Stationary mKdV hierarchy and integrability of the Dirac equations by quadratures

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Using the Lie's infinitesimal method we establish that the Dirac equation in one variable is integrable by quadratures if the potential $V(x)$ is a solution of one of the equations of the stationary mKdV hierarchy.

Consider the eigenvalue problem for the Dirac operator $\mathcal{L} = i\sigma_1\, d/dx - V(x)\sigma_2$,

$$(\mathcal{L} - \lambda)\boldsymbol{u} \equiv i\sigma_1 \frac{d\boldsymbol{u}}{dx} - (V(x)\sigma_2 + \lambda)\boldsymbol{u} = \boldsymbol{0}, \tag{1}$$

where σ_1, σ_2 are 2×2 Pauli matrices, $\boldsymbol{u} = (u_1(x), u_2(x))^T$, $V(x)$ is a real-valued function and λ is a real parameter. We remind that Eq. (1) is one of two equations composing the Lax pair for the mKdV equation,

$$v_t + v_{xxx} - 6v^2 v_x = 0, \tag{2}$$

integrable by the inverse scattering method (see, e.g., Refs. [1, 2]). Next, as the identity

$$(\mathcal{L} - \lambda)(\mathcal{L} + \lambda) = -\frac{d^2}{dx^2} + V^2 - \sigma_3 \frac{dV}{dx} - \lambda^2,$$

holds, components of the vector-function $\boldsymbol{u}$ fulfill the stationary Schrödinger equation,

$$\frac{d^2 u_i}{dx^2} + \left((-1)^{i+1}\frac{dV}{dx} - V^2 + \lambda^2\right) u_i = 0, \quad i = 1, 2. \tag{3}$$

The aim of the present Letter is to show that there exists an initimate connection between integrability of system (1) (in what follows we will call it the Dirac equation) by quadratures and solutions of the stationary mKdV hierarchy.

Integrability of system (1) will be studied with the use of its Lie symmetries. As usual, we call a first-order differential operator

$$X = \xi(x)\frac{d}{dx} + \eta(x),$$

where ξ is a real-valued function and η is a 2×2 matrix complex-valued function, a Lie symmetry of system (1) if commutation relation

$$[\mathcal{L}, X] = R(x)\mathcal{L}, \tag{4}$$

holds with some 2×2 matrix function $R(x)$ (for details, see, e.g., Ref. [3]).

A simple computation shows that if X is a Lie symmetry of system (1), then an operator $X + r(x)\mathcal{L}$ with a smooth function $r(x)$ is its Lie symmetry as well. Hence we conclude that without loss of generality we can look for Lie symmetries within the

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class of matrix operators $X = \eta(x)$. Furthermore, inserting $X = \eta(x)$ into Eq. (4) and computing the commutator yield that the matrix $\eta(x)$ is necessarily of the form

$$\eta = \left\| \begin{array}{cc} f(x) & g(x) \\ h(x) & -f(x) \end{array} \right\|, \tag{5}$$

where $f(x)$, $g(x)$, $h(x)$ are arbitrary solutions of the following system of ordinary differential equations,

$$\frac{df}{dx} = i\lambda(g-h), \quad \frac{dg}{dx} = 2i\lambda f + 2gV, \quad \frac{dh}{dx} = -2i\lambda f - 2hV. \tag{6}$$

With a solution of system (6) in hand we can integrate the initial equations (1) by quadratures using the classical results by Elie Cartan [4]. Since these results are well-known we will give them without derivation in the form of the following lemma.

Lemma 1. *Let the functions $f(x)$, $g(x)$, $h(x)$ satisfy system (6). Then the general solution of the Dirac equation is given by the formulae*

$$\begin{aligned} u_1(x) &= C_1(R(x)+f(x))(h(x))^{-1/2}(R^2(x)-\Delta)^{-1/2}, \\ u_2(x) &= C_1(h(x))^{1/2}(R^2(x)-\Delta)^{-1/2}, \end{aligned} \tag{7}$$

where $\Delta = f^2(x) + g(x)h(x)$ is constant on the solution variety of system (6),

$$R(x) = \begin{cases} \sqrt{\Delta}\tanh\left(C_2 - i\lambda\sqrt{\Delta}\displaystyle\int\frac{dx}{g(x)}\right), & \Delta > 0, \\ \left(C_2 - i\lambda\displaystyle\int\frac{dx}{g(x)}\right)^{-1}, & \Delta = 0, \\ \sqrt{-\Delta}\tan\left(C_2 + i\lambda\sqrt{-\Delta}\displaystyle\int\frac{dx}{g(x)}\right), & \Delta < 0, \end{cases}$$

and C_1, C_2 are arbitrary complex constants.

However, solving system of ordinary differential equations (6) is by no means easier than solving the initial Dirac equation. This is a common problem in applying Lie symmetries to integration of ordinary differential equations. The key idea of our approach is to restrict a priori the class within which Lie symmetries are looked for and suppose that they are polynomials in λ with variable matrix coefficients.

Introducing the new dependent variables $\psi_1(x)$, $\psi_2(x)$,

$$\begin{aligned} f(x) &= \frac{i}{4\lambda}\left(-\frac{d\psi_1}{dx} + 2V\psi_2\right), \\ g(x) &= \frac{1}{2}(\psi_1(x)+\psi_2(x)), \quad h(x) = \frac{1}{2}(\psi_1(x)-\psi_2(x)), \end{aligned} \tag{8}$$

we rewrite Eq. (6) in the following equivalent form,

$$\frac{d^2\psi_1}{dx^2} = -4\lambda^2\psi_1 + 2V\frac{d\psi_2}{dx} + 2\psi_2\frac{dV}{dx}, \quad \frac{d\psi_2}{dx} = 2V\psi_1. \tag{9}$$

As mentioned above solutions of system (9) are looked for as polynomials in λ, namely

$$\psi_1(x) = \sum_{k=1}^{n} p_k(x)(2\lambda)^{2k}, \quad \psi_2(x) = \sum_{k=1}^{n} r_k(x)(2\lambda)^{2k}. \tag{10}$$

Inserting the expressions (10) into (9) and equating the coefficients by the powers of λ yield $p_n = 0$ and

$$\frac{dr_k}{dx} = 2Vp_k, \quad k = 1, \dots, n \tag{11}$$

$$\frac{d^2 p_k}{dx^2} = 2V\frac{dr_k}{dx} + 2\frac{dV}{dx} r_k - p_{k-1}, \quad k = 1, \dots, n-1, \tag{12}$$

where we set by definition $p_{-1}(x) = 0$. Eliminating from Eqs. (11), (12) the functions $r_k(x)$, we get recurrent relations for the functions $p_k(x)$,

$$p_{k-1}(x) = \underbrace{\left\{ -\frac{d^2}{dx^2} + 4\frac{dV}{dx} D_x^{-1} V + 4V^2 \right\}}_{\mathcal{Q}} p_k(x), \quad k = n, n-1, \dots, 0. \tag{13}$$

Here D_x^{-1} denotes integration by x.

A reader familiar with the theory of solitons will immediately recognize the operator $\mathcal{Q}$ as the recursion operator for the mKdV equation (2) (see, e.g., Refs. [5, 6]). Acting repeatedly with this operator on the trivial symmetry $S_0 = 0$ yields an infinite number of higher symmetries $S_1, S_2, \dots$ admitted by the mKdV equation [5]. Hence it is not difficult to derive that the functions p_k, $k = 0, \dots, n-1$ are linear combinations of the higher symmetries $S_1, \dots, S_n$ with arbitrary constant coefficients C_i,

$$p_{n-k}(x) = \sum_{i=1}^{k} C_i S_{k+1-i}, \quad k = 1, \dots, n, \tag{14}$$

where S_i are determined by the recurrent relations

$$S_{i+1}(x) = -\frac{d^2 S_i(x)}{dx^2} + 4\frac{dV}{dx}\int_{x_0}^{x} V(y) S_i(y) dy + 4V^2 S_i(x), \quad i = 1, \dots, n-1,$$

with $S_1 \stackrel{\text{def}}{=} dV/dx$.

The above formulae (14) give the general solution of the first n equations from Eq. (13). Inserting these into the last equation yields equation for the function $V(x)$ of the form

$$\sum_{k=1}^{n+1} C_k S_{n+2-k} = 0. \tag{15}$$

As $S_1 = dV/dx$, Eq. (15) is nothing else than an equation of the stationary mKdV hierarchy, which is obtained from the higher mKdV equations by setting $v(t,x) = v(x + Ct)$, $C = \text{const}$.

Integrating Eqs. (11) yields

$$r_k(x) = 2\sum_{i=1}^{k} C_i \int_{x_0}^{x} V(y) S_{k+1-i}(y) dy + \tilde{C}_k, \quad k = 1, \dots, n, \tag{16}$$

where $\tilde{C}_i$ are arbitrary complex constants.

Thus, the formulae (10), (14), (15), (16) give the general solution of the system of determining equations (11), (12) within the class of functions of the form (10). This means, in particular, that provided the function $V(x)$ is a solution of Eq. (15) with some fixed n and $C_1, \ldots, C_n$, the Dirac equation possesses a Lie symmetry. Hence we conclude that it is integrable by quadratures due to Lemma 1. Consequently, we have proved the following remarkable fact.

Theorem 1. *Let $V(x)$ be a solution of an equation of the mKdV hierarchy of the form (15). Then the Dirac equation (1) is integrable by quadratures.*

Note that the equations of the stationary mKdV hierarchy are transformed to the equations of the stationary KdV hierarchy with the help of the Miura transformation and the latter are integrated in θ-functions with any $n \in \mathbb{N}$ [7].

There is a deep relationship of the above results with those obtained by Novikov in Ref. [8], where it was established, in particular, that periodical solutions of the stationary KdV hierarchy give rise to the integrable stationary Schrödinger equations (3). This relationship is established via the Lax representation for higher KdV equations. Since we consider the stationary KdV equations, the Lax representation reduces to the condition that there exists an Nth-order differential operator

$$Q = \sum_{i=0}^{N} q_i(x) \frac{d^i}{dx^i},$$

commuting with the Schrödinger operator $d^2/dx^2 - W(x)$, provided $W(x)$ is a solution of the corresponding higher stationary KdV equation. Consequently, Q is the higher symmetry of the Schrödinger equation in a sense of [3].

On the set of solutions of the Schrödinger equation (3) we can reduce the operator Q to a first-order Lie symmetry of the form (for more details, see Ref. [9])

$$\tilde{Q} = \xi(x, \lambda) \frac{d}{dx} + \eta(x, \lambda),$$

where ξ, η are polynomials in λ. This gives us the ansatz for a Lie symmetry used at the beginning of this Letter.

Thus, the approach to integrating ordinary differential equations suggested here is based on their high-order Lie symmetry. To the best of our knowledge, the high-order Lie symmetries were not used until now for integrating ordinary differential equations.

It is important to note that within the framework of the Lie approach one always deals with the set of solutions as a whole. This means that specific properties of subsets of solutions (like periodicity) are not taken into account. To study these one needs more subtle analytic methods. On the other hand, the Lie approach has the merit of being a universal tool applicable to a wide range of ordinary differential equations having the same algebraic-theoretical properties. For example, it is not difficult to generalize the technique developed for integrating the Dirac equation (1) in order to integrate an arbitrary system of ordinary differential equations of the form

$$i\Omega_1 \frac{d\boldsymbol{u}}{dx} - (V(x)\Omega_3 + \lambda)\boldsymbol{u} = \boldsymbol{0}, \tag{17}$$

where Ω_1, Ω_2 are arbitrary finite- or infinite-dimensional constant matrices forming, together with the matrix $\Omega_3 = -i[\Omega_1, \Omega_2]$, a basis of the Lie algebra $su(2)$. The result will be the same, namely, if $V = V(x)$ is a solution of an equation of the stationary mKdV hierarchy, then the system of ordinary differential equations (17) is integrable by quadratures.

In conclusion let us demonstrate how the above procedure works for the simplest case $n = 1$. With this choice of n, Eq. (15) reads

$$\frac{C_2}{C_1}\frac{dV}{dx} - \frac{d^3V}{dx^3} + 6V^2\frac{dV}{dx} = 0, \tag{18}$$

which is exactly the stationary mKdV equation and is obtained from Eq. (2) via the ansatz $v(t,x) = V(C_2x - C_1t)$.

A simple computation yield the form of the coefficients of the Lie symmetry (5),

$$\begin{aligned} f(x) &= -\frac{i}{4\lambda}\left(C_1\frac{d^2V}{dx^2} - 2C_1V^3 - C_2 - 4C_1\lambda^2)\right), \\ g(x) &= \frac{1}{2}\left(C_1\frac{dV}{dx} - C_1V^2 - \frac{1}{2}C_2 - 2C_1\lambda^2\right), \\ h(x) &= \frac{1}{2}\left(C_1\frac{dV}{dx} + C_1V^2 + \frac{1}{2}C_2 + 2C_1\lambda^2\right), \end{aligned} \tag{19}$$

which satisfy the determining equations (6) inasmuch as the function $V(x)$ is a solution of the stationary mKdV equation.

Thus, the Dirac equation with potential $V(x)$ satisfying the stationary mKdV equation (18) is integrable by quadratures and its general solution is given by formulae (7) and (19).

Note that due to the remark made at the very beginning of the paper components of the function $\boldsymbol{u}$ fulfill the stationary Schrödinger equation (3). This is in a good accordance with results of Ref. papers [9].

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