On the classification of subalgebras of the conformal algebra with respect to inner automorphisms

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We give a complete justification of the classification of inequivalent subalgebras of the conformal algebra with respect to the inner automorphisms of the conformal group, and we perform the classification of the subalgebras of the conformal algebra $AC(1,3)$.

1 Introduction

The necessity of classifying the subalgebras of the conformal algebra is motivated by many problems in mathematics and mathematical physics [1, 2]. The conformal algebra $AC(1,n)$ of Minkowski space $\mathbb{R}_{1,n}$ contains the extended Poincaré algebra $A\tilde{P}(1,n)$ and the full Galilei algebra $AG_4(n-1)$ (also known as the optical algebra). The classification of the subalgebras of the conformal algebra $AC(l,n)$ is almost reducible to the classification of the subalgebras of the algebras $A\tilde{P}(1,n)$ and $AG_4(n-1)$.

Patera, Winternitz and Zassenhaus [1] have given a general method for the classification of the subalgebras of inhomogeneous transformations. Using this method, the classification of the subalgebras $AP(1,n)$, $A\tilde{P}(1,n)$, and $AG_4(n-1)$ was carried out in Refs. [1–9] for $n=2,3,4$. In Refs. [7–11], this general method was supplemented by many structural results which made possible the algorithmization of the classification of the subalgebras of the Euclidean, Galilean, and Poincaré algebras for spaces of arbitrary dimensions. Indeed, this was done in Refs. [9] and [10], where the subalgebras of $AC(1,n)$ were classified up to conjugation under the conformal group $C(1,n)$ for $n=2,3,4$.

In order to perform the symmetry reduction of differential equations, it is necessary to identify the subalgebras of the symmetry algebra (of the equation) which give the same systems of basic invariants. This observation has led to the introduction in Ref. [12] of the concept of I-maximal subalgebras: a subalgebra F is said to be I-maximal if it contains every subalgebra of the symmetry algebra with the same invariants as F. In Ref. [13], all I-maximal subalgebras of $AC(1,4)$, classified up to $C(1,4)$-conjugation, were found in the representation defined on the solutions of the eikonal equation. Using these subalgebras, reductions of the eikonal and Hamilton–Jacobi equations to differential equations of lower order were obtained in Refs. [9] and [12]. We note that the list of I-maximal subalgebras for a given algebra can differ according to the equation being investigated.

In the above works, the question of the connection between conjugation of the subalgebras of the algebra $A\tilde{P}(1,n)$ under the group $\tilde{P}(1,n)$ (or the group $\mathrm{Ad}\,A\tilde{P}(1,n)$ of inner automorphisms of the algebra $A\tilde{P}(1,n)$) and the conjugacy of these subalgebras under the group $C(1,n)$ was not dealt with. This, and the same problem for

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subalgebras of the Galilei algebra $AG_4(n-1)$, is the problem we address in the present article.

Since the group analysis of differential equations is of a local nature, we concentrate on conjugacy of the subalgebras under the group of inner automorphisms of the algebra $AC(1,n)$. Going over to conjugacy under $C(1,n)$ is not complicated, and requires only a further identification of the subalgebras under the action of at most three discrete symmetries. The results of this paper allow us to obtain a full classification of the subalgebras of $AC(1,n)$ for low values of n. On the basis of these results, we give at the end of this paper a classification of the algebra $AC(1,3)$ with respect to its group of inner automorphisms. The list of subalgebras obtