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EXISTENCE THEOREMS FOR MULTIDIMENSIONAL GENERALIZED MOMENT REPRESENTATIONS

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We establish the conditions for the existence of multidimensional generalized moment representations.

In 1981, Dzyadyk [1] proposed a method of generalized moment representations, which later turned into an efficient tool for the construction and investigation of rational approximations of special functions (see [2]).

Definition 1 [1]. *A generalized moment representation of the number sequence of complex numbers* $\{s_k\}_{k\in\mathbb{Z}_+}$ *on the product of linear spaces* $\mathcal{X}\times\mathcal{Y}$ *is defined as a two-parameter collection of equalities*

$$s_{k+j} = \langle x_k, y_j\rangle, \quad k, j \in \mathbb{Z}_+, \tag{1}$$

where $\{x_k\}_{k\in\mathbb{Z}_+}\subset\mathcal{X},\ \{y_j\}_{j\in\mathbb{Z}_+}\subset\mathcal{Y}$, *and* $\langle .,.\rangle$ *is a bilinear form defined on* $\mathcal{X}\times\mathcal{Y}$.

The following result concerning the conditions of existence of mappings of the form (1) was established in [3]:

Theorem 1 [3, 4]. *Suppose that* $\mathcal{H}$ *is an infinite-dimensional separable Hilbert space and* $\{e_k\}_{k\in\mathbb{Z}_+}$ *is an orthonormal basis in this space. In order that a sequence* $\{s_k\}_{k\in\mathbb{Z}_+}$ *have a generalized moment representation of the form (1), where*

$$\langle x, y\rangle = \sum_{m=0}^{\infty}(x, e_m)(y, e_m)$$

and the elements $x_k, k\in\mathbb{Z}_+$, *and* $y_j, j\in\mathbb{Z}_+$, *have the form*

$$x_k = \sum_{m=0}^{k}\alpha_m^{(k)}e_m, \quad \alpha_k^{(k)}\neq 0, \quad k\in\mathbb{Z}_+; \qquad y_j = \sum_{m=0}^{j}\beta_m^{(j)}e_m, \quad \beta_j^{(j)}\neq 0, \quad j\in\mathbb{Z}_+, \tag{2}$$

it is necessary and sufficient that all Hankel determinants of this sequence

$$H_N := H_{0,N} = \det\|s_{k+j}\|_{k,j=0}^{N}, \quad N\in\mathbb{Z}_+,$$

be nonzero.

Moreover, the relations

$$\alpha_p^{(p)}\beta_p^{(p)} = \frac{H_p}{H_{p-1}}, \qquad p\in\mathbb{Z}_+, \quad H_{-1} := 1, \tag{3}$$

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are true and if the sequences of nonzero numbers $\{\alpha_p^{(p)}\}_{p\in\mathbb{Z}_+}$ *and* $\{\beta_p^{(p)}\}_{p\in\mathbb{Z}_+}$ *satisfying (3) are fixed, then the remaining coefficients in (2) are uniquely determined by the formulas*

$$\alpha_p^{(k)} = \alpha_k^{(k)} \frac{S_k\begin{pmatrix} 0 & 1 & \dots & p-1 & k \\ 0 & 1 & \dots & p-1 & p \end{pmatrix}}{H_p}, \qquad p = \overline{0,k}, \quad k \in \mathbb{Z}_+, \tag{4}$$

$$\beta_p^{(j)} = \beta_j^{(j)} \frac{S_j\begin{pmatrix} 0 & 1 & \dots & p-1 & p \\ 0 & 1 & \dots & p-1 & j \end{pmatrix}}{H_p}, \qquad p = \overline{0,j}, \quad j \in \mathbb{Z}_+, \tag{5}$$

where $S_N\begin{pmatrix} l_0 & l_1 & \dots & l_r \\ n_0 & n_1 & \dots & n_r \end{pmatrix}$ *are minors of the matrix*

$$S_N = \|s_{k+j}\|_{k,j=0}^N = \begin{Vmatrix} s_0 & s_1 & \dots & s_N \\ s_1 & s_2 & \dots & s_{N+1} \\ \dots & \dots & \dots & \dots \\ s_N & s_{N+1} & \dots & s_{2N} \end{Vmatrix}, \quad N \in \mathbb{Z}_+,$$

with the numbers of columns $l_0, l_1, \dots, l_r$ *and the numbers of rows* $n_0, n_1, \dots, n_r$ *for* $l_m \le N$ *and* $n_m \le N$, $m = \overline{0,r}$.

Later, the method of generalized moment representations was developed for two- and multidimensional sequences [5, 6]. This led to the problem of determination of the conditions of existence of generalized multidimensional moment representations.

Definition 2 [5]. *A generalized moment representation of the two-dimensional number sequence*

$$\{s_{k,m}\}_{k,m\in\mathbb{Z}_+}$$

on the product of linear spaces $\mathcal{X} \times \mathcal{Y}$ *is defined as a collection of equalities*

$$s_{k+j,m+n} = \langle x_{k,m}, y_{j,n} \rangle, \quad k, j, m, n \in \mathbb{Z}_+,$$

where $\{x_{k,m}\}_{k,m\in\mathbb{Z}_+} \subset \mathcal{X}$, $\{y_{j,n}\}_{j,n\in\mathbb{Z}_+} \subset \mathcal{Y}$, *and* $\langle \cdot,\cdot \rangle$ *is a bilinear form on* $\mathcal{X} \times \mathcal{Y}$.

Prior to formulating the corresponding result, we recall that the Cantor numbering function

$$c(x,y) = \frac{(x+y)^2 + x + 3y}{2}$$

bijectively maps $\mathbb{Z}_+^2$ onto $\mathbb{Z}_+$ (see, e.g., [7, p. 13]). Moreover, there exist inverse functions $l, r\colon \mathbb{Z}_+ \to \mathbb{Z}_+$, such that

$$c(l(n), r(n)) \equiv n, \qquad l(c(m,n)) = m, \qquad r(c(m,n)) = n \quad \forall m,n \in \mathbb{Z}_+.$$

By using the two-dimensional sequence $\{s_{k,m}\}_{k,m\in\mathbb{Z}_+}$, we can define a one-dimensional sequence $\{\tilde{s}_p\}_{p\in\mathbb{Z}_+}$ such that

$$s_{k,m} = \tilde{s}_{c(k,m)}, \quad (k,m) \in \mathbb{Z}_+^2,$$
$$\tilde{s}_p = s_{l(p),r(p)}, \quad p \in \mathbb{Z}_+. \tag{6}$$

We also construct the sequence of matrices

$$\tilde{S}_N = \left\| s_{l(k)+l(j),r(k)+r(j)} \right\|_{k,j=0}^{N}, \quad N \in \mathbb{Z}_+. \tag{7}$$

In the indicated terms, we can formulate the following result:

Theorem 2. *Suppose that $\mathcal{H}$ is an infinite-dimensional separable Hilbert space and that $\{e_k\}_{k\in\mathbb{Z}_+}$ is an orthonormal basis in this space. In order that a sequence $\{s_{k,m}\}_{k,m\in\mathbb{Z}_+}$ possess a generalized moment representation of the form*

$$s_{k+j,m+n} = \langle x_{k,m}, y_{j,n} \rangle, \quad k,j,m,n \in \mathbb{Z}_+, \tag{8}$$

where

$$\langle x, y \rangle = \sum_{m=0}^{\infty} (x, e_m)(y, e_m),$$

and the elements $\{x_{k,m}\}_{k,m\in\mathbb{Z}_+} \subset \mathcal{X}$ and $\{y_{j,n}\}_{j,n\in\mathbb{Z}_+} \subset \mathcal{X}$ have the form

$$x_{k,m} = \sum_{p=0}^{c(k,m)} \alpha_p^{(k,m)} e_p, \qquad \alpha_{c(k,m)}^{(k,m)} \neq 0, \quad (k,m) \in \mathbb{Z}_+^2, \tag{9}$$

$$y_{j,n} = \sum_{p=0}^{c(j,n)} \beta_p^{(j,n)} e_p, \qquad \beta_{c(j,n)}^{(j,n)} \neq 0, \quad (j,n) \in \mathbb{Z}_+^2, \tag{10}$$

it is necessary and sufficient that all determinants $\tilde{H}_N = \det \tilde{S}_N$, $N \in \mathbb{Z}_+$, of the matrices given by relations (7) be nonzero. Moreover, the relations

$$\alpha_{c(k,m)}^{(k,m)} \beta_{c(k,m)}^{(k,m)} = \frac{\tilde{H}_{c(k,m)}}{\tilde{H}_{c(k,m)-1}}, \qquad (k,m) \in \mathbb{Z}_+^2, \quad \tilde{H}_{-1} := 1, \tag{11}$$

are true and if the sequences of nonzero numbers

$$\left\{ \alpha_p^{(l(p),r(p))} \right\}_{p\in\mathbb{Z}_+} \quad \text{and} \quad \left\{ \beta_p^{(l(p),r(p))} \right\}_{p\in\mathbb{Z}_+}$$

satisfying (11) are fixed, then the other coefficients in (9) and (10) are uniquely determined by the relations

$$\alpha_p^{(k,m)} = \alpha_{c(k,m)}^{(k,m)} \frac{\tilde{S}_{c(k,m)} \begin{pmatrix} 0 & 1 & \ldots & p-1 & c(k,m) \\ 0 & 1 & \ldots & p-1 & p \end{pmatrix}}{\tilde{H}_{c(k,m)}}, \qquad p = \overline{0, c(k,m)}, \quad (k,m) \in \mathbb{Z}_+^2, \tag{12}$$

$$\beta_p^{(j,n)} = \beta_{c(j,n)}^{(j,n)} \frac{\tilde{S}_{c(j,n)}\begin{pmatrix} 0 & 1 & \dots & p-1 & p \\ 1 & 2 & \dots & p-1 & c(j,n) \end{pmatrix}}{\tilde{H}_{c(j,n)}}, \qquad p = \overline{0, c(j,n)}, \quad (j,n) \in \mathbb{Z}_+^2. \tag{13}$$

Proof. It is easy to see that, in view of (9) and (10), equalities (8) are equivalent to the equalities

$$s_{k+j,m+n} = \sum_{p=0}^{\min\{c(k,m),\, c(j,n)\}} \alpha_p^{(k,m)} \beta_p^{(j,n)}, \quad k,m,j,n \in \mathbb{Z}_+, \tag{14}$$

and, in turn, equalities (14) are equivalent to a family of matrix equalities

$$\tilde{S}_N = A_N \cdot B_N, \quad N = \overline{0,\infty},$$

where A_N is a lower triangular matrix of the form

$$A_N = \|a_{j,k}\|_{k,j=0}^{N}, \qquad a_{j,k} = \begin{cases} \alpha_j^{(l(k),r(k))} & \text{for } k \geq j, \\ 0 & \text{for } k < j, \end{cases}$$

and B_N is an upper triangular matrix of the form

$$B_N = \|b_{j,k}\|_{k,j=0}^{N}, \qquad b_{j,k} = \begin{cases} 0 & \text{for } k > j, \\ \beta_j^{(l(k),r(k))} & \text{for } k \leq j. \end{cases}$$

Therefore,

$$\tilde{H}_N = \det \tilde{S}_N = \prod_{p=0}^{N} \alpha_p^{(l(p),r(p))} \cdot \prod_{q=0}^{N} \beta_q^{(l(q),r(q))} \neq 0.$$

This yields the *necessity* of the assertion of the theorem. Its *sufficiency* is a corollary of the theorem on factorization of a nonsingular matrix in triangular factors (see [8, p. 50]).

Similarly, we can establish a condition for the existence of d-dimensional generalized moment representations.

Definition 3 [6]. *A generalized moment representation of a d-dimensional number sequence $\{s_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^d}$ on the product of linear spaces $\mathcal{X} \times \mathcal{Y}$ is defined as a collection of equalities*

$$s_{\mathbf{k}+\mathbf{j}} = \langle x_{\mathbf{k}}, y_{\mathbf{j}} \rangle, \quad \mathbf{k}, \mathbf{j} \in \mathbb{Z}_+^d,$$

where $\{x_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^d} \subset \mathcal{X}$, $\{y_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}_+^d} \subset \mathcal{Y}$, and $\langle\cdot,\cdot\rangle$ is a bilinear form on $\mathcal{X} \times \mathcal{Y}$.

It is known (see [7, p. 14]) that one can find a function

$$c^d \colon \mathbb{Z}_+^d \to \mathbb{Z}_+$$

that bijectively maps $\mathbb{Z}_+^d$ onto $\mathbb{Z}_+$ and, in addition, the inverse functions $l_1, l_2, \dots, l_d \colon \mathbb{Z}_+ \to \mathbb{Z}_+$ are uniquely defined and such that

$$c^d\big(l_1(n), l_2(n), \dots, l_d(n)\big) \equiv n, \qquad l_i\big(c^d(n_1, \dots, n_i, \dots, n_d)\big) = n_i, \quad i = \overline{1,d}, \quad n \in \mathbb{Z}_+.$$

Thus, for any d-dimensional number sequence $\{s_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^d}$, we can construct a sequence of matrices

$$\tilde{S}_N = \left\| s_{l_1(k)+l_1(j),l_2(k)+l_2(j),\ldots,l_d(k)+l_d(j)} \right\|_{k,j=0}^{N}, \quad N \in \mathbb{Z}_+. \tag{15}$$

