On covariant realizations of the Euclid group

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We classify realizations of the Lie algebras of the rotation $O(3)$ and Euclid $E(3)$ groups within the class of first-order differential operators in arbitrary finite dimensions. It is established that there are only two distinct realizations of the Lie algebra of the group $O(3)$ which are inequivalent within the action of a diffeomorphism group. Using this result we describe a special subclass of realizations of the Euclid algebra which are called covariant ones by analogy to similar objects considered in the classical representation theory. Furthermore, we give an exhaustive description of realizations of the Lie algebra of the group $O(4)$ and construct covariant realizations of the Lie algebra of the generalized Euclid group $E(4)$.

1 Introduction

The standard approach to constructing linear relativistic motion equations contains as a subproblem the one of describing inequivalent matrix representations of the Poincaré group $P(1,3)$. So that if one succeeds in obtaining an exhaustive (in some sense) description of all inequivalent representations of the latter, then it is possible to construct all possible Poincaré-invariant linear wave equations (for more details see, e.g. [1–3]). It would be only natural to apply the same approach to describing *nonlinear* relativistically-invariant models with the help of the Lie's infinitesimal technique. However, in the overwhelming majority of the papers devoted to symmetry classification of nonlinear differential equations admitting some Lie transformation group G the realization of the group was fixed *a priory*. As a result, only particular classes of partial differential equations invariant with respect to a prescribed group G were obtained. One of the possible reasons for this is that the problem of describing inequivalent realizations of a given Lie transformation group reduces to constructing general solution of some over-determined system of *nonlinear partial differential equations* (in contrast to the case of the classical matrix representation theory where one has to solve *nonlinear matrix equations*).

We recall that given a fixed realization of a Lie transformation group G, the problem of describing partial differential equations invariant under the group G is reduced with the help of the infinitesimal Lie method to integrating some over-determined linear system of partial differential equations (called determining equations) [4–7]. However, to solve the problem of constructing *all* differential equations admitting the transformation group G whose realization is not fixed *a priori* one has

- to construct all inequivalent (in some sense) realizations of the Lie transformation group G,
- to solve the determining equations for each realization obtained.

And what is more, the first problem, in contrast to the second one, reduces to solving nonlinear systems of partial differential equations. In this respect one should mention the Lie's classification of integrable ordinary differential equations based on his

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classification of complex Lie algebras of first-order differential operators in one and two variables [8]. However, it seems impossible to give an exhaustive description of all Lie algebras of first-order differential operators. Till now there is no complete classification of them even for the case of first-order differential operators in three variables, though a partial classification was obtained by Lie a century ago [8].

The classification problem is substantially simplified if we are looking for inequivalent realizations of a specific Lie algebra. It has been completely solved by Rideau and Winternitz [9], Zhdanov and Fushchych [10] for the generalized Galilei (Schrödinger) group $G_2(1,1)$ acting in the space of two dependent and two independent variables.

Yehorchenko [11] and Fushchych, Tsyfra and Boyko [12] have constructed new (nonlinear) realizations of the Poincaré groups $P(1,2)$ and $P(1,3)$, correspondingly (see also [13, 14]). Some new realizations of the Galilei group $G(1,3)$ were suggested in [15]. A complete description of covariant realizations of the conformal group $C(n,m)$ in the space of $n+m$ independent and one dependent variables was obtained by Fushchych, Zhdanov and Lahno [16, 17] (see, also [18]). It has been established, in particular, that any covariant realization of the Poincaré group $P(n,m)$ with $\max\{n,m\} \geq 3$ in the case of one dependent variable is equivalent to the standard realization. But given the condition $\max\{n,m\} < 3$, there exist essentially new realizations of the corresponding Poincaré groups.

The present paper is devoted mainly to classification of inequivalent realizations of the Euclid group $E(3)$, which is a semi-direct product of the three-parameter rotation group $O(3)$ and of the three-parameter Abelian translation group $T(3)$, acting in the space of three independent (x_1,x_2,x_3) and $n \in \mathbb{N}$ dependent $(u_1,\ldots,u_n)$ variables. Being a subgroup of such fundamental groups as the Poincaré and Galilei groups, the Euclid group plays an exceptional role in modern mathematical and theoretical physics, since it is admitted both by equations of relativistic and non-relativistic theories. In particular, group $E(3)$ is an invariance group of the Klein–Gordon–Fock, Maxwell, heat, Schrödinger, Dirac, Weyl, Navier–Stokes, Lamé and Yang–Mills equations.

The paper is organized as follows. The second section contains the necessary notations, conventions and definitions used throughout the paper. In the third section we give an exhaustive classification of inequivalent realizations of the Lie algebra of the rotation group $O(3)$ within the class of first-order differential operators. The fourth section is devoted to description of covariant realizations of the Euclid algebra $AE(3)$. We give a complete classification of them and, furthermore, demonstrate how to reduce the realizations of $AE(3)$ realized on the sets of solutions of the Navier–Stokes, Lamè, Weyl, Maxwell and Dirac equations to one of the two canonical forms. In the forth section the results obtained are applied to describe covariant realizations of the Lie algebra of the generalized Euclid group $AE(4)$.

2 Basic notations and definitions

It is a common knowledge that investigation of realizations of a Lie transformation group G is reduced to study of realizations of its Lie algebra AG whose basis elements are the first-order differential operators (Lie vector fields) of the form

$$Q = \xi_\alpha(x,u)\partial_{x_\alpha} + \eta_i(x,u)\partial_{u_i}, \tag{1}$$

where ξ_α, η_i are some real-valued smooth functions of $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$ and $u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$, $\partial_{x_\alpha} = \frac{\partial}{\partial x_\alpha}$, $\partial_{u_i} = \frac{\partial}{\partial u_i}$, $\alpha = 1, 2, \ldots, m$, $i = 1, 2, \ldots, n$. Hereafter, a summation over the repeated indices is understood.

In the above formulae we have two "sorts" of variables. The variables $x_1, x_2, \ldots, x_m$ and $u_1, u_2, \ldots, u_n$ will be referred to as independent and dependent variables, respectively. The difference between these becomes essential when we consider AG as an invariance algebra of some system of partial differential equations for $u_1(x), \ldots, u_n(x)$.

Due to properties of the corresponding Lie transformation group G basis operators Q_a, $a = 1, \ldots, N$ of a Lie algebra AG satisfy commutation relations

$$[Q_a, Q_b] = C^c_{ab} Q_c, \quad a, b = 1, \ldots, N, \tag{2}$$

where $[Q_a, Q_b] \equiv Q_a Q_b - Q_b Q_a$ is the commutator.

In (2) $C^c_{ab} = \mathrm{const} \in \mathbb{R}$ are structure constants which determine uniquely the Lie algebra AG. A fixed set of Lie vector fields (LVFs) Q_a satisfying (2) is called a realization of the Lie algebra AG.

Thus the problem of description of all realizations of a given Lie algebra AG reduces to solving the relations (2) with some fixed structure constants C^c_{ab} within the class of LVFs (1).

It is easy to check that the relations (2) are not altered with an arbitrary invertible transformation of variables x, u

$$\begin{aligned} y_\alpha &= f_\alpha(x, u), \quad \alpha = 1, \ldots, m, \\ v_i &= g_i(x, u), \quad i = 1, \ldots, n, \end{aligned} \tag{3}$$

where f_α, g_i are smooth functions. That is why we can introduce on the set of realizations of a Lie algebra AG the following relation: two realizations $\langle Q_1, \ldots, Q_N \rangle$ and $\langle Q'_1, \ldots, Q'_N \rangle$ are called equivalent if they are transformed one into another by means of an invertible transformation (3). As invertible transformations of the form (3) form a group (called diffeomorphism group), the relation above is an equivalence relation. It divides the set of all realizations of a Lie algebra AG into equivalence classes $A_1, \ldots, A_r$. Consequently, to describe all possible realizations of AG it suffices to construct one representative of each equivalence class A_j, $j = 1, \ldots, r$.

Definition 1. *First-order linearly-independent differential operators*

$$\begin{aligned} P_a &= \xi^{(1)}_{ab}(x, u)\partial_{x_b} + \eta^{(1)}_{ai}(x, u)\partial_{u_i}, \\ J_a &= \xi^{(2)}_{ab}(x, u)\partial_{x_b} + \eta^{(2)}_{ai}(x, u)\partial_{u_i}, \end{aligned} \tag{4}$$

where the indices a, b take the values 1, 2, 3 and the index i takes the values $1, 2, \ldots, n$, form a realization of the Euclid algebra $AE(3)$ provided the following commutation relations are fulfilled:

$$[P_a, P_b] = 0, \tag{5}$$

$$[J_a, P_b] = \varepsilon_{abc} P_c, \tag{6}$$

$$[J_a, J_b] = \varepsilon_{abc} J_c, \tag{7}$$

where

$$\varepsilon_{abc} = \begin{cases} 1, & (abc) = \text{cycle}\,(123), \\ -1, & (abc) = \text{cycle}\,(213), \\ 0, & \text{in the remaining cases.} \end{cases}$$

Definition 2. *Realization of the Euclid algebra within the class of LVFs (4) is called covariant if coefficients of the basis elements P_a satisfy the following condition:*

$$\text{rank} \left\| \begin{matrix} \xi^{(1)}_{11} & \xi^{(1)}_{12} & \xi^{(1)}_{13} & \eta^{(1)}_{11} & \dots & \eta^{(1)}_{1n} \\ \xi^{(1)}_{21} & \xi^{(1)}_{22} & \xi^{(1)}_{23} & \eta^{(1)}_{21} & \dots & \eta^{(1)}_{2n} \\ \xi^{(1)}_{31} & \xi^{(1)}_{32} & \xi^{(1)}_{33} & \eta^{(1)}_{31} & \dots & \eta^{(1)}_{3n} \end{matrix} \right\| = 3. \tag{8}$$

3 Realizations of the Lie algebra of the rotation group $O(3)$

It is well-known from the classical representation theory that there are infinitely many inequivalent matrix representations of the Lie algebra of the rotation group $O(3)$ [1]. A natural equivalence relation on the set of matrix representations of $AO(3)$ is defined as follows

$$J_a \to V J_a V^{-1}$$

with an arbitrary constant nonsingular matrix V. If we represent the matrices J_a as the first-order differential operators (see, e.g. [7])

$$\mathcal{J}_a = -\{J_a \boldsymbol{u}\}_\alpha \partial_{u_\alpha}, \tag{9}$$

where $\boldsymbol{u}$ is a vector-column of the corresponding dimension, then the above equivalence relation means that the representations of the algebra $AO(3)$ are looked within the class of LVFs (9) up to invertible *linear* transformations

$$\boldsymbol{u} \to \boldsymbol{v} = V\boldsymbol{u}.$$

It is proved below that provided realizations of $AO(3)$ are classified within arbitrary invertible transformations of variables

$$v_i = F_i(u), \quad i = 1, \dots, n, \tag{10}$$

there are only two inequivalent realizations.

Theorem 1. *Let first-order differential operators*

$$\mathcal{J}_a = \eta_{ai}(u)\partial_{u_i}, \quad a = 1, 2, 3 \tag{11}$$

satisfy the commutation relations of the Lie algebra of the rotation group $O(3)$ (7). Then either all of them are equal to zero, i.e.

$$\mathcal{J}_a = 0, \quad a = 1, 2, 3 \tag{12}$$

or there exists a transformation (10) reducing these operators to one of the following forms:

$$\begin{aligned} 1.\quad & \mathcal{J}_1 = -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2}, \\ & \mathcal{J}_2 = -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2}, \\ & \mathcal{J}_3 = \partial_{u_1}; \end{aligned} \tag{13}$$

$$\begin{aligned} 2.\quad & \mathcal{J}_1 = -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2} + \sin u_1 \sec u_2 \partial_{u_3}, \\ & \mathcal{J}_2 = -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2} + \cos u_1 \sec u_2 \partial_{u_3}, \\ & \mathcal{J}_3 = \partial_{u_1}. \end{aligned} \tag{14}$$

Proof. If at least one of the operators $\mathcal{J}_a$ (say $\mathcal{J}_3$) is equal to zero, then due to the commutation relations (7) two other operators $\mathcal{J}_2$, $\mathcal{J}_3$ are also equal to zero and we arrive at the formulae (12).

Let $\mathcal{J}_3$ be a non-zero operator. Then, using a transformation (10) we can always reduce the operator $\mathcal{J}_3$ to the form $\mathcal{J}_3 = \partial_{v_1}$ (we should write $\mathcal{J}_3'$ but to simplify the notations we omit hereafter the primes). Next, from the commutation relations $[\mathcal{J}_3, \mathcal{J}_1] = \mathcal{J}_2$, $[\mathcal{J}_3, \mathcal{J}_2] = -\mathcal{J}_1$ it follows that coefficients of the operators $\mathcal{J}_1$, $\mathcal{J}_2$ satisfy the system of ordinary differential equations with respect to v_1,

$$\eta_{2iv_1} = \eta_{3i}, \quad \eta_{3iv_1} = -\eta_{2i}. \quad i = 1, \ldots, n.$$

Solving the above system yields

$$\eta_{2i} = f_i \cos v_1 + g_i \sin v_1, \quad \eta_{3i} = g_i \cos v_1 - f_i \sin v_1, \tag{15}$$

where f_i, g_i are arbitrary smooth functions of $v_2, \ldots, v_n$, $i = 1, \ldots, n$.

Case 1. $f_j = g_j = 0$, $j \geq 2$. In this case operators $\mathcal{J}_1$, $\mathcal{J}_2$ read

$$\mathcal{J}_1 = f \cos v_1 \partial_{v_1}, \quad \mathcal{J}_2 = -f \sin v_1 \partial_{v_1}$$

with an arbitrary smooth function $f = f(v_2, \ldots, v_n)$.

Inserting the above expressions into the remaining commutation relation $[\mathcal{J}_1, \mathcal{J}_2] = \mathcal{J}_3$ and computing the commutator on the left-hand side we arrive at the equality $f^2 = -1$ which can not be satisfied by a real-valued function.

Case 2. Not all f_j, g_j, $j \geq 2$ are equal to 0. Making a change of variables

$$w_1 = v_1 + V(v_2, \ldots, v_n), \quad w_j = v_j, \quad j = 2, \ldots, n$$

we transform operators $\mathcal{J}_a$, $a = 1, 2, 3$ with coefficients (15) as follows

$$\begin{aligned} \mathcal{J}_1 &= \tilde{f} \sin w_1 \partial_{w_1} + \sum_{j=2}^{n} (\tilde{f}_j \cos w_1 + \tilde{g}_j \sin w_1) \partial_{w_j}, \\ \mathcal{J}_2 &= \tilde{f} \cos w_1 \partial_{w_1} + \sum_{j=2}^{n} (\tilde{g}_j \cos w_1 - \tilde{f}_j \sin w_1) \partial_{w_j}, \\ \mathcal{J}_3 &= \partial_{w_1}. \end{aligned} \tag{16}$$

Here $\tilde{f}$, $\tilde{f}_j$, $\tilde{g}_j$ are some functions of $w_2, \ldots, w_n$.

Subcase 2.1. Not all $\tilde{f}_j$ are equal to 0. Making a transformation

$$z_1 = w_1, \quad z_j = W_j(w_2, \dots, w_n), \quad j = 2, \dots, n,$$

where W_2 is a particular solution of partial differential equation

$$\sum_{j=2}^{n} \tilde{f}_j \partial_{w_j} W_2 = 1$$

and $W_3, \dots, W_n$ are functionally-independent first integrals of partial differential equation

$$\sum_{j=2}^{n} \tilde{f}_j \partial_{w_j} W = 0,$$

we reduce the operators (16) to be

$$\begin{aligned} \mathcal{J}_1 &= F \sin z_1 \partial_{z_1} + \cos z_1 \partial_{z_2} + \sum_{j=2}^{n} G_j \sin z_1 \partial_{z_j}, \\ \mathcal{J}_2 &= F \cos z_1 \partial_{z_1} - \sin z_1 \partial_{z_2} + \sum_{j=2}^{n} G_j \cos z_1 \partial_{w_j}, \\ \mathcal{J}_3 &= \partial_{z_1}. \end{aligned} \tag{17}$$

Substituting operators (17) into the commutation relation $[\mathcal{J}_1, \mathcal{J}_2] = \mathcal{J}_3$ and equating coefficients of the linearly-independent operators $\partial_{z_1}, \dots, \partial_{z_n}$ we arrive at the following system of partial differential equations for the functions $F, G_2, \dots, G_n$:

$$F_{z_2} - F^2 = 1, \quad G_{jz_2} - FG_j = 0, \quad j = 2, \dots, n.$$

Integrating the above equations yields

$$F = \tan(z_2 + c_1), \quad G_j = \frac{c_j}{\cos(z_2 + c_1)},$$

where $c_1, \dots, c_n$ are arbitrary smooth functions of $z_3, \dots, z_n$, $j = 2, \dots, n$.

Changing, if necessary, z_2 by $z_2 + c_1(z_3, \dots, z_n)$ we may put c_1 equal to zero. Next, making a transformation

$$\begin{aligned} y_a &= z_a, \quad a = 1, 2, 3, \\ y_k &= Z_k(z_3, \dots, z_n), \quad k = 4, \dots, n, \end{aligned}$$

where Z_k are functionally-independent first integrals of partial differential equation

$$\sum_{j=3}^{n} G_j \partial_{z_j} Z = 0,$$

we can put $G_k = 0$, $k = 4, \dots, n$.

With these remarks the operators (17) take the form

$$\begin{aligned} \mathcal{J}_1 &= \sin y_1 \tan y_2 \partial_{y_1} + \cos y_1 \partial_{y_2} + \frac{\sin y_1}{\cos y_2}(f\partial_{y_2} + g\partial_{y_3}), \\ \mathcal{J}_2 &= \cos y_1 \tan y_2 \partial_{y_1} - \sin y_1 \partial_{y_2} + \frac{\cos y_1}{\cos y_2}(f\partial_{y_2} + g\partial_{y_3}), \\ \mathcal{J}_3 &= \partial_{y_1}, \end{aligned} \tag{18}$$

where f, g are arbitrary smooth functions of $y_3, \dots, y_n$.

If $g \equiv 0$, then making a transformation

$$\tilde{u}_1 = y_1 - \arctan \frac{f}{\cos y_2}, \quad \tilde{u}_2 = -\arctan \frac{\sin y_2}{\sqrt{\cos^2 y_2 + f^2}}, \quad \tilde{u}_k = y_k,$$

where $k = 3, \ldots, n$, we reduce the operators (18) to the form (13).

