0(R_z(A)x_0), \qquad (4)$$

where $R_z(A) = (I - zA)^{-1}$ - the resolvent of the operator A (see [4]).

In this paper we construct Padé approximants of orders $[N-1/N]$, $n \geq 1$, for functions:

$$f_1(z) = \frac{2(2+z)}{z\sqrt{4-z^2}} \arctan \frac{z}{\sqrt{4-z^2}},$$

$$f_2(z) = \frac{\tan \sqrt{z}}{\sqrt{z}},$$

$$f_3(z) = \frac{\sin z + 1 - \cos z}{z \cos z}.$$

2. *Padé Approximants for Function $f_1(z)$.*

Theorem 1. The Padé approximants of orders $[N-1/N]$, $N \geq 1$ for the function

$$f_1(z) = \frac{2(2+z)}{z\sqrt{4-z^2}} \arctan \frac{z}{\sqrt{4-z^2}},$$

may be represented in the form

$$[N-1/N]_{f_1}(z) = \frac{P_{N-1}(z)}{Q_N(z)},$$

where

$$P_{N-1}(z) = \sum_{m=1}^{N} z^{N-m}(-1)^{[m/2]} \frac{1}{[(m-1)/2]!} \times$$

$$\times \sum_{k=m}^{N} l_k^{(N)} \frac{(k-[m/2]-1)!}{(k-m)!} \sum_{j=0}^{m-1} \frac{[(j+1)/2]![j/2]!}{(j+1)!} z^j,$$

$$Q_N(z) = l_0^{(N)} z^N + \sum_{m=1}^{N} (-1)^{[m/2]} \frac{1}{[(m-1)/2]!} \sum_{k=m}^{N} l_k^{(N)} \frac{(k-[m/2]-1)!}{(k-m)!} z^{N-m},$$

and $l_k^{(N)}$, $k = \overline{0,N}$ are the coefficients of shifted orthonormal on $[0,1]$ Legendre polynomial

$$L_N^*(t) = \sum_{k=0}^{N} l_k^{(N)} t^k.$$

Here and further by $[p]$ entire part of number p is denoted.

Proof. Let us consider in the space $C[0,1]$ of continuous on $[0,1]$ functions linear bounded operator

$$(A\phi)(t) = t\phi(1-t).$$

It is easy seen that its second degree is representable in the form

$$(A^2\phi)(t) = t(1-t)\phi(t). \tag{5}$$

The resolvent of operator A^2 has the form:

$$[R_z(A^2)\phi](t) = \sum_{k=0}^{\infty} z^k (A^{2k}\phi)(t) = \frac{\phi(t)}{1 - zt(1-t)}. \tag{6}$$

Obviously:

$$R_z(A^2) = R_{-\sqrt{z}}(A) R_{\sqrt{z}}(A),$$

and, consequently,

$$R_{\sqrt{z}}(A) = (I + \sqrt{z}A) R_z(A^2).$$

Thus, because of (6):

$$[R_z(A)\phi](t) = \frac{\phi(t) + zt\phi(1-t)}{1 - z^2 t(1-t)}.$$

Let us assume now:

$$x_0(t) \equiv 1, l_0(x) = \int_0^1 x(t)dt,$$

and construct the function of the form (4):

$$f_1(z) = \int_0^1 \frac{1+zt}{1 - z^2 t(1-t)} dt = \frac{2(2+z)}{z\sqrt{4-z^2}} \arctan \frac{z}{\sqrt{4-z^2}}.$$

Its Padé approximant of order $[N-1/N]$, $N \geq 1$ according to [2] may be written in the form:

$$[N-1/N]_{f_1}(z) = \frac{P_{N-1}(z)}{Q_N(z)},$$

where

$$P_{N-1}(z) = \sum_{m=1}^{N} c_m^{(N)} z^{N-m} \sum_{k=0}^{m-1} s_k z^k, \tag{7}$$

$$Q_N(z) = \sum_{m=0}^{N} c_m^{(N)} z^{N-m}, \tag{8}$$

and coefficients $c_m^{(N)}$, $m = \overline{0, N}$ are defined from bi-orthogonality relations for generalized polynomial:

$$L_N = \sum_{m=0}^{N} c_m^{(N)} l_m$$

of the form:

$$L_N(x_k) = 0, k = \overline{0, N-1},$$

and s_k, $k=\overline{0,\infty}$ - Maclaurin coefficients of the function $f_1(z)$.