The following assertion is true:

Theorem 3. *Suppose that $\mathcal{H}$ is an infinite-dimensional separable Hilbert space and $\{e_k\}_{k\in\mathbb{Z}_+}$ is an orthonormal basis in this space. In order that a sequence $\{s_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^d}$ possess a generalized moment representation of the form*

$$s_{\mathbf{k}+\mathbf{j}} = \langle x_{\mathbf{k}}, y_{\mathbf{j}} \rangle, \quad \mathbf{k}, \mathbf{j} \in \mathbb{Z}_+^d, \tag{16}$$

where

$$\langle x, y \rangle = \sum_{m=0}^{\infty} (x, e_m)(y, e_m)$$

and elements $\{x_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^d}$ and $\{y_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}_+^d}$ have the form

$$x_{\mathbf{k}} = \sum_{p=0}^{c^d(\mathbf{k})} \alpha_p^{(\mathbf{k})} e_p, \qquad \alpha_{c^d(\mathbf{k})}^{(\mathbf{k})} \neq 0, \quad \mathbf{k} \in \mathbb{Z}_+^d, \tag{17}$$

$$y_{\mathbf{j}} = \sum_{p=0}^{c^d(\mathbf{j})} \beta_p^{(\mathbf{j})} e_p, \qquad \beta_{c^d(\mathbf{j})}^{(\mathbf{j})} \neq 0, \quad \mathbf{j} \in \mathbb{Z}_+^d, \tag{18}$$

it is necessary and sufficient that all determinants $\tilde{H}_p = \det \tilde{S}_N$, $N \in \mathbb{Z}_+$, of the matrices $\tilde{S}_N$ given by relations (15) be nonzero.

Moreover, the relations

$$\alpha_{c^d(\mathbf{k})}^{(\mathbf{k})} \beta_{c^d(\mathbf{k})}^{(\mathbf{k})} = \frac{\tilde{H}_{c^d(\mathbf{k})}}{\tilde{H}_{c^d(\mathbf{k})-1}}, \qquad \tilde{H}_{-1} := 1, \quad \mathbf{k} \in \mathbb{Z}_+^d, \tag{19}$$

are true and if the sequences of nonzero numbers

$$\{\alpha_p^{(\mathbf{l}(p))}\}_{p\in\mathbb{Z}_+} \quad \textit{and} \quad \{\beta_p^{(\mathbf{l}(p))}\}_{p\in\mathbb{Z}_+},$$

where $\mathbf{l}(p) = \big(l_1(p), l_2(p), \ldots, l_d(p)\big)$, satisfying (19) are fixed, then the remaining coefficients in (17) and (18) are uniquely determined by the relations

$$\alpha_p^{(\mathbf{k})} = \alpha_{c^d(\mathbf{k})}^{(\mathbf{k})} \frac{\tilde{S}_{c^d(\mathbf{k})} \begin{pmatrix} 0 & 1 & \ldots & p-1 & c^d(\mathbf{k}) \\ 0 & 1 & \ldots & p-1 & p \end{pmatrix}}{\tilde{H}_{c^d(\mathbf{k})}}, \qquad p = \overline{0, c^d(\mathbf{k})}, \quad \mathbf{k} \in \mathbb{Z}_+^d, \tag{20}$$

$$\beta_p^{(\mathbf{j})} = \beta_{c^d(\mathbf{j})}^{(\mathbf{j})} \frac{\tilde{S}_{c^d(\mathbf{j})} \begin{pmatrix} 0 & 1 & \ldots & p-1 & p \\ 0 & 1 & \ldots & p-1 & c^d(\mathbf{j}) \end{pmatrix}}{\tilde{H}_{c^d(\mathbf{j})}}, \qquad p = \overline{0, c^d(\mathbf{j})}, \quad \mathbf{j} \in \mathbb{Z}_+^d. \tag{21}$$

It is known (see [2]) that the problem of generalized moment representations can be formulated in terms of linear operators. Indeed, if we have a generalized moment representation of the form (1) and, in the space $\mathcal{X}$, one can find a linear operator $A\colon\ \mathcal{X} \to \mathcal{X}$ such that

$$Ax_k = x_{k+1}, \quad k \in \mathbb{Z}_+, \tag{22}$$

whereas in the space $\mathcal{Y}$, there exists a linear operator $A^*\colon\ \mathcal{Y} \to \mathcal{Y}$ adjoint to the operator A with respect to the bilinear form $\langle \cdot, \cdot \rangle$ in a sense that

$$\langle Ax, y \rangle = \langle x, A^* y \rangle \qquad \forall x \in \mathcal{X} \quad \forall y \in \mathcal{Y},$$

then the representation of the form (1) is equivalent to the representation

$$s_k = \langle A^k x_0, y_0 \rangle, \quad k \in \mathbb{Z}_+. \tag{23}$$

If, in addition, the spaces $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, the bilinear form $\langle \cdot, \cdot \rangle$ is separately continuous, and the operator A is bounded, then the series

$$\sum_{k=0}^{\infty} s_k z^k$$

converges in a neighborhood of the origin to an analytic function f, which can be represented in the form

$$f(z) = \langle \mathcal{R}_z(A) x_0, y_0 \rangle, \tag{24}$$

where $\mathcal{R}_z(A) = (I - zA)^{-1}$ is the resolvent function of the operator A.

This leads to the problem of existence of representations of the form (23), (24). Actually, this problem was solved in [9] prior to the appearance of the method of generalized moment representations.

Theorem 4 [9]. *For any function f analytic in the disk $K_R = \{z\colon\ |z| \le R\}$, $0 < R < \infty$, and any infinite-dimensional separable Hilbert space $\mathcal{H}$, there exist elements $x_0, y_0 \in \mathcal{H}$ and a linear bounded operator $A\colon \mathcal{H} \to \mathcal{H}$ with the norm $\|A\| < \dfrac{1}{R}$ such that, for any $z \in K_R$,*

$$f(z) = \big(\mathcal{R}_z(A) x_0, y_0\big). \tag{25}$$

Remark. Representation (25) is equivalent to representation (24) with

$$\langle x, y \rangle = \sum_{m=0}^{\infty} (x, e_m)(y, e_m)$$

playing the role of bilinear form, where $\{e_p\}_{p \in \mathbb{Z}_+}$ is an orthonormal basis in the space $\mathcal{H}$.

A similar result for entire functions was obtained in [4].

Theorem 5 [4]. *For any entire function f and any infinite-dimensional separable Hilbert space $\mathcal{H}$, there exist elements $x_0, y_0 \in \mathcal{H}$ and a linear bounded operator $A\colon\ \mathcal{H} \to \mathcal{H}$ whose spectral radius is equal to zero such that the representation*

$$f(z) = \big(\mathcal{R}_z(A) x_0, y_0\big) \tag{26}$$

is true.

In addition, if the entire function has the order $\rho > 0$, then the operator A can be chosen so that, for any $n \in \mathbb{N}$,

$$\sqrt[n]{\|A^n\|} \leq \frac{C}{n^{\frac{1}{\rho}}}, \tag{27}$$

where C is a constant.

These problems were also investigated in [10] where, in particular, the author considered representations of the form (25) with unbounded operators A.

As in the one-dimensional case, the problem of generalized moment representations for greater dimensions can be formulated in terms of linear operators (see [5, 6]). Indeed, if we have a generalized moment representation of the form (16) and, in the space $\mathcal{X}$, one can find commuting linear operators $A_j\colon \mathcal{X} \to \mathcal{X},\ j = \overline{1,d}$, such that

$$A_j x_{\mathbf{k}} = x_{\mathbf{k}+\boldsymbol{\delta}_{\mathbf{j}}}, \qquad j = \overline{1,d}, \quad \mathbf{k} \in \mathbb{Z}_+^d,$$

where

$$\boldsymbol{\delta}_j = (\delta_{j,1}, \delta_{j,2}, \ldots, \delta_{j,d}), \qquad \delta_{j,k} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases}$$

and, in the space $\mathcal{Y}$, there exist linear operators $A_j^*\colon \mathcal{Y} \to \mathcal{Y},\ j = \overline{1,d}$, adjoint to the operators $A_j,\ j = \overline{1,d}$, with respect to the bilinear form $\langle \cdot, \cdot \rangle$, then the representation of the form (16) is equivalent to the representation

$$s_{\mathbf{k}} = \langle A_1^{k_1} A_2^{k_2} \ldots A_d^{k_d} x_{\mathbf{0}}, y_{\mathbf{0}} \rangle, \quad \mathbf{k} \in \mathbb{Z}_+^d. \tag{28}$$

Moreover, if $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, the bilinear form $\langle \cdot, \cdot \rangle$ is separably continuous, and the operators $A_j,\ j = \overline{1,d}$, are bounded, then the series

$$\sum_{\mathbf{k} \in \mathbb{Z}_+^d} s_{\mathbf{k}} \mathbf{z}^{\mathbf{k}} = \sum_{\mathbf{k} \in \mathbb{Z}_+^d} s_{\mathbf{k}} z_1^{k_1} z_2^{k_2} \ldots z_d^{k_d}$$

converges in a neighborhood of the origin of coordinates to an analytic function f of d variables, which can be represented in the form

$$f(\mathbf{z}) = \langle \mathcal{R}_{z_1}(A_1) \mathcal{R}_{z_2}(A_2) \ldots \mathcal{R}_{z_d}(A_d) x_0, y_0 \rangle.$$

Theorems 4 and 5 can be generalized to the case of functions of several variables.

Theorem 6. *For any function f analytic in a semidisk*

$$K_{\mathbf{R}} = K_{R_1} \times K_{R_2} \times \ldots \times K_{R_d}, \quad 0 < R_j < \infty, \quad j = \overline{1,d},$$

and any infinite-dimensional separable Hilbert space $\mathcal{H}$, there exist elements $x_0, y_0 \in \mathcal{H}$ and commuting linear bounded operators $A_j\colon \mathcal{H} \to \mathcal{H}$ with the norms

$$\|A_j\| < \frac{1}{R_j}, \quad j = \overline{1,d},$$

such that, for any $\mathbf{z} \in K_{\mathbf{R}}$,

$$f(\mathbf{z}) = \big\langle \mathcal{R}_{z_1}(A_1) \mathcal{R}_{z_2}(A_2) \ldots \mathcal{R}_{z_d}(A_d) x_0, y_0 \big\rangle. \tag{29}$$

Proof. Assume that a function f can be expanded in a power series

$$f(\mathbf{z}) = \sum_{\mathbf{k}\in\mathbb{Z}_+^d} s_{\mathbf{k}} z_1^{k_1} z_2^{k_2} \dots z_d^{k_d}$$

in the neighborhood of the origin.

Under the conditions of the theorem, by the Cauchy–Hadamard inequality, we get

$$|s_{\mathbf{k}}| \le \frac{M}{(R_1+\varepsilon_1)^{k_1}(R_2+\varepsilon_2)^{k_2}\dots(R_d+\varepsilon_1)^{k_d}},$$

where

$$M = \sup_{\mathbf{z}\in K_{\mathbf{R}}} |f(\mathbf{z})|, \qquad \varepsilon_1, \varepsilon_2, \dots, \varepsilon_d > 0$$

(see [11, p. 62]).

We fix certain numbers $\tilde{R}_j \in (R_j, R_j+\varepsilon_j)$, $j=\overline{1,d}$, and, for any orthonormal basis $\{e_p\}_{p\in\mathbb{Z}_+}$ of the space $\mathcal{H}$, consider a d-dimensional sequence of elements

$$x_{\mathbf{k}} = \frac{1}{\tilde{R}_1^{k_1}\tilde{R}_2^{k_2}\dots\tilde{R}_d^{k_d}} e_{c^d(\mathbf{k})}, \quad \mathbf{k}\in\mathbb{Z}_+^d.$$

We now define the action of the linear operators A_j, $j=\overline{1,d}$, upon the elements of the basis $\{e_p\}_{p\in\mathbb{Z}_+}$:

$$A_j e_m = \frac{1}{\tilde{R}_j} e_{c^d(l_1(m), l_2(m), \dots, l_j(m)+1, \dots, l_d(m))}, \quad m\in\mathbb{Z}_+.$$

We see that, first,

$$A_j x_{\mathbf{k}} = x_{\mathbf{k}+\boldsymbol{\delta}_j}, \quad \mathbf{k}\in\mathbb{Z}_+^d,$$

and, second,

$$\|A_j\| = \frac{1}{\tilde{R}_j} < \frac{1}{R_j}.$$

In addition, it is clear that the operators A_j, $j=\overline{1,d}$, are commuting.

We now define an element $y_0\in\mathcal{H}$ in the form of a series

$$y_0 = \sum_{p=0}^{\infty} \tilde{R}_1^{l_1(p)} \dots \tilde{R}_d^{l_d(p)} \overline{s}_{l_1(p), l_2(p), \dots, l_d(p)} e_p.$$

We check that $y_0\in\mathcal{H}$. Indeed,

$$\|y_0\|^2 = \sum_{p=0}^{\infty} \tilde{R}_1^{2l_1(p)} \dots \tilde{R}_d^{2l_d(p)} |s_{l_1(p), l_2(p), \dots, l_d(p)}|^2 < \infty.$$

On the other hand,

$$(x_{\mathbf{k}}, y_0) = \left(\frac{1}{\tilde{R}_1^{k_1} \tilde{R}_2^{k_2} \dots \tilde{R}_d^{k_d}} e_{c^d(\mathbf{k})}, \sum_{p=0}^{\infty} \tilde{R}_1^{l_1(p)} \dots \tilde{R}_d^{l_d(p)} \overline{s}_{l_1(p), l_2(p), \dots, l_d(p)} e_p \right) = s_{\mathbf{k}}, \quad \mathbf{k} \in \mathbb{Z}_+^d,$$

and, hence, representation (28) is true.

Example. Assume that a function f admits the following representation:

$$f(\mathbf{z}) = \int_{\mathbb{I}^d} \prod_{p=1}^{d} \frac{1}{1 - \dfrac{z_p t_p}{R_p}} d\mu(\mathbf{t}), \tag{30}$$

where $\mathbb{I}^d = [0,1]^d$ and μ is a Borel measure on $\mathbb{I}^d$.

As operators $A_j, j = \overline{1,d}$, we can take the operators of multiplication by independent variables

$$(A_j \varphi)(\mathbf{t}) = \frac{t_j}{R_j} \varphi(\mathbf{t})$$

with the norms

$$\|A_j\| = \frac{1}{R_j}.$$

Theorem 7. *For any entire function f of d variables and any infinite-dimensional separable Hilbert space $\mathcal{H}$, there exist elements $x_0, y_0 \in \mathcal{H}$ and commuting linear bounded operators $A_j \colon \mathcal{H} \to \mathcal{H},\ j = \overline{1,d}$, with spectral radius equal to zero and such that*

$$f(\mathbf{z}) = \big(\mathcal{R}_{z_1}(A_1) \mathcal{R}_{z_2}(A_2) \dots \mathcal{R}_{z_d}(A_d) x_0, y_0 \big).$$

Moreover, if the orders of increase of the function f (see [11, p. 390]) with respect to the variables $z_j,\ j = \overline{1,d}$, are equal to $\rho_j > 0,\ j = \overline{1,d}$, respectively, then the operators $A_j,\ j = \overline{1,d}$, can be chosen so that

$$\sqrt[p]{\|A_j^p\|} \le \frac{C_j}{p^{\frac{1}{\rho_j}}}, \quad j = \overline{1,d}, \tag{31}$$

for any $p \in \mathbb{N}$, where $C_j,\ j = \overline{1,d}$, are constants.

Proof. Assume that the function f can be expanded in a power series

$$f(\mathbf{z}) = \sum_{\mathbf{k} \in \mathbb{Z}_+^d} s_{\mathbf{k}} \mathbf{z}^{\mathbf{k}}.$$

Under the conditions of the theorem, we obtain

$$\limsup_{|\mathbf{k}| \to \infty} \sqrt[|\mathbf{k}|]{|s_{\mathbf{k}}|} = 0, \qquad \text{where} \quad |\mathbf{k}| = k_1 + k_2 + \dots + k_d.$$

Therefore, we can choose a sequence of positive numbers $\{\gamma_p\}_{p\in\mathbb{Z}_+}$ monotonically decreasing to zero and such that, for any $\mathbf{k}\in\mathbb{Z}_+^d$,

$$\sqrt[|\mathbf{k}|]{|s_{\mathbf{k}}|}\le\gamma_{|\mathbf{k}|},$$

and, hence,

$$|s_{\mathbf{k}}|\le\left(\gamma_{|\mathbf{k}|}\right)^{|\mathbf{k}|}.$$

For any orthonormal basis $\{e_p\}_{p\in\mathbb{Z}_+}$ any any $\lambda>1,$ we consider a d-dimensional sequence of elements

$$x_{\mathbf{k}}=\left(\lambda\gamma_{|\mathbf{k}|}\right)^{|\mathbf{k}|}e_{c^d(\mathbf{k})},\quad \mathbf{k}\in\mathbb{Z}_+^d.$$

On the elements of the basis $\{e_p\}_{p\in\mathbb{Z}_+}$, we define linear operators $A_j,\ j=\overline{1,d}$, as follows:

$$A_je_p=\lambda\frac{\left(\gamma_{|\mathbf{l}(p)|+1}\right)^{|\mathbf{l}(p)|+1}}{\left(\gamma_{|\mathbf{l}(p)|}\right)^{|\mathbf{l}(p)|}}e_{c^d(\mathbf{l}(p)+\boldsymbol{\delta}_j)}. \tag{32}$$

Thus, we get

$$A_jx_{\mathbf{k}}=x_{\mathbf{k}+\boldsymbol{\delta}_j},\qquad \mathbf{k}\in\mathbb{Z}_+^d,\quad j=\overline{1,d}.$$

It follows from equality (32) that

$$A_j^me_p=\lambda^m\frac{\left(\gamma_{|\mathbf{l}(p)|+m}\right)^{|\mathbf{l}(p)|+m}}{\left(\gamma_{|\mathbf{l}(p)|}\right)^{|\mathbf{l}(p)|}}e_{c^d(\mathbf{l}(p)+m\boldsymbol{\delta}_j)}$$

and, hence,

$$\|A_j^m\|=\sup_{p\in\mathbb{Z}_+}\lambda^m\frac{\left(\gamma_{|\mathbf{l}(p)|+m}\right)^{|\mathbf{l}(p)|+m}}{\left(\gamma_{|\mathbf{l}(p)|}\right)^{|\mathbf{l}(p)|}}\le(\lambda\gamma_m)^m$$

and

$$\sqrt[m]{\|A_j^m\|}\le\lambda\gamma_m\to0\qquad\text{as}\quad m\to\infty,$$

i.e., the spectral radius of the operators $A_j,\ j=\overline{1,d}$, is equal to zero.

Setting

$$y_0=\sum_{p=0}^{\infty}\frac{1}{(\lambda\gamma_p)^p}s_{\mathbf{l}(p)}e_p,$$

we obtain

$$\|y_0\|^2=\sum_{p=0}^{\infty}\frac{1}{(\lambda\gamma_p)^{2p}}|s_{\mathbf{l}(p)}|^2\le\sum_{p=0}^{\infty}\frac{1}{\lambda^{2p}}=\frac{\lambda^2}{\lambda^2-1}<\infty.$$

Therefore, $y_0\in\mathcal{H}$.

On the other hand,

$$(x_{\mathbf{k}}, y_0) = \left((\lambda\gamma_{\mathbf{k}})^{\mathbf{k}} e_{c^d(\mathbf{k})}, \sum_{p=0}^{\infty} \frac{1}{\lambda\gamma_p{}^p} s_{\mathbf{l}(p)} e_p \right) = s_{\mathbf{k}}, \quad \mathbf{k} \in \mathbb{Z}_+^d,$$

and, hence, representation (28) is true.

If the orders of increase of the function f with respect to the variables $z_j,\ j = \overline{1,d}$, are equal to $\rho_j,\ j = \overline{1,d}$, respectively, then

$$|s_{\mathbf{k}}| \leq \frac{C_j^{\mathbf{k}}}{|\mathbf{k}|^{\left(\frac{k_1}{\rho_1} + \frac{k_2}{\rho_2} + \ldots + \frac{k_d}{\rho_d}\right)}} \leq \prod_{j=1}^{d} \left(\frac{C_j}{k_j^{\frac{1}{\rho_j}}} \right)^{k_j}, \quad \mathbf{k} \in \mathbb{Z}_+^d,$$

where $C_j > 0,\ j = \overline{1,d}$, are constants.

For a certain fixed $\lambda > 1$, we set

$$x_{\mathbf{k}} = \lambda^{|\mathbf{k}|} \prod_{j=1}^{d} \left(\frac{1}{k_j^{\frac{1}{\rho_j}}} \right)^{k_j} e_{c^d(\mathbf{k})}.$$

On the vectors of the basis $\{e_p\}_{p \in \mathbb{Z}_+}$, we set

$$A_j e_p = \lambda \left(\frac{l_j(p)^{l_j(p)}}{(l_j(p)+1)^{l_j(p)+1}} \right)^{\frac{1}{\rho_j}} e_{c^d(\mathbf{l}(p) + \boldsymbol{\delta}_j)}.$$

Thus, we get

$$A_j x_{\mathbf{k}} = x_{\mathbf{k}+\boldsymbol{\delta}_j}, \qquad \mathbf{k} \in \mathbb{Z}_+^d, \quad j = \overline{1,d},$$

$$A_j^m e_p = \lambda^m \left(\frac{l_j(p)^{l_j(p)}}{(l_j(p)+m)^{l_j(p)+m}} \right)^{\frac{1}{\rho_j}} e_{c^d(\mathbf{l}(p) + m\boldsymbol{\delta}_j)}$$

and, hence,

$$\|A_j^m\| = \sup_{p \in \mathbb{Z}_+} \lambda^m \left(\frac{l_j(p)^{l_j(p)}}{(l_j(p)+m)^{l_j(p)+m}} \right)^{\frac{1}{\rho_j}} \leq \left(\frac{\lambda}{m^{\rho_j}} \right)^m.$$

This yields inequality (31).

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