If in (18) $g \not\equiv 0$, then changing y_3 to $\tilde{y}_3 = \int g^{-1} dy_3$ and y_2 to $\tilde{y}_2 = -y_2$ we transform the above operators to become

$$\begin{aligned}
\mathcal{J}_1 &= -\sin \tilde{y}_1 \tan \tilde{y}_2 \partial_{\tilde{y}_1} - \left(\cos \tilde{y}_1 - \alpha \frac{\sin \tilde{y}_1}{\cos \tilde{y}_2} \right) \partial_{\tilde{y}_2} + \frac{\sin \tilde{y}_1}{\cos \tilde{y}_2} \partial_{\tilde{y}_3}, \\
\mathcal{J}_2 &= -\cos \tilde{y}_1 \tan \tilde{y}_2 \partial_{\tilde{y}_1} + \left(\sin \tilde{y}_1 + \alpha \frac{\cos \tilde{y}_1}{\cos \tilde{y}_2} \right) \partial_{\tilde{y}_2} + \frac{\cos \tilde{y}_1}{\cos \tilde{y}_2} \partial_{\tilde{y}_3}, \\
\mathcal{J}_3 &= \partial_{\tilde{y}_1}.
\end{aligned} \tag{19}$$

Here α is an arbitrary smooth function of $\tilde{y}_3, \ldots, \tilde{y}_n$.

Finally, making the transformation

$$\tilde{u}_1 = \tilde{y}_1 + f, \quad \tilde{u}_2^2 = g, \quad \tilde{u}_3 = h, \quad \tilde{u}_k = \tilde{y}_k,$$

where $k = 3, \ldots, n$ and $f(\tilde{y}_2, \ldots, \tilde{y}_n)$, $g(\tilde{y}_2, \ldots, \tilde{y}_n)$, $h(\tilde{y}_2, \ldots, \tilde{y}_n)$ satisfy the compatible over-determined system of nonlinear partial differential equations

$$\begin{aligned}
&f_{\tilde{y}_2} = \sin f \tan g, \quad f_{\tilde{y}_3} = \sin \tilde{y}_2 - \alpha \sin f \tan g - \cos \tilde{y}_2 \cos f \tan g, \\
&g_{\tilde{y}_2} = \cos f, \quad g_{\tilde{y}_3} = \sin f \cos \tilde{y}_2 - \alpha \cos f, \\
&h_{\tilde{y}_2} = -\sin f \sec g, \quad h_{\tilde{y}_3} = (\cos f \cos \tilde{y}_2 + \alpha \sin f) \sec g,
\end{aligned}$$

reduces operators (19) to the form (14).

Subcase 2.2. $f_j = 0$, $j = 2, \ldots, n$. Substituting the operators (16) under $f_j = 0$ into the commutation relation $[\mathcal{J}_1, \mathcal{J}_2] = \mathcal{J}_3$ and equating coefficients of the linearly-independent operators $\partial_{z_1}, \ldots, \partial_{z_n}$ yield system of algebraic equations

$$-f^2 = 1, \quad f g_j = 0, \quad j = 2, \ldots, n.$$

As the function f is a real-valued one, the system obtained is inconsistent.

Thus we have proved that the formulae (13)–(12) give all possible inequivalent realizations of the Lie algebra (7) within the class of first-order differential operators (11). The theorem is proved. ■

If we realize the rotation group as the group of transformations of the space of spherical functions, then the basis elements of its Lie algebra are exactly of the form (13) [1]. Hence it follows that the realization space $\mathcal{V}$ of the Lie algebra (13) is a direct sum of subspaces $\mathcal{V}_{2l+1}$ of spherical functions of the order l. Furthermore, if we consider $O(3)$ as the group of transformations of the space of generalized spherical functions [1], then the operators (14) are the basis elements of the corresponding Lie algebra.

4 Realizations of the algebra $AE(3)$

First we will prove an auxiliary assertion giving inequivalent realizations of Lie algebras of the translation $T(3)$ group within the class of LVFs.

Lemma 1. *Let mutually commuting LVFs*

$$P_a = \xi^{(1)}_{ab}(x,u)\partial_{x_b} + \eta^{(1)}_{ai}(x,u)\partial_{u_i},$$

where $a, b = 1,\dots,N$, satisfy the relation

$$\operatorname{rank}\left\| \begin{matrix} \xi^{(1)}_{11} & \dots & \xi^{(1)}_{1N} & \eta^{(1)}_{11} & \dots & \eta^{(1)}_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi^{(1)}_{N1} & \dots & \xi^{(1)}_{NN} & \eta^{(1)}_{N1} & \dots & \eta^{(1)}_{Nn} \end{matrix} \right\| = N. \tag{20}$$

Then there exists a transformation of the form (3) reducing operators P_a to become $P'_a = \partial_{y_a}$, $a = 1,\dots,N$.

Proof. To avoid unessential technicalities we will give the detailed proof of the lemma for the case $N = 3$.

Given a condition $N = 3$, relation (20) reduces to the form (8). Due to the latter $P_a \neq 0$ for all $a = 1, 2, 3$. It is well-known that a non-zero operator

$$P_1 = \xi^{(1)}_{1b}(x,u)\partial_{x_b} + \eta^{(1)}_{1i}(x,u)\partial_{u_i}$$

can always be reduced to the form $P'_1 = \partial_{y_1}$ by a transformation (3) with $m = 3$. If we denote by P'_2, P'_3 the operators P_2, P_3 written in the new variables y, v, then owing to the commutation relations (5) they commute with the operator $P'_1 = \partial_{y_1}$. Hence, we conclude that their coefficients are independent of y_1.

Furthermore due to the condition (8) at least one of the coefficients $\xi'^{(1)}_{22}$, $\xi'^{(1)}_{23}$, $\eta'^{(1)}_{21}$, …, $\eta'^{(1)}_{2n}$ of the operator P'_2 is not equal to zero.

Summing up, we conclude that the operator P'_2 is of the form

$$P'_2 = \xi'^{(1)}_{2b}(y_2, y_3, v)\partial_{y_b} + \eta'^{(1)}_{2i}(y_2, y_3, v)\partial_{v_i} \neq 0,$$

not all the functions $\xi'^{(1)}_{22}$, $\xi'^{(1)}_{23}$, $\eta'^{(1)}_{21}$, …, $\eta'^{(1)}_{2n}$ being identically equal to zero.

Making a transformation

$$\begin{aligned} &z_1 = y_1 + F(y_2, y_3, v), \\ &z_2 = G(y_2, y_3, v), \\ &z_3 = \omega_0(y_2, y_3, v), \\ &w_i = \omega_i(y_2, y_3, v), \quad i = 1,\dots,n, \end{aligned} \tag{21}$$

where the functions F, G are particular solutions of differential equations

$$\begin{aligned} &\xi'^{(1)}_{22}(y_2, y_3, v)F_{y_2} + \xi'^{(1)}_{22}(y_2, y_3, v)F_{y_3} + \eta'^{(1)}_{2i}(y_2, y_3, v)F_{u_i} + \xi'^{(1)}_{21}(y_2, y_3, v) = 0, \\ &\xi'^{(1)}_{22}(y_2, y_3, v)G_{y_2} + \xi'^{(1)}_{22}(y_2, y_3, v)G_{y_3} + \eta'^{(1)}_{2i}(y_2, y_3, v)G_{u_i} = 1 \end{aligned}$$

and $\omega_0, \omega_1, \dots, \omega_n$ are functionally-independent first integrals of the Euler–Lagrange system

$$\frac{dy_2}{\xi'^{(1)}_{22}} = \frac{dy_3}{\xi'^{(1)}_{23}} = \frac{dv_1}{\eta'^{(1)}_{21}} = \dots = \frac{dv_n}{\eta'^{(1)}_{2n}},$$

which has exactly $n+1$ functionally-independent integrals, we reduce the operator P_2' to the form $P_2'' = \partial_{z_2}$. It is easy to check that the transformation (21) does not alter form of the operator P_1'. Being rewritten in the new variables z, w it reads $P_1'' = \partial_{z_1}$.

As the right-hand sides of (21) are functionally-independent by construction, the transformation (21) is invertible. Consequently, operators P_a are equivalent to operators P_a'', where $P_1'' = \partial_{z_1}$, $P_2'' = \partial_{z_2}$ and

$$P_3'' = \xi_{3b}''^{(1)}(z_3, w)\partial_{y_b} + \eta_{3i}''^{(1)}(z_3, w)\partial_{v_i} \neq 0.$$

(Coefficients of the above operator are independent of z_1, z_2 because of the fact that it commutes with the operators P_1'', P_2''.) And what is more, due to (8) at least one of the coefficients $\xi_{33}''^{(1)}$, $\eta_{31}''^{(1)}$, ..., $\eta_{3i}''^{(1)}$ of the operator P_3'' is not identically equal to zero.