Let us determine the functions

$$x_k(t)=(A^k x_0)(t), k=\overline{0,\infty}.$$

From (5) it is seen that for even $k=2m$:

$$x_{2m}(t)=t^m(1-t)^m, m=\overline{0,\infty}. \tag{9}$$

Applying operator A to (9) we will obtain:

$$x_{2m+1}(t)=t^{m+1}(1-t)^m, m=\overline{0,\infty}.$$

Similarly we now determine linear functionals $l_k=A^{*k}l_0$, $k=\overline{0,\infty}$:

$$l_k(x)=\int_0^1 x(t)y_k(t)dt,$$

where

$$y_k(t)=\begin{cases} t^m(1-t)^m & \text{for } k=2m \\ t^m(1-t)^{m+1} & \text{for } k=2m+1. \end{cases}$$

Thus, the construction of bi-orthogonal polynomial L_N is reduced to bi-orthogonalization of systems of functions $\{x_k(t)\}_{k=0}^N$ and $\{y_k(t)\}_{k=0}^N$ on interval $[0,1]$. Because $x_k(t)$ and $y_k(t)$ are algebraic polynomials of degree equal exactly to k, then such bi-orthogonalization inevitably will lead us to construction up to constant multiplyer which is unessential in our reasoning of shifted orthonormal on $[0,1]$ Legendre polynomials $L_N^*(t)$ (see, for example, [5]):

$$X_N(t)=\sum_{m=0}^N c_m^{(N)}x_m(t)=L_N^*(t). \tag{10}$$

In order to calculate coefficients $c_m^{(N)}$ it is necessary to represent functions t^k, $k=\overline{0,\infty}$ by means of functions $x_k(t)$, $k=\overline{0,\infty}$. Let us write required representation with indeterminate coefficients:

$$t^{2k}=\sum_{m=0}^k \alpha_m^{(k)}x_{2m}(t)+\sum_{m=0}^{k-1}\beta_m^{(k)}x_{2m+1}(t),\ k=\overline{0,\infty}, \tag{11}$$

$$t^{2k+1}=\sum_{m=1}^k \gamma_m^{(k)}x_{2m}(t)+\sum_{m=0}^{k}\delta_m^{(k)}x_{2m+1}(t),\ k=\overline{0,\infty}, \tag{12}$$

and consider generating functions:

$$A(z,w)=\sum_{k=0}^\infty z^k\sum_{m=0}^k \alpha_m^{(k)}w^m,$$

$$B(z,w)=\sum_{k=1}^{\infty} z^k \sum_{m=0}^{k-1} \beta_m^{(k)} w^m,$$

$$\Gamma(z,w)=\sum_{k=1}^{\infty} z^k \sum_{m=1}^{k} \gamma_m^{(k)} w^m,$$

$$\Delta(z,w)=\sum_{k=0}^{\infty} z^k \sum_{m=0}^{k} \delta_m^{(k)} w^m.$$

Multiplying equality (11) by t we will obtain:

$$t^{2k+1}=\sum_{m=0}^{k} \alpha_m^{(k)} x_{2m+1}(t)+\sum_{m=0}^{k-1} \beta_m^{(k)} x_{2m+1}(t)-\sum_{m=0}^{k-1} \beta_m^{(k)} x_{2m+2}(t)=$$

$$=\sum_{m=0}^{k} \alpha_m^{(k)} x_{2m+1}(t)+\sum_{m=0}^{k-1} \beta_m^{(k)} x_{2m+1}(t)-\sum_{m=1}^{k} \beta_{m-1}^{(k)} x_{2m}(t). \qquad (13)$$

Since functions $x_k(t)$ are linearly independent, and right sides of (12) and (13) coinside, then their equality will not be broken if we instead of functions $x_{2m}(t)$ substitute w^m, and instead of functions $x_{2m+1}(t)$ substitute zeros. We will receive:

$$\sum_{m=1}^{k} \gamma_m^{(k)} w^m=-\sum_{m=1}^{k} \beta_{m-1}^{(k)} w^m. \qquad (14)$$

Let us multiply (14) by z^k, and sum by k from 1 to ∞. We will obtain:

$$\Gamma(z,w)=-wB(z,w). \qquad (15)$$

Similarly we will establish the relations:

$$A(z,w)=1-zw\Delta(z,w), \qquad (16)$$

$$B(z,w)=z\Delta(z,w)+z\Gamma(z,w), \qquad (17)$$

$$\Delta(z,w)=A(z,w)+B(z,w). \qquad (18)$$

Solving the system of linear algebraic equations (15)-(18) we will receive:

$$A(z,w)=\frac{1+zw-z}{(1+zw)^2-z},$$

$$B(z,w)=\frac{z}{(1+zw)^2-z},$$

$$\Gamma(z,w)=\frac{-wz}{(1+zw)^2-z},$$

$$\Delta(z,w)=\frac{1+zw}{(1+zw)^2-z}.$$

From this formulae we have:

$$A(z,w)=\frac{1+zw-z}{(1+zw)^2-z}=\frac{(1-\sqrt{z})/2}{1+zw-\sqrt{z}}+\frac{(1+\sqrt{z})/2}{1+zw+\sqrt{z}}=$$

$$=1/2\sum_{k=0}^{\infty}(-1)^k\frac{z^kw^k}{(1-\sqrt{z})^k}+1/2\sum_{k=0}^{\infty}(-1)^k\frac{z^kw^k}{(1+\sqrt{z})^k}=$$

$$=1/2\sum_{k=0}^{\infty}(-1)^kz^kw^k[\sum_{m=0}^{\infty}\frac{(k+m-1)!}{(k-1)!m!}z^{m/2}+\sum_{m=0}^{\infty}\frac{(k+m-1)!}{(k-1)!m!}(-1)^mz^{m/2}]=$$

$$=\sum_{k=0}^{\infty}(-1)^kw^k\sum_{m=k}^{\infty}\frac{(2m-k-1)!}{(k-1)!(2m-2k)!}z^m=\sum_{m=0}^{\infty}z^m\sum_{k=0}^{m}(-1)^kw^k\frac{(2m-k-1)!}{(k-1)!(2m-2k)!},$$

whence

$$\alpha_m^{(k)}=(-1)^m\frac{(2k-m-1)!}{(m-1)!(2k-2m)!}. \tag{19}$$

Similarly we will obtain:

$$\beta_m^{(k)}=(-1)^m\frac{(2k-m-1)!}{m!(2k-2m-1)!}, \tag{20}$$

$$\gamma_m^{(k)}=(-1)^m\frac{(2k-m)!}{(m-1)!(2k-2m+1)!}, \tag{21}$$

$$\delta_m^{(k)}=(-1)^m\frac{(2k-m)!}{m!(2k-2m)!}. \tag{22}$$

Substituting (19)-(22) in (11)-(12), and combining these equalities, we will receive:

$$t^k=\sum_{m=1}^{k}(-1)^{[m/2]}\frac{(k-[m/2]-1)!}{[(m-1)/2]!(k-m)!}x_m(t) \text{ for } k\ge 1 \tag{23}$$

and $t^0=1=x_0(t)$. From (10) and (23) we will obtain:

$$c_m^{(N)}=(-1)^{[m/2]}\frac{1}{[(m-1)/2]!}\sum_{k=m}^{N}l_k^{(N)}\frac{(k-[m/2]-1)!}{(k-m)!} \text{ for } m=\overline{1,N} \tag{24}$$

and $c_0^{(N)}=l_0^{(N)}$.

Substituting (24) in (7) and (8) we will receive the statement of the Theorem 1.

Remark. Similarly it is possible to construct Padé approximants for function:

$$f(x)=\frac{2}{z\sqrt{1-\alpha^2}}\sqrt{\frac{2+(1-\alpha)z}{2-(1+\alpha)z}}\arctan\frac{z\sqrt{1-\alpha^2}}{\sqrt{(2-(\alpha+1)z)(2-(\alpha-1)z)}}$$

for $\alpha \neq \pm 1$ (for $\alpha = 0$ we will obtain function $f_1(z)$). For this it is necessary to consider in space $C[0,1]$ operator

$$(A\phi)(t) = \alpha t\phi(t) + t\phi(1-t).$$

3. *PadéApproximants for function $f_2(z)$.*

Theorem 2. Padé approximants of orders $[N-1/N]$, $N \geq 1$ for the function:

$$f_2(z) = \frac{\tan\sqrt{z}}{\sqrt{z}}$$

are representable in the form:

$$[N-1/N]_{f_2}(z) = \frac{P_{N-1}(z)}{Q_N(z)},$$

where

$$P_{N-1}(z) = \sum_{k=1}^{N}(-1)^k \sum_{m=k}^{N} \kappa_m^{(N)} \frac{(2m)!}{(2m-2k)!} z^{N-k} \sum_{j=0}^{k-1} \frac{2^{2j+2}(2^{2j+2}-1)B_{j+1}}{(2j+2)!} z^j,$$

$$Q_N(z) = \sum_{k=0}^{N}(-1)^k \sum_{m=k}^{N} \kappa_m^{(N)} \frac{(2m)!}{(2m-2k)!} z^{N-k},$$

and by $\kappa_m^{(N)}$ the coefficients of shifted orthonormal on $[0,1]$ with weight $t^{-1/2}$ Jacobi polynomial

$$R_N^{(0,-1/2)}(t) = \sum_{m=0}^{N} \kappa_m^{(N)} t^m$$

are denoted, and B_j - Bernoulli numbers, defined by formulae:

$$B_j = \frac{(2j)!}{\pi^{2j}2^{2j-1}}[1 + \frac{1}{2^{2j}} + \frac{1}{3^{2j}} + \frac{1}{4^{2j}} + ...]. \quad (25)$$

Proof. Let us consider in space $C[0,1]$ linear bounded operator

$$(A\phi)(t) = \int_0^{1-t} \phi(\tau)d\tau.$$

Its second degree may be represented in the form:

$$(A^2\phi)(t) = (1-t)\int_0^t \phi(\tau)d\tau + \int_t^1 \phi(\tau)(1-\tau)d\tau.$$

Let us assume $x_0(t) \equiv 1$ and find $[R_z(A^2)x_0](t)$ from operator equation:

$$[(I - zA^2)\phi](t) = \phi(z) - z(1-t)\int_0^t \phi(\tau)d\tau - z\int_t^1 \phi(\tau)(1-\tau)d\tau = 1. \quad (26)$$

Successive double differentiation of the equality (26) gives:

$$\phi'(t) + z\int_0^t \phi(\tau)d\tau = 0, \tag{27}$$

$$\phi''(t) + z\phi(t) = 0. \tag{28}$$

General solution of equation (28) is representable in the form:

$$\phi(t) = C_1 \cos\sqrt{z}t + C_2 \sin\sqrt{z}t. \tag{29}$$

From (26) and (27) we will obtain boundary conditions:

$$\phi(1) = 1, \ \ \phi'(0) = 0. \tag{30}$$

Taking into account (29) and (30), we will receive:

$$[R_z(A^2)x_0](t) = \frac{\cos\sqrt{z}t}{\cos\sqrt{z}}.$$

Let us assume now $l_0(x) = \int\limits_0^1 x(\tau)d\tau$, and construct function:

$$f_2(x) = l_0[R_z(A^2)x_0] = \int\limits_0^1 \frac{\cos\sqrt{z}t}{\cos\sqrt{z}}dt = \frac{\tan\sqrt{z}}{\sqrt{z}}.$$

Let us assume:

$$x_{2k}(t) = (A^{2k}x_0)(t).$$

Taking into account the equality:

$$[R_z(A^2)x_0](t) = \sum_{k=0}^{\infty} z^k (A^{2k}x_0)(t) = \sum_{k=0}^{\infty} z^k x_{2k}(t),$$

as well as expansion:

$$\frac{\cos\sqrt{z}t}{\cos\sqrt{z}} = \cos\sqrt{z}t \sec\sqrt{z} = \sum_{k=0}^{\infty} \frac{(-1)^k z^k t^{2k}}{(2k)!} \sum_{k=0}^{\infty} \frac{E_k z^k}{(2k)!} =$$

$$= \sum_{k=0}^{\infty} z^k \sum_{m=0}^{k} \frac{(-1)^m t^{2m} E_{k-m}}{(2m)!(2k-2m)!},$$

where E_k are Euler numbers defined by formulae:

$$E_k = \frac{2^{2k+2}(2k)!}{\pi^{2k+1}}[1 - \frac{1}{3^{2k+1}} + \frac{1}{5^{2k+1}} - \frac{1}{7^{2k+1}} + ...], \tag{31}$$

we will obtain:

$$x_{2k}(t) = \sum_{m=0}^{k} \frac{(-1)^m t^{2m} E_{k-m}}{(2m)!(2k-2m)!},$$

i.e. functions $x_{2k}(t)$ are even algebraic polynomials of degree equal exactly to $2k$. Let us take into account also that

$$l_{2k}(x) = A^{*2k} l_0(x) = l_0(A^{2k}x) = \int\limits_0^1 (A^{2k}x)(t)dt = \int\limits_0^1 \int\limits_0^{1-t} (A^{2k-1}x)(\tau)d\tau dt =$$

$$= \int\limits_0^1 \int\limits_0^t (A^{2k-1}x)(\tau)d\tau dt = \int\limits_0^1 (A^{2k-1}x)(t)(1-t)dt = ... = \int\limits_0^1 x(t)x_{2k}(t)dt. \quad (32)$$

According to [2] Padé approximant for function $f_2(z)$ of order $[N-1/N]$, $N \geq 1$ may be written in the form:

$$[N-1/N]_{f_2}(z) = \frac{P_{N-1}(z)}{Q_N(z)},$$

where

$$P_{N-1}(z) = \sum_{m=1}^{N} c_m^{(N)} z^{N-m} \sum_{k=0}^{m-1} s_k z^k, \quad (33)$$

$$Q_N(z) = \sum_{m=0}^{N} c_m^{(N)} z^{N-m}, \quad (34)$$

and coefficients $c_m^{(N)}$, $m = \overline{0, N}$ are defined from bi-orthogonality relations for generalized polynomial:

$$L_{2N} = \sum_{m=0}^{N} c_m^{(N)} l_{2m}$$

of the form:

$$L_{2N}(x_{2k}) = 0, k = \overline{0, N-1},$$

and s_k, $k = \overline{0, \infty}$ - Maclaurin coefficients of the function $f_2(z)$.

Keeping in mind (32) we conclude that the construction of polynomial L_{2N} is equivalent to construction of polynomial

$$X_{2N}(t) = \sum_{m=0}^{N} c_m^{(N)} x_{2m}(t),$$

having bi-orthogonality properties

$$\int\limits_0^1 x_{2k}(t)X_{2N}(t)dt = 0, \; k = \overline{0, N-1}.$$

Taking into account that $x_{2k}(t)$ are even algebraic polynomials one can write:

$$X_{2N}(t) = U_N(t^2),$$

where $U_N(t)$ is algebraic polynomial of degree equal exactly to N such that

$$\int\limits_0^1 U_N(t^2)t^{2k}dt = 0,\ k = \overline{0, N-1}.$$

Fulfilling the substitution $v = t^2$ in the last integral we see that $U_N(v)$ is shifted orthonormal on $[0, 1]$ with the weight $v^{-1/2}$ Jacobi polynomial up to constant multiplyer (see, for example [5])

$$U_N(v) = \sum_{m=0}^{N} \kappa_m^{(N)} v^m = R_N^{(0,-1/2)}(v).$$

In order to determine coefficients $c_m^{(N)}$ of the polynomial $X_{2N}(t)$ we need, therefore, to find the expression of even degrees of variable by means of functions $x_{2k}(t)$. We have:

$$\frac{\cos\sqrt{z}t}{\cos\sqrt{z}} = \sum_{k=0}^{\infty} z^k x_{2k}(t).$$

Hence

$$\cos\sqrt{z}t = \cos\sqrt{z}\sum_{k=0}^{\infty} z^k x_{2k}(t)$$

or

$$\sum_{k=0}^{\infty} \frac{z^k(-1)^k t^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{z^k(-1)^k}{(2k)!} \sum_{k=0}^{\infty} z^k x_{2k}(t) = \sum_{k=0}^{\infty} z^k \sum_{m=0}^{k} x_{2m}(t)\frac{(-1)^{k-m}}{(2k-2m)!}.$$

From here we obtain

$$t^{2k} = \sum_{m=0}^{k} x_{2m}(t)\frac{(-1)^m(2k)!}{(2k-2m)!}.$$

Thus,

$$X_{2N}(t) = U_N(t^2) = \sum_{k=0}^{N} \kappa_k^{(N)} t^{2k} = \sum_{k=0}^{N} \kappa_k^{(N)} \sum_{m=0}^{k} x_{2m}(t)\frac{(-1)^m(2k)!}{(2k-2m)!} =$$

$$= \sum_{m=0}^{N} x_{2m}(t)(-1)^m \sum_{k=m}^{N} \kappa_k^{(N)} \frac{(2k)!}{(2k-2m)!},$$

whence

$$c_m^{(N)} = (-1)^m \sum_{k=m}^{N} \kappa_k^{(N)} \frac{(2k)!}{(2k-2m)!}. \tag{35}$$

Substituting (35) in (33)-(34) and taking account of well-known formula for Maclaurin coefficients of function $f_2(z)$, we will obtain the statement of the Theorem 2.

Remark. Let us note that Padé approximants for $f_2(z)$ by another way were constructed in [1].

4. *Padé Approximants for function* $f_3(z)$.
Theorem 3. Padé approximants of orders $[N-1/N]$, $N \geq 1$ for function

$$f_3(z) = \frac{\sin z + 1 - \cos z}{z \cos z}$$

are representable in the form:

$$[N-1/N]_{f_2}(z) = \frac{P_{N-1}(z)}{Q_N(z)},$$

where

$$P_{N-1}(z) = \sum_{k=1}^{N} (-1)^{[k/2]} \sum_{m=k}^{N} l_m^{(N)} \frac{m!}{(m-k)!} [\epsilon_m + \delta_{k,m}(1-\epsilon_m)] z^{N-k} \sum_{j=0}^{k-1} s_j z^j,$$

$$Q_N(z) = \sum_{k=0}^{N} (-1)^{[k/2]} \sum_{m=k}^{N} l_m^{(N)} \frac{m!}{(m-k)!} [\epsilon_m + \delta_{k,m}(1-\epsilon_m)] z^{N-k},$$

and by $l_k^{(N)}$, $k = \overline{0, N}$ the coefficients of shifted orthonormal on $[0,1]$ Legendre polynomial are denoted,

$$\epsilon_m = \begin{cases} 1, & \text{if } m \text{ is even}, \\ 0, & \text{if } m \text{ is odd}, \end{cases}$$

Kronecker symbol $\delta_{k,m}$ is defined by formula:

$$\delta_{k,m} = \begin{cases} 1, & \text{if } k = m, \\ 0, & \text{if } k \neq m, \end{cases}$$

and s_j, $j = \overline{0, \infty}$ are Maclaurin coefficients of function $f_3(z)$:

$$s_j = \begin{cases} \frac{2^{2k+2}(2^{2k+2}-1)B_{k+1}}{(2k+2)!}, & \text{if } j = 2k, \\ \frac{E_{k+1}}{(2k+2)!}, & \text{if } j = 2k+1 \end{cases}$$

(Bernoulli numbers B_k and Euler numbers E_k are defined respectively by formulae (25) and (31)).

Proof. Let us use the same operator A as in proof of the Theorem 2. We have established that

$$[R_z(A^2)x_0](t) = \frac{\cos\sqrt{z}t}{\cos\sqrt{z}}.$$

Hence

$$[R_z(A)x_0](t) = \{(I + zA)R_{z^2}(A^2)x_0\}(t) =$$

$$= \frac{\cos zt}{\cos z} + z \int_0^{1-t} \frac{\cos z\tau}{\cos z} d\tau = \frac{\cos zt + \sin z(1-t)}{\cos z}.$$

Assuming $l_0(x) = \int_0^1 x(\tau)d\tau$, we receive the function

$$f_3(z) = l_0[R_z(A)x_0] = \int_0^1 \frac{\cos zt + \sin z(1-t)}{\cos z} dt = \frac{\sin z + 1 - \cos z}{z \cos z}.$$

While proving the Theorem 2 we also have obtained that

$$x_{2k}(t) = (A^{2k}x_0)(t) = \sum_{m=0}^{k} \frac{(-1)^m t^{2m} E_{k-m}}{(2m)!(2k-2m)!}. \quad (36)$$

Hence

$$x_{2k+1}(t) = (Ax_{2k})(t) = \sum_{m=0}^{k} \frac{(-1)^m (1-t)^{2m+1} E_{k-m}}{(2m+1)!(2k-2m)!}. \quad (37)$$

Formulae (36) and (37) ensure that $x_k(t)$ are algebraic polynomials of degrees equal exactly to k.

According to [2] Padé approximant for function $f_3(z)$ of order $[N-1/N]$, $N \geq 1$ may be written in the form:

$$[N-1/N]_{f_1}(z) = \frac{P_{N-1}(z)}{Q_N(z)},$$

where

$$P_{N-1}(z) = \sum_{m=1}^{N} c_m^{(N)} z^{N-m} \sum_{k=0}^{m-1} s_k z^k, \quad (38)$$

$$Q_N(z) = \sum_{m=0}^{N} c_m^{(N)} z^{N-m}, \quad (39)$$

and coefficients $c_m^{(N)}$, $m = \overline{0, N}$ are defined from bi-orthogonality relations for generalized polynomial:

$$L_N = \sum_{m=0}^{N} c_m^{(N)} l_m$$

of the form:

$$L_N(x_k) = 0, k = \overline{0, N-1},$$

and s_k, $k = \overline{0, \infty}$ - Maclaurin coefficients of the function $f_3(z)$.

As before we conclude that construction of polynomials L_N is equivalent to construction of the polynomial

$$X_N(t) = \sum_{m=0}^{N} c_m^{(N)} x_m(t),$$

having bi-orthogonality properties

$$\int_0^1 x_k(t)X_N(t)dt = 0,\ k = \overline{0, N-1},$$

but this construction taking into account stated above will give us as well as in Theorem 1 shifted orthonormal on $[0,1]$ Legendre polynomials $L_N^*(t)$ (up to constant mulriplyer). In order to obtain coefficients $c_m^{(N)}$ of polynomial $X_N(t)$ let us first find expressions of functions t^k, $k = \overline{0,\infty}$ by means of functions $x_k(t)$, $k = \overline{0,\infty}$. For even degrees these expressions are received in the proof of the Theorem 2:

$$t^{2k} = \sum_{m=0}^{k} x_{2m}(t)\frac{(-1)^m(2k)!}{(2k-2m)!}.$$

For odd degrees let us write expression with indeterminate coefficients:

$$t^{2k+1} = \sum_{m=0}^{k} \alpha_m^{(k)} x_{2m}(t) + \sum_{m=0}^{k} \beta_m^{(k)} x_{2m+1}(t). \tag{40}$$

Let us apply operator A^2 to (40). We will obtain:

$$\frac{1-t^{2k+3}}{(2k+2)(2k+3)} = \sum_{m=0}^{k} \alpha_m^{(k)} x_{2m+2}(t) + \sum_{m=0}^{k} \beta_m^{(k)} x_{2m+3}(t). \tag{41}$$

From other hand

$$\frac{1-t^{2k+3}}{(2k+2)(2k+3)} = \frac{1}{(2k+2)(2k+3)}[x_0(t) - \sum_{m=0}^{k+1} \alpha_m^{(k+1)} x_{2m}(t) -$$

$$- \sum_{m=0}^{k+1} \beta_m^{(k+1)} x_{2m+1}(t)]. \tag{42}$$

Comparing right sides of (41) and (42) and taking into account linear independence of functions $x_k(t)$, $k = \overline{0,\infty}$, we will receive

$$\alpha_0^{(k+1)} = 1,$$

$$\alpha_m^{(k+1)} = -(2k+2)(2k+3)\alpha_{m-1}^{(k)} = ... = (-1)^m \frac{(2k+3)!}{(2k-2m+3)!}\alpha_0^{(k-m+1)},$$

whence

$$\alpha_m^{(k)} = (-1)^m \frac{(2k+1)!}{(2k-2m+1)!},$$

and also

$$\beta_0^{(k+1)} = 0,$$

$$\beta_m^{(k+1)} = -(2k+2)(2k+3)\beta_{m-1}^{(k)} = ... = (-1)^m \frac{(2k+3)!}{(2k-2m+3)!}\beta_0^{(k-m+1)},$$

whence

$$\beta_m^{(k)} = 0, \text{ if } m < k,$$

$$\beta_k^{(k)} = (-1)^k (2k+1)!\beta_0^{(0)} = -(-1)^k (2k+1)!.$$

We obtain the representation:

$$t^{2k+1} = \sum_{m=0}^{k} (-1)^k \frac{(2k+1)!}{(2k-2m+1)!} x_{2m}(t) - (-1)^k (2k+1)! x_{2k+1}(t). \qquad (43)$$

Combining formulae (40) and (43) we will receive:

$$t^k = \sum_{m=0}^{[k/2]} (-1)^m \frac{k!}{(k-2m)!} x_{2m}(t) - (1-\epsilon_k)(-1)^{(k-1)/2} k! x_k(t), \qquad (44)$$

where

$$\epsilon_m = \begin{cases} 1, & \text{if } m \text{ is even,} \\ 0, & \text{if } m \text{ is odd.} \end{cases}$$

From (44) we have:

$$X_N(t) = \sum_{k=0}^{N} c_k^{(N)} x_k(t) = L_N^*(t) = \sum_{k=0}^{N} l_k^{(N)} t^k =$$

$$= \sum_{k=o}^{N} l_k^{(N)} [\sum_{m=0}^{[k/2]} (-1)^m \frac{k!}{(k-2m)!} x_{2m}(t) - \epsilon_k (-1)^{(k-1)/2} k! x_k(t)] =$$

$$= \sum_{m=0}^{[N/2]} (-1)^m x_{2m}(t) \sum_{k=m}^{[N/2]} l_{2k}^{(N)} \frac{(2k)!}{(2k-2m)!} +$$

$$+ \sum_{m=0}^{[(N-1)/2]} (-1)^m x_{2m}(t) \sum_{k=m}^{[(N-1)/2]} l_{2k+1}^{(N)} \frac{(2k+1)!}{(2k-2m+1)!} -$$

$$- \sum_{k=0}^{[(N-1)/2]} (-1)^k x_{2k+1}(t) l_{2k+1}^{(N)} (2k+1)!.$$

Thus for $N = 2M$ being even we will obtain

$$c_{(2m)}^{2M} = (-1)^m [\sum_{k=m}^{M} l_{2k}^{(2M)} \frac{(2k)!}{(2k-2m)!} + (1-\delta_{m,M}) \sum_{k=m}^{M-1} l_{2k+1}^{(2M)} \frac{(2k+1)!}{(2k-2m+1)!},$$

$$c_{2m+1}^{(2M)} = (-1)^m l_{2m+1}^{(2M)} (2m+1)!. \qquad (45)$$

For $N = 2M+1$ being odd

$$c_{2m}^{(2M+1)} = (-1)^m[\sum_{k=m}^{M} l_{2k}^{(2M+1)} \frac{(2k)!}{(2k-2m)!} + \sum_{k=m}^{M} l_{2k+1}^{(2M+1)} \frac{(2k+1)!}{(2k-2m+1)!}],$$

$$c_{2m+1}^{(2M+1)} = (-1)^m l_{2m+1}^{(2M+1)} (2m+1)!. \tag{46}$$

Substituting ((45)-(46) to (38)-(39) we will receive the statement of the Theorem 3.

Remark. Continuing the reasoning used in proofs of the Theorem 2 and Theorem 3 it is possible to construct also Padé approximants of orders $[N - 1/N]$, $N \geq 1$ for function $f(z) = (sec\sqrt{z} - 1)/z$, which is representable in the form $f(z) = l_1(R_z(A^2)x_0)$, where A_0, x_0 and l_1 are just the same as in mentioned theorems. This result is equivalent to construction of diagonal Padé approximants for function $\cos z$ carried out in [6]. Besides that if in the proof of the Theorem 3 instead of operator $(A\phi)(t) = \int\limits_0^{1-t} \phi(\tau)d\tau$ one consider operator

$$(A\phi)(t) = \alpha \int\limits_0^t \phi(\tau)d\tau + \int\limits_0^{1-t} \phi(\tau)d\tau$$

(for $\alpha \neq 1$) it is possible to construct Padé approximants for function

$$f(z) = \frac{(1-\alpha)\frac{\sin z\sqrt{1-\alpha^2}}{\sqrt{1-\alpha^2}} - \cos z\sqrt{1-\alpha^2} + 1}{z[\cos z\sqrt{1-\alpha^2} - \alpha]}.$$

References

1. Luke Y., *Mathematical Functions and Their Approximations*, Academic Press, New York (1975).

2. Dzyadyk,V.K., *A Generalization of Moment Problem*, Dokl.Akad.Nauk Ukr.SSR, Ser.A, No.6 (1981), 8-12.

3. Baker G.A., Graves-Morris P.R., *Padé Approximants*, Parts I and II, Addison-Wesley, Reading, Mass.(1981).

4. Golub A.P., *Generalized Moment Representations and Rational Approximants*, Institute of Mathematics of the Academy of Sciences of Ukrainian SSR, Preprint No.25, (1987).

5. Suetin P.K., *Classical Orthogonal Polynomials*, Nauka Publishers, Moscow (1979).

6. Dzyadyk, V.K. *On Asymptotics of Diagonal Padé Approximants of functions* $\sin z$*,* $\cos z$*,* $\sinh z$ *and* $\cosh z$, Mat.Sbornik, **108** (1979), 247-267.