Making a transformation

$$Z_1 = z_1 + F(z_3, w),$$
$$Z_2 = z_2 + G(z_3, w),$$
$$Z_3 = H(z_3, w),$$
$$W_i = \Omega_i(z_3, w), \quad i = 1, \dots, n,$$

where F, G, H are particular solutions of partial differential equations

$$\xi_{33}''^{(1)}(z_3, w)F_{z_3} + \eta_{3i}''^{(1)}(z_3, w)F_{w_i} = -\xi_{31}''^{(1)}(z_3, w),$$
$$\xi_{33}''^{(1)}(z_3, w)G_{z_3} + \eta_{3i}''^{(1)}(z_3, w)G_{w_i} = -\xi_{32}''^{(1)}(z_3, w),$$
$$\xi_{33}''^{(1)}(z_3, w)H_{z_3} + \eta_{3i}''^{(1)}(z_3, w)H_{w_i} = 1,$$

and $\Omega_1, \dots, \Omega_n$ are functionally-independent first integrals of the Euler–Lagrange system

$$\frac{dz_3}{\xi_{33}''^{(1)}} = \frac{dw_1}{\eta_{31}''^{(1)}} = \dots = \frac{dw_n}{\eta_{3n}''^{(1)}},$$

we reduce the operators P_a'', $a = 1, 2, 3$ to the form $P_a''' = \partial_{Z_a}$, $a = 1, 2, 3$, the same as what was to be proved.

Note 1. In the papers [9, 17] mentioned above a classification of realizations of the groups $G_2(1,1)$, $C(n,m)$ was carried out under assumption that mutually commuting LVFs

$$Q_a = \xi_{a\alpha}(x)\partial_{x_\alpha}, \quad a = 1, \dots, N$$

can be simultaneously reduced by the map

$$y_\alpha = f_\alpha(x), \quad \alpha = 1, \dots, n \tag{22}$$

to the form $Q_a' = \partial_{y_a}$.

It is not difficult to become convinced of the fact that this is possible if and only if the condition

$$\operatorname{rank} \|\xi_{a\alpha}\|_{a=1}^{N}{}_{\alpha=1}^{n} = N \tag{23}$$

holds.

The sufficiency of the above statement is a consequence of Lemma 1. The necessity follows from the fact that function-rows of coefficients of operators $Q'_1, \dots, Q'_N$ transformed according to formulae (22) are obtained by multiplying function-rows of coefficients of the operators $Q_1, \dots, Q_N$ by a Jacobi matrix of the map (22), i.e.

$$\xi'_{a\alpha} = \xi_{a\beta} f_{\alpha x_\beta}, \quad a = 1, \dots, N, \quad \alpha = 1, \dots, n$$

which leaves the relation (23) invariant.

Consequently, in [9, 17] only covariant realizations of the corresponding Lie algebras were considered, which, generally speaking, do not exhaust a set of all possible realizations.

Now we can prove a principal theorem giving a description of all inequivalent covariant realizations of the Euclid algebra $AE(3)$.

Theorem 2. *Any covariant realization of the algebra $AE(3)$ within the class of first-order differential operators is equivalent to one of the following realizations:*

1. $$P_a = \partial_{x_a}, \quad J_a = -\varepsilon_{abc} x_b \partial_{x_c}, \quad a = 1, 2, 3; \tag{24}$$

2. $$\begin{aligned} &P_a = \partial_{x_a}, \quad a = 1, 2, 3, \\ &J_1 = -x_2\partial_{x_3} + x_3\partial_{x_2} + f\partial_{x_1} - f_{u_2} \sin u_1 \partial_{x_3} - \\ &\qquad - \sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2}, \\ &J_2 = -x_3\partial_{x_1} + x_1\partial_{x_3} + f\partial_{x_2} - f_{u_2} \cos u_1 \partial_{x_3} - \\ &\qquad - \cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2}, \\ &J_3 = -x_1\partial_{x_2} + x_2\partial_{x_1} + \partial_{u_1}; \end{aligned} \tag{25}$$

3. $$\begin{aligned} &P_a = \partial_{x_a}, \quad a = 1, 2, 3, \\ &J_1 = -x_2\partial_{x_3} + x_3\partial_{x_2} + g\partial_{x_1} - (\sin u_1 g_{u_2} + \cos u_1 \sec u_2 g_{u_3})\partial_{x_3} - \\ &\qquad - \sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2} + \sin u_1 \sec u_2 \partial_{u_3}, \\ &J_2 = -x_3\partial_{x_1} + x_1\partial_{x_3} + g\partial_{x_2} - (\cos u_1 g_{u_2} - \sin u_1 \sec u_2 g_{u_3})\partial_{x_3} - \\ &\qquad - \cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2} + \cos u_1 \sec u_2 \partial_{u_3}, \\ &J_3 = -x_1\partial_{x_2} + x_2\partial_{x_1} + \partial_{u_1}. \end{aligned} \tag{26}$$

Here $f = f(u_2, \dots, u_n)$ is given by the formula

$$f = \alpha \sin u_2 + \beta \left(\sin u_2 \ln \frac{\sin u_2 + 1}{\cos u_2} - 1 \right), \tag{27}$$

α, β are arbitrary smooth functions of $u_3, \dots, u_n$ and $g = g(u_2, \dots, u_n)$ is a solution of the following linear partial differential equation:

$$\cos^2 u_2\, g_{u_2u_2} + g_{u_3u_3} - \sin u_2 \cos u_2\, g_{u_2} + 2\cos^2 u_2\, g = 0. \tag{28}$$

Proof. Due to Lemma 1 operators P_a can always be reduced to the form $P_a = \partial_{x_a}$ by means of a properly chosen transformation (3). Inserting the operators

$$P_a = \partial_{x_a}, \quad J_a = \xi_{ab}(x, u)\partial_{x_b} + \eta_{ai}(x, u)\partial_{u_i}$$

into the commutation relations (6) and equating the coefficients of the linearly-independent operators ∂_{x_1}, ∂_{x_2}, ∂_{x_3}, $\partial_{u_1}, \ldots, \partial_{u_n}$ we arrive at the system of partial differential equations for the functions $\xi_{ab}(x,u)$, $\eta_{ai}(x,u)$,

$$\xi_{acx_b} = -\varepsilon_{abc}, \quad \eta_{aix_b} = 0, \quad a,b,c = 1,2,3, \quad i = 1\ldots,n.$$

Integrating the above system we conclude that the operators J_a have the form

$$J_a = -\varepsilon_{abc} x_b \partial_{x_c} + j_{ab}(u)\partial_{x_b} + \tilde{\eta}_{ai}(u)\partial_{u_i}, \quad a = 1,2,3, \tag{29}$$

where j_{ab}, $\tilde{\eta}_{ab}$ are arbitrary smooth functions.

Inserting (29) into the commutation relations (7) and equating coefficients of $\partial_{u_1}, \ldots, \partial_{u_n}$ show that the operators $\mathcal{J}_a = \tilde{\eta}_{ai}\partial_{u_i}$, $a = 1,2,3$ have to fulfill (7) with $J_a \to \mathcal{J}_a$. Hence, taking into account Theorem 1 we conclude that any covariant realization of the algebra $AE(3)$ is equivalent to the following one:

$$P_a = \partial_{x_a}, \quad J_a = -\varepsilon_{abc} x_b \partial_{x_c} + j_{ab}(u)\partial_{x_b} + \mathcal{J}_a, \quad a = 1,2,3, \tag{30}$$

operators $\mathcal{J}_a$ being given by one of the formulae (12)–(14).

Making a transformation

$$y_a = x_a + F_a(u), \quad v_i = u_i, \quad a = 1,2,3, \quad i = 1,\ldots,n,$$

we reduce operators J_a from (30) to be

$$\begin{aligned}
J_1 &= -y_2\partial_{y_3} + y_3\partial_{y_2} + A\partial_{y_1} + B\partial_{y_2} + C\partial_{y_3} + \mathcal{J}_1,\\
J_2 &= -y_3\partial_{y_1} + y_1\partial_{y_3} + F\partial_{y_2} + G\partial_{y_3} + \mathcal{J}_2,\\
J_3 &= -y_1\partial_{y_2} + y_2\partial_{y_1} + H\partial_{y_3} + \mathcal{J}_3,
\end{aligned} \tag{31}$$

where A, B, C, F, G, H are arbitrary smooth functions of $v_1, \ldots, v_n$.

Substituting the operators (31) into (7) and equating coefficients of linearly-independent operators ∂_{y_1}, ∂_{y_2}, ∂_{y_3}, $\partial_{v_1}, \ldots, \partial_{v_n}$ result in the following system of partial differential equations:

$$\begin{array}{ll}
1)\ \mathcal{J}_2 A = -C, & 6)\ \mathcal{J}_3 C - \mathcal{J}_1 H = G,\\
2)\ \mathcal{J}_3 F = -B, & 7)\ \mathcal{J}_1 G - \mathcal{J}_2 C = H - A - F,\\
3)\ \mathcal{J}_3 A = B, & 8)\ \mathcal{J}_3 B = F - A - H,\\
4)\ \mathcal{J}_1 F - \mathcal{J}_2 B = G, & 9)\ A - F - H = 0.\\
5)\ \mathcal{J}_2 H - \mathcal{J}_3 G = C, &
\end{array} \tag{32}$$

Case 1. All operators $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$ are equal to zero. Then, (32) reduces to the system of linear algebraic equations

$$B = C = G = 0, \quad H - A - F = 0, \quad F - A - H = 0, \quad A - F - H = 0,$$

whence it follows immediately that $A = F = G = 0$. Substituting the above results into formulae (31) we arrive at the realization (24).

Case 2. Suppose now that not all operators $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$ vanish. Then, they are given either by formulae (13) or (14), where one should replace $u_1, \ldots, u_n$ by $v_1, \ldots, v_n$. As for the both cases $\mathcal{J}_3 = \partial_{v_1}$, a subsystem of equations 2, 3, 8, 9 forms a system of linear ordinary differential equations for functions A, B, F, H with respect to v_1. Integrating it we have

$$\begin{aligned} &A = B_0 + B_1 \sin 2v_1 - B_2 \cos 2v_1, \quad B = 2B_1 \cos 2v_1 + 2B_2 \sin 2v_1, \\ &F = B_0 + B_2 \cos 2v_1 - B_1 \sin 2v_1, \quad H = 2B_1 \sin 2v_1 - 2B_2 \cos 2v_1, \end{aligned} \tag{33}$$

where B_0, B_1, B_2 are arbitrary smooth functions of $v_2, \ldots, v_n$.

Subcase 2.1. Let the operators $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$ be of the form (13). Then, making a transformation

$$\begin{aligned} &z_1 = y_1 + R_1 \cos v_1 + R_2 \sin v_1, \\ &z_2 = y_2 + R_2 \cos v_1 - R_1 \sin v_1, \\ &z_3 = y_3 + \frac{1}{2}(R_{2v_2} + \tan v_2 R_2) \cos 2v_1 - \frac{1}{2}(R_{1v_2} + \tan v_2 R_1) \sin 2v_1 + \\ &\qquad + \frac{1}{2}(\tan v_2 R_2 - R_{2v_2}), \end{aligned}$$

where the functions R_1, R_2 are solutions of the system of partial differential equations

$$R_{1v_2} + \frac{1}{2} \tan v_2 R_1 = -2B_2, \quad R_{2v_2} + \frac{1}{2} \tan v_2 R_2 = 2B_1,$$

we reduce the operators (31) with A, B, F, H given by (33) to the form

$$\begin{aligned} &J_1 = -z_2\partial_{z_3} + z_3\partial_{z_2} + \widetilde{A}\partial_{z_1} + \widetilde{C}\partial_{z_3} + \mathcal{J}_1, \\ &J_2 = -z_3\partial_{z_1} + z_1\partial_{z_3} + \widetilde{A}\partial_{z_2} + \widetilde{G}\partial_{z_3} + \mathcal{J}_2, \\ &J_3 = -z_1\partial_{z_2} + z_2\partial_{z_1} + \mathcal{J}_3. \end{aligned} \tag{34}$$

Here $\widetilde{A}$, $\widetilde{C}$, $\widetilde{G}$ are arbitrary smooth functions of $v_1, \ldots, v_n$, and what is more, $\widetilde{A}$ does not depend on v_1.

Given such a form of operators J_a, system (32) reduces to three differential equations

$$\mathcal{J}_2\widetilde{A} = -\widetilde{C}, \quad \mathcal{J}_1\widetilde{A} = \widetilde{G}, \quad \mathcal{J}_1\widetilde{G} - \mathcal{J}_2\widetilde{C} = -2\widetilde{A}. \tag{35}$$

Inserting expressions for the operators $\mathcal{J}_1$, $\mathcal{J}_2$ from (13) into the first two equations we have

$$\widetilde{C} = -\sin v_1 \widetilde{A}_{v_2}, \quad \widetilde{G} = -\cos v_1 \widetilde{A}_{v_2}.$$

Substituting the above formulae into the third equation of the system (35) we conclude that it is equivalent to the differential equation

$$\widetilde{A}_{v_2v_2} - \tan v_2 \widetilde{A}_{v_2} + 2\widetilde{A} = 0,$$

whose general solution is given by (27). At last, inserting the results obtained into (34) we get the formulae (25).

Subcase 2.2. Let the operators $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$ be of the form (14). Then, making a transformation

$$\begin{aligned} &z_1 = y_1 + R_1 \cos v_1 + R_2 \sin v_1, \\ &z_2 = y_2 + R_2 \cos v_1 - R_1 \sin v_1, \end{aligned}$$

$$
\begin{aligned}
z_3 = y_3 + \frac{1}{2}\,(R_{2v_2} - \sec v_2 R_{1v_3} + \tan v_2 R_2)\cos 2v_1 - \\
- \frac{1}{2}(R_{1v_2} + \sec v_2 R_{2v_3} + \tan v_2 R_1)\sin 2v_1 + \\
+ \frac{1}{2}(\tan v_2 R_2 - \sec v_2 R_{1v_3} - R_{2v_2}),
\end{aligned}
$$

where the functions R_1, R_2 are solutions of the system of partial differential equations

$$
\begin{aligned}
2B_1 &= R_{2v_2} - \sec v_2 R_{1v_3} + \tan v_2 R_2,\\
2B_2 &= -R_{1v_2} - \sec v_2 R_{2v_3} - \tan v_2 R_1,
\end{aligned}
$$

we reduce the operators (31) with A, B, F, H given by (33) to the form (34), where $\widetilde{A}$, $\widetilde{C}$, $\widetilde{G}$ are arbitrary smooth functions, and what is more, $\widetilde{A}$ does not depend on v_1.

Given such a form of the operators J_a, system (32) reduces to three differential equations (35). Inserting expressions for the operators $\mathcal{J}_1$, $\mathcal{J}_2$ from (13) into the first two equations of (35) we have

$$
\begin{aligned}
\widetilde{C} &= -\cos v_1 \widetilde{A}_{v_2} + \sin v_1 \sec v_2 \widetilde{A}_{v_3},\\
\widetilde{G} &= -\sin v_1 A_{v_2} - \cos v_1 \sec v_2 \widetilde{A}_{v_3}.
\end{aligned}
\tag{36}
$$

Substituting the above formulae into the third equation of (35) after some algebra we arrive at the conclusion that it is equivalent to equation (28). Inserting (36) into (34) yields formulae (26).

Thus we have proved that if LVFs P_a, J_a realize a covariant realization of the Euclid algebra $AE(3)$, then they can be reduced to one of the forms (24)–(26) by means of an invertible transformation (3). The theorem is proved. ■

While proving Theorem 1, we have established, in particular, that any realization of the Euclid algebra satisfying the condition (8) can be transformed to become

$$
P_a = \partial_{x_a}, \quad J_a = -\varepsilon_{abc} x_b \partial_{x_c} + j_{ab}(u)\partial_{x_b} + \tilde{\eta}_{ai}(u)\partial_{u_i}, \quad a = 1,2,3.
$$

If we choose in the above formulae

$$
j_{ab}(u) = 0, \quad \eta_{ai}(u) = -\Lambda_{aij} u_j, \quad a,b = 1,2,3, \quad i = 1,\ldots,n,
$$

where $\Lambda_{aij} = \text{const}$, then the following realization

$$
P_a = \partial_{x_a}, \quad J_a = -\varepsilon_{abc} x_b \partial_{x_c} + \mathcal{J}_a, \quad a = 1,2,3 \tag{37}
$$

with $\mathcal{J}_a = -\Lambda_{aij} u_j \partial_{u_i}$ is obtained.

A realization of the Euclid algebra with generators of the form (37) is called in the classical linear representation theory a *covariant realization*. That is why it is natural to preserve for a realization of the algebra $AE(3)$ within the class of LVFs obeying (8) the same terminology.

As an illustration to Theorem 2 we will demonstrate how to reduce realizations of the Euclid algebras realized on sets of solutions of the heat, wave, Laplace, Navier–Stokes, Lamè, Weyl, Dirac and Maxwell equations to one of the three canonical forms (24)–(26). First of all, we note that the realization (24) is exactly the one realized on the sets of solutions of the linear and nonlinear heat (Schrödinger), wave, Laplace equations.

Symmetry algebras of the Navier–Stokes and Lamè equations contain as a subalgebra the Euclid algebra having basis elements (37), where (see, e.g. [6])

$$\mathcal{J}_a = -\varepsilon_{abc}\, v_b \partial_{v_c}, \quad a = 1, 2, 3. \tag{38}$$

The change of variables

$$v_1 = u_3 \sin u_1 \cos u_2, \quad v_2 = u_3 \cos u_1 \cos u_2, \quad v_3 = u_3 \sin u_2$$

reduce these LVFs to the form (25) with $f = 0$.

Next, if we consider the Weyl equation as the system of four real equations for four real-valued functions v_1, v_2, w_1, w_2, then on the set of its solutions realization (37) of the algebra $AE(3)$ is realized, where [3, 7]

$$\begin{aligned}
\mathcal{J}_1 &= \frac{1}{2}(w_2\partial_{v_1} - v_1\partial_{w_2} + w_1\partial_{v_2} - v_2\partial_{w_1}),\\
\mathcal{J}_2 &= \frac{1}{2}(v_2\partial_{v_1} - v_1\partial_{v_2} + w_2\partial_{w_1} - w_1\partial_{w_2}),\\
\mathcal{J}_3 &= \frac{1}{2}(w_1\partial_{v_1} - v_1\partial_{w_1} + v_2\partial_{w_2} - w_2\partial_{v_2}).
\end{aligned} \tag{39}$$

Making the change of variables

$$\begin{aligned}
v_1 &= u_4\left(\sin\frac{u_1}{2}\sin\frac{u_2}{2}\cos\frac{u_3}{2} + \cos\frac{u_1}{2}\cos\frac{u_2}{2}\sin\frac{u_3}{2}\right),\\
v_2 &= u_4\left(\cos\frac{u_1}{2}\cos\frac{u_2}{2}\cos\frac{u_3}{2} - \sin\frac{u_1}{2}\sin\frac{u_2}{2}\sin\frac{u_3}{2}\right),\\
w_1 &= u_4\left(\cos\frac{u_1}{2}\sin\frac{u_2}{2}\cos\frac{u_3}{2} - \sin\frac{u_1}{2}\cos\frac{u_2}{2}\sin\frac{u_3}{2}\right),\\
w_2 &= u_4\left(\sin\frac{u_1}{2}\cos\frac{u_2}{2}\cos\frac{u_3}{2} + \cos\frac{u_1}{2}\sin\frac{u_2}{2}\sin\frac{u_3}{2}\right)
\end{aligned}$$

reduces the above LVFs to the form (26) with $g = 0$.

On the solution set of the Maxwell equations the realization of the Euclid algebra (37), where

$$\mathcal{J}_a = -\varepsilon_{abc}\left(E_b\partial_{E_c} + H_b\partial_{H_c}\right), \quad a = 1, 2, 3,$$

is realized [19].

This realization is reduced to the form (26) under $g = 0$ with the help of the change of variables

$$\begin{aligned}
E_1 &= u_6 \sin u_1 \cos u_2,\\
E_2 &= u_6 \cos u_1 \cos u_2,\\
E_3 &= u_6 \sin u_2,\\
H_1 &= u_4(\cos u_1 \sin u_3 + \sin u_1 \sin u_2 \cos u_3) + u_5 \sin u_1 \cos u_2,\\
H_2 &= u_4(\cos u_1 \sin u_2 \cos u_3 - \sin u_1 \sin u_3) + u_5 \cos u_1 \cos u_2,\\
H_3 &= -u_4 \cos u_2 \cos u_3 + u_5 \sin u_2.
\end{aligned}$$

Taking the Dirac matrices γ_μ in the Majorana representation we can represent the Dirac equation as the system of eight real equations for eight real-valued functions

$\psi_1^0, \ldots, \psi_1^3$, $\psi_2^0, \ldots, \psi_2^3$ (for details, see e.g. [7]). With this choice of γ-matrices, on the set of solutions of the Dirac equation realization of the Euclid algebra (37) with

$$\mathcal{J}_1 = -\frac{1}{2}\big(\psi_1^3\partial_{\psi_1^0} + \psi_1^2\partial_{\psi_1^1} - \psi_1^1\partial_{\psi_1^2} - \psi_1^0\partial_{\psi_1^3} + \psi_2^3\partial_{\psi_2^0} + \psi_2^2\partial_{\psi_2^1} - \psi_2^1\partial_{\psi_2^2} - \psi_2^0\partial_{\psi_2^3}\big),$$
$$\mathcal{J}_2 = \frac{1}{2}\big(- \psi_1^2\partial_{\psi_1^0} + \psi_1^3\partial_{\psi_1^1} + \psi_1^0\partial_{\psi_1^2} - \psi_1^1\partial_{\psi_1^3} - \psi_2^2\partial_{\psi_2^0} + \psi_2^3\partial_{\psi_2^1} + \psi_2^0\partial_{\psi_2^2} - \psi_2^1\partial_{\psi_2^3}\big),$$
$$\mathcal{J}_3 = -\frac{1}{2}\big(\psi_1^1\partial_{\psi_1^0} - \psi_1^0\partial_{\psi_1^1} + \psi_1^3\partial_{\psi_1^2} - \psi_1^2\partial_{\psi_1^3} + \psi_2^1\partial_{\psi_2^0} - \psi_2^0\partial_{\psi_2^1} + \psi_2^3\partial_{\psi_2^2} - \psi_2^2\partial_{\psi_2^3}\big)$$

is realized on the set of solutions of the Dirac equation.

Making the change of variables

$$\psi_1^0 = u_4\left(\cos\frac{u_1}{2}\cos\frac{u_2}{2}\sin\frac{u_3}{2} + \sin\frac{u_1}{2}\sin\frac{u_2}{2}\cos\frac{u_3}{2}\right),$$
$$\psi_1^1 = u_4\left(\sin\frac{u_1}{2}\cos\frac{u_2}{2}\sin\frac{u_3}{2} - \cos\frac{u_1}{2}\sin\frac{u_2}{2}\cos\frac{u_3}{2}\right),$$
$$\psi_1^2 = -u_4\left(\cos\frac{u_1}{2}\cos\frac{u_2}{2}\cos\frac{u_3}{2} - \sin\frac{u_1}{2}\sin\frac{u_2}{2}\sin\frac{u_3}{2}\right),$$
$$\psi_1^3 = -u_4\left(\sin\frac{u_1}{2}\cos\frac{u_2}{2}\cos\frac{u_3}{2} + \cos\frac{u_1}{2}\sin\frac{u_2}{2}\sin\frac{u_3}{2}\right),$$
$$\psi_2^0 = u_5\left(\sin\frac{u_1}{2}\sin\frac{u_2}{2}\sin\frac{u_3+u_6}{2} - \cos\frac{u_1}{2}\cos\frac{u_2}{2}\cos\frac{u_3+u_6}{2}\right) + {}$$
$$+ u_7\left(\sin\frac{u_1}{2}\cos\frac{u_2}{2}\sin\frac{u_3+u_8}{2} - \cos\frac{u_1}{2}\sin\frac{u_2}{2}\cos\frac{u_3+u_8}{2}\right),$$
$$\psi_2^1 = -u_5\left(\sin\frac{u_1}{2}\cos\frac{u_2}{2}\cos\frac{u_3+u_6}{2} + \cos\frac{u_1}{2}\sin\frac{u_2}{2}\sin\frac{u_3+u_6}{2}\right) - {}$$
$$- u_7\left(\sin\frac{u_1}{2}\sin\frac{u_2}{2}\cos\frac{u_3+u_8}{2} - \cos\frac{u_1}{2}\cos\frac{u_2}{2}\sin\frac{u_3+u_8}{2}\right),$$
$$\psi_2^2 = -u_5\left(\cos\tfrac{u_1}{2}\cos\tfrac{u_2}{2}\sin\tfrac{u_3+u_6}{2} + \sin\tfrac{u_1}{2}\sin\tfrac{u_2}{2}\cos\tfrac{u_3+u_6}{2}\right)$$
$$+ u_7\left(\cos\frac{u_1}{2}\sin\frac{u_2}{2}\sin\frac{u_3+u_8}{2} + \sin\frac{u_1}{2}\cos\frac{u_2}{2}\cos\frac{u_3+u_8}{2}\right),$$
$$\psi_2^3 = u_5\left(\cos\frac{u_1}{2}\sin\frac{u_2}{2}\cos\frac{u_3+u_6}{2} - \sin\frac{u_1}{2}\cos\frac{u_2}{2}\sin\frac{u_3+u_6}{2}\right) - {}$$
$$- u_7\left(\cos\frac{u_1}{2}\cos\frac{u_2}{2}\cos\frac{u_3+u_8}{2} - \sin\frac{u_1}{2}\sin\frac{u_2}{2}\sin\frac{u_3+u_8}{2}\right)$$

reduces the above realization to the form (26) with $g = 0$.

5 Covariant realizations of the Lie algebra of the group $E(4)$

We recall that the basis elements of the Lie algebra of the Euclid group $E(4)$ fulfill the following commutation relations:

$$[P_\alpha,\ P_\beta] = 0, \tag{40}$$

$$[J_{\mu\nu},\ P_\alpha] = \delta_{\mu\alpha}P_\nu - \delta_{\nu\alpha}P_\mu, \tag{41}$$

$$[J_{\alpha\beta},\ J_{\mu\nu}] = \delta_{\alpha\mu}J_{\beta\nu} + \delta_{\beta\nu}J_{\alpha\mu} - \delta_{\alpha\nu}J_{\beta\mu} - \delta_{\beta\mu}J_{\alpha\nu}, \tag{42}$$

where α, β, $\mu, \nu = 1, 2, 3, 4$.

Using the results of the previous sections and the fact that the Lie algebra of the rotation group $O(4)$ is the direct sum of two algebras $AO(3)$ we will obtain a description of covariant realizations of the Lie algebra (40)–(42) within the class of LVFs

$$P_\mu = \xi_{\mu\nu}(x,u)\partial_{x_\nu} + \eta_{\mu i}(x,u)\partial_{u_i},$$
$$J_{\mu\nu} = \xi_{\mu\nu\alpha}(x,u)\partial_{x_\alpha} + \eta_{\mu\nu i}(x,u)\partial_{u_i}$$

with $J_{\mu\nu} = -J_{\nu\mu}$. Here the indices μ, ν, α take the values $1, 2, 3, 4$ and the index i takes the values $1, \ldots, n$.

As we consider covariant realizations, mutually commuting operators P_μ satisfy (20) with $N = 4$. Hence due to Lemma 1 it follows that they can be reduced to the form $P_\mu = \partial_{x_\mu}$, $\mu = 1, 2, 3, 4$. Next, using the commutation relations (41) we establish that the operators $J_{\mu\nu}$ have the following structure:

$$J_{\mu\nu} = x_\nu\partial_{x_\mu} - x_\mu\partial_{x_\nu} + f_{\mu\nu\alpha}(u)\partial_{x_\alpha} + g_{\mu\nu i}(u)\partial_{u_i} \tag{43}$$

with arbitrary sufficiently smooth $f_{\mu\nu\alpha}$, $g_{\mu\nu i}$.

In what follows we will restrict our considerations to the case when in (43) $f_{\mu\nu\alpha} \equiv 0$. This means geometrically that the transformation groups generated by the operators $J_{\mu\nu}$ in the space of independent variables are standard rotations in the planes (x_μ, x_ν). With this restriction LVFs $J_{\mu\nu}$ take the form

$$J_{\mu\nu} = x_\nu\partial_{x_\mu} - x_\mu\partial_{x_\nu} + \mathcal{J}_{\mu\nu}, \tag{44}$$

where

$$\mathcal{J}_{\mu\nu} = g_{\mu\nu i}(u)\partial_{u_i} \tag{45}$$

and, furthermore, $g_{\mu\nu i}(u) = -g_{\nu\mu i}(u)$.

Inserting LVFs (44) into (42) we come to conclusion that the operators $\mathcal{J}_{\mu\nu}$ satisfy the commutation relations of the Lie algebra of the rotation group $O(4)$

$$[\mathcal{J}_{\alpha\beta},\ \mathcal{J}_{\mu\nu}] = \delta_{\alpha\mu}\mathcal{J}_{\beta\nu} + \delta_{\beta\nu}\mathcal{J}_{\alpha\mu} - \delta_{\alpha\nu}\mathcal{J}_{\beta\mu} - \delta_{\beta\mu}\mathcal{J}_{\alpha\nu}. \tag{46}$$

An exhaustive description of inequivalent realizations of the above Lie algebra within the class of LVFs (45) is given below. It is based on results of Section 2 and on the well-known fact that the algebra $AO(4)$ is decomposed into the direct sum of two algebras $AO(3)$. This is achieved by choosing the basis of $AO(4)$ in the following way:

$$\mathcal{J}_a^\pm = \frac{1}{2}\left(\frac{1}{2}\varepsilon_{abc}\mathcal{J}_{bc} \pm \mathcal{J}_{a4}\right), \tag{47}$$

where the indices a, b, c take the values 1, 2, 3. Due to (46) LVFs $\mathcal{J}_a^-$, $\mathcal{J}_a^+$ fulfill the following commutation relations:

$$[\mathcal{J}_a^+, \mathcal{J}_b^+] = \varepsilon_{abc}\mathcal{J}_c^+, \tag{48}$$

$$[\mathcal{J}_a^+,\ \mathcal{J}_b^-] = 0, \tag{49}$$

$$[\mathcal{J}_a^-,\ \mathcal{J}_b^-] = \varepsilon_{abc}\mathcal{J}_c^-, \tag{50}$$

which is the same as what was required. Now we are ready to formulate an assertion giving an exhaustive description of LVFs (45) satisfying commutation relations (46) or, equivalently, (48)–(50).

Theorem 3. *Any realization of the Lie algebra $AO(4)$ within the class of LVFs (45) is given by the formulae (47) and by one of the formulae 1–6 presented below.*

1. $\mathcal{J}_1^+ = -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2},$
   $\mathcal{J}_2^+ = -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2},$
   $\mathcal{J}_3^+ = \partial_{u_1},$
   $\mathcal{J}_1^- = -\sin u_3 \tan u_4 \partial_{u_3} - \cos u_3 \partial_{u_4},$
   $\mathcal{J}_2^- = -\cos u_3 \tan u_4 \partial_{u_3} + \sin u_3 \partial_{u_4},$
   $\mathcal{J}_3^- = \partial_{u_3};$
2. $\mathcal{J}_1^+ = -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2},$
   $\mathcal{J}_2^+ = -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2},$
   $\mathcal{J}_3^+ = \partial_{u_1},$
   $\mathcal{J}_1^- = -\sin u_3 \tan u_4 \partial_{u_3} - \cos u_3 \partial_{u_4} - \sin u_3 \sec u_4 \partial_{u_5},$
   $\mathcal{J}_2^- = -\cos u_3 \tan u_4 \partial_{u_3} + \sin u_3 \partial_{u_4} - \cos u_3 \sec u_4 \partial_{u_5},$
   $\mathcal{J}_3^- = \partial_{u_3};$
3. $\mathcal{J}_1^+ = -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2} - \sin u_1 \sec u_2 \partial_{u_3},$
   $\mathcal{J}_2^+ = -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2} - \cos u_1 \sec u_2 \partial_{u_3},$
   $\mathcal{J}_3^+ = \partial_{u_1},$
   $\mathcal{J}_1^- = \sec u_2 \cos u_3 \partial_{u_1} + \sin u_3 \partial_{u_2} - \tan u_2 \cos u_3 \partial_{u_3},$
   $\mathcal{J}_2^- = -\sec u_2 \sin u_3 \partial_{u_1} + \cos u_3 \partial_{u_2} + \tan u_2 \sin u_3 \partial_{u_3},$
   $\mathcal{J}_3^- = \partial_{u_3};$
4. $\mathcal{J}_1^+ = -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2} - \sin u_1 \sec u_2 \partial_{u_3},$
   $\mathcal{J}_2^+ = -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2} - \cos u_1 \sec u_2 \partial_{u_3},$
   $\mathcal{J}_3^+ = \partial_{u_1},$
   $\mathcal{J}_1^- = -\sin u_4 \tan u_5 \partial_{u_4} - \cos u_4 \partial_{u_5} - \sin u_4 \sec u_5 \partial_{u_6},$
   $\mathcal{J}_2^- = -\cos u_4 \tan u_5 \partial_{u_4} + \sin u_4 \partial_{u_5} - \cos u_4 \sec u_5 \partial_{u_6},$
   $\mathcal{J}_3^- = \partial_{u_4};$
5. $\mathcal{J}_1^+ = -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2} - \sin u_1 \sec u_2 \partial_{u_3},$
   $\mathcal{J}_2^+ = -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2} - \cos u_1 \sec u_2 \partial_{u_3},$
   $\mathcal{J}_3^+ = \partial_{u_1},$
   $\mathcal{J}_1^- = k \sin u_4 \sec u_5 \partial_{u_3} - \sin u_4 \tan u_5 \partial_{u_4} - \cos u_4 \partial_{u_5},$
   $\mathcal{J}_2^- = k \sin u_4 \sec u_5 \partial_{u_3} - \cos u_4 \tan u_5 \partial_{u_4} + \sin u_4 \partial_{u_5},$
   $\mathcal{J}_3^- = \partial_{u_4};$
6. $\mathcal{J}_1^+ = -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2} - \sin u_1 \sec u_2 \partial_{u_3},$
   $\mathcal{J}_2^+ = -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2} - \cos u_1 \sec u_2 \partial_{u_3},$

$$
\begin{aligned}
\mathcal{J}_3^+ &= \partial_{u_1},\\
\mathcal{J}_1^- &= u_6 \sin u_4 \sec u_5 \partial_{u_3} - \sin u_4 \tan u_5 \partial_{u_4} - \cos u_4 \partial_{u_5},\\
\mathcal{J}_2^- &= u_6 \sin u_4 \sec u_5 \partial_{u_3} - \cos u_4 \tan u_5 \partial_{u_4} + \sin u_4 \partial_{u_5},\\
\mathcal{J}_3^- &= \partial_{u_4},
\end{aligned}
$$

where $k = \mathrm{const}$, $k \neq 0$.

Proof. We will give the principal steps of the proof omitting intermediate computations.

According to Theorem 1, there are two inequivalent realizations of the algebra $AO(3)$ with basis elements $\mathcal{J}_1^+$, $\mathcal{J}_2^+$, $\mathcal{J}_3^+$

$$
\begin{aligned}
1.\quad \mathcal{J}_1^+ &= -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2},\\
\mathcal{J}_2^+ &= -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2},\\
\mathcal{J}_3^+ &= \partial_{u_1};\\
2.\quad \mathcal{J}_1^+ &= -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2} - \sin u_1 \sec u_2 \partial_{u_3},\\
\mathcal{J}_2^+ &= -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2} - \cos u_1 \sec u_2 \partial_{u_3},\\
\mathcal{J}_3^+ &= \partial_{u_1}.
\end{aligned}
\tag{51}
$$

To complete a classification of inequivalent realization of $AO(4)$ we have to find all triplets of operators $\mathcal{J}_1^-, \mathcal{J}_2^-, \mathcal{J}_3^-$ which together with the operators (51) satisfy (49), (50).

Analyzing the commutation relations (49) we arrive at the following expressions for operators $\mathcal{J}_1^-$, $\mathcal{J}_2^-$, $\mathcal{J}_3^-$:

$$
1.\quad \mathcal{J}_a^- = \sum_{i=3}^{n} f_{ai}(u_3, \dots, u_n)\partial_{u_i},
$$

$$
2.\quad \mathcal{J}_a^- = \sum_{b=1}^{3} f_{ab}(u_4, \dots, u_n)\mathcal{Q}_b + \sum_{i=4}^{n} f_{ai}(u_4, \dots, u_n)\partial_{u_i},
$$

where f_{ij} are arbitrary smooth functions and

$$
\begin{aligned}
\mathcal{Q}_1 &= \sec u_2 \cos u_3 \partial_{u_1} + \sin u_3 \partial_{u_2} - \tan u_2 \cos u_3 \partial_{u_3},\\
\mathcal{Q}_2 &= -\sec u_2 \sin u_3 \partial_{u_1} + \cos u_3 \partial_{u_2} + \tan u_2 \sin u_3 \partial_{u_3},\\
\mathcal{Q}_3 &= \partial_{u_3}.
\end{aligned}
$$

Note that the operators $\mathcal{Q}_a$ fulfill the commutation relations of the algebra $AO(3)$.

Hence, we conclude that for the case 1 from (51) the operators $\mathcal{J}_a^-$ are given by the formulae (51), where one should replace u_i by u_{i+2}, correspondingly.

Let us turn now to the second realization of the algebra $AO(3)$ from (51).

Case 1. $f_{ai} = 0$, $a = 1, 2, 3$, $i = 4, \dots, n$. In this case we can reduce $\mathcal{J}_1^-$ to the form

$$
\mathcal{J}_1^- = \tilde{r}(u_4, \dots, n)\mathcal{Q}_1
$$

with the help of equivalence transformation

$$
X \to \tilde{X} = \mathcal{V} X \mathcal{V}^{-1}, \quad \mathcal{V} = \exp\left\{\sum_{a=1}^{3} F_a \mathcal{Q}_a\right\}, \tag{52}
$$

where F_a are some functions of $u_4, \ldots, u_n$. Note that transformation (52) does not change the form of the operators $\mathcal{J}_a^+$, since $[\mathcal{J}_a^+, \mathcal{Q}_b] = 0$, $a, b = 1, 2, 3$.

From commutation relations (50) it follows that $\tilde{r} = 1$ and furthermore $\mathcal{J}_2^- = \mathcal{Q}_2$, $\mathcal{J}_3^- = \mathcal{Q}_3$. Thus we get the following forms of the operators $\mathcal{J}_a^-$:

$$\begin{aligned} \mathcal{J}_1^- &= \sec u_2 \cos u_3 \partial_{u_1} + \sin u_3 \partial_{u_2} - \tan u_2 \cos u_3 \partial_{u_3}, \\ \mathcal{J}_2^- &= -\sec u_2 \sin u_3 \partial_{u_1} + \cos u_3 \partial_{u_2} + \tan u_2 \sin u_3 \partial_{u_3}, \\ \mathcal{J}_3^- &= \partial_{u_3}. \end{aligned}$$

Case 2. Not all f_{ai} vanish. Then the operators $\mathcal{J}_1^-$, $\mathcal{J}_2^-$, $\mathcal{J}_3^-$ can be transformed to become

$$\mathcal{J}_a^- = f_a(u_4, \ldots, u_n)\mathcal{Q}_1 + g_a(u_4, \ldots, u_n)\mathcal{Q}_2 + h_a(u_4, \ldots, u_n)\mathcal{Q}_3 + \mathcal{Z}_a,$$

where $a = 1, 2, 3$, and

$$\begin{aligned} \mathcal{Z}_1 &= -\sin u_4 \tan u_5 \partial_{u_4} - \cos u_4 \partial_{u_5} - \varepsilon \sin u_4 \sec u_5 \partial_{u_6}, \\ \mathcal{Z}_2 &= -\cos u_4 \tan u_5 \partial_{u_4} + \sin u_4 \partial_{u_5} - \varepsilon \cos u_4 \sec u_5 \partial_{u_6}, \\ \mathcal{Z}_3 &= \partial_{u_4}, \end{aligned}$$

and $\varepsilon = 0, 1$.

Now using the transformation (52) we reduce the operator $\mathcal{J}_3^-$ to the form $\mathcal{Z}_3 = \partial_{u_4}$. Next, from commutation relations

$$[\mathcal{J}_3^-, \mathcal{J}_1^-] = \mathcal{J}_2^-, \quad [\mathcal{J}_3^-, \mathcal{J}_2^-] = -\mathcal{J}_1^-$$

we get

$$\begin{aligned} \mathcal{J}_1^- &= \sum_{a=1}^{3} (G_a \cos u_4 + H_a \sin u_4)\mathcal{Q}_a + \mathcal{Z}_1, \\ \mathcal{J}_2^- &= \sum_{a=1}^{3} (H_a \cos u_4 - G_a \sin u_4)\mathcal{Q}_a + \mathcal{Z}_2, \end{aligned}$$

where G_a, H_a are arbitrary smooth functions of $u_5, \ldots, u_n$.

Making use of the equivalence transformation (52) with F_a being functions of $u_5, \ldots, u_n$ we can cancel coefficients G_a. The remaining commutation relation $[\mathcal{J}_1^-, \mathcal{J}_2^-] = \mathcal{J}_3^-$ yields equations for H_1, H_2, H_3

$$H_{au_5} - \tan u_5 H_a = 0,$$

whence

$$H_a = \tilde{H}_a \sec u_5, \quad a = 1, 2, 3,$$

$\tilde{H}_a$ being arbitrary functions of $u_6, \ldots, u_n$. Consequently, the operators $\mathcal{J}_a^-$ read

$$\begin{aligned} \mathcal{J}_1^- &= \sum_{a=1}^{3} \sin u_4 \sec u_5 \tilde{H}_a \mathcal{Q}_a + \mathcal{Z}_1, \\ \mathcal{J}_2^- &= \sum_{a=1}^{3} \cos u_4 \sec u_5 \tilde{H}_a \mathcal{Q}_a + \mathcal{Z}_2, \\ \mathcal{J}_3^- &= \mathcal{Z}_3. \end{aligned}$$

If $\varepsilon = 1$, then using the transformation (52) with F_a being functions of $u_6, \ldots, u_n$ we can cancel $\tilde{H}_a$, thus getting $\mathcal{J}_a^- = \mathcal{Z}_a$, $a = 1, 2, 3$. If $\varepsilon = 0$, then making use of the transformation (52) with F_a being functions of $u_6, \ldots, u_n$ we can put $\tilde{H}_1 = \tilde{H}_2 = 0$.

Provided $\tilde{H}_3 = 0$, we get the realization which is reduced to that given by the formulae 2 from the statement of the theorem.

Provided $\tilde{H}_3 = \text{const} \neq 0$, we get the formulae 5. At last, if $\tilde{H}_3 \neq \text{const}$, then performing a proper change of variables we arrive at the realization given by the formulae 6 from the statement of the theorem. The theorem is proved. ■

It follows from the above theorem that formulae (47) and 1–6 of the statement of Theorem 3 give six inequivalent realizations of the Lie algebra of the Euclid group $E(4)$ having the basis elements $P_\mu = \partial_{x_\mu}$ and (44), (45). To get all possible realizations of the algebra in question belonging to the above class it is necessary to add to the list of realizations of the algebra $AO(4)$ obtained in Theorem 3 the following three realizations of the operators $\mathcal{J}_a^-$, $\mathcal{J}_a^+$:

1. $\mathcal{J}_1^+ = -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2}$,
   $\mathcal{J}_2^+ = -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2}$,
   $\mathcal{J}_3^+ = \partial_{u_1}, \quad \mathcal{J}_a^- = 0$;
2. $\mathcal{J}_1^+ = -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2} - \sin u_1 \sec u_2 \partial_{u_3}$,
   $\mathcal{J}_2^+ = -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2} - \cos u_1 \sec u_2 \partial_{u_3}$,
   $\mathcal{J}_3^+ = \partial_{u_1}, \quad \mathcal{J}_a^- = 0$;
3. $\mathcal{J}_a^+ = 0, \quad \mathcal{J}_a^- = 0$,

where $a = 1, 2, 3$. This yields nine inequivalent realizations of the Lie algebra of the group $E(4)$.

In particular, the basis generators of the Euclid groups realized on the sets of solutions of the Dirac and self-dual Yang–Mills equations in the Euclidean space $\mathbb{R}^4$ are reduced to such a form that the generators of the rotation groups are given by (44), (45), $\mathcal{J}_{\mu\nu}$ being adduced in the formulae 4 of the statement of Theorem 3.

6 Concluding remarks

Summarizing the results of Sections 3 and 4 yields the following structure of realizations of the Lie algebra of rotation group by LVFs in n variables:

- If n=1, then there are no realizations.
- As there is no realization of $AO(3)$ by real non-zero 2×2 matrices, the only realization for the case $n = 2$ is given by (13). Furthermore, this realization is essentially nonlinear (i.e., it is not equivalent to a realization of the form (9)).
- In the case $n = 3$ there are two more realizations (38) (which is equivalent to (13)) and by formula (14). The latter realization is essentially nonlinear.
- Provided $n > 3$, there is no new realizations of $AO(3)$ and, furthermore, any realization can be reduced to a linear one (say, to (39)).

An evident (and very important) consequence of Theorem 1 is that there are only two inequivalent classes of $O(3)$-invariant partial differential equations of order r.

They are obtained via differential invariants of the order not higher than r of the Lie algebras having the basis elements (13), (14). In particular, the Weyl, Maxwell, Dirac equations are the special cases of the general system of first-order partial differential equations in $n \geq 8$ dependent variables invariant with respect to the algebra (14). We intend to devote one of our future publications to description of first-order differential invariants of the Lie algebra of the Euclid group $E(3)$ having the basis elements (13), (14) and (37). Let us note that this problem has been completely solved provided basis elements of $AE(3)$ are given by formulae (12) [20].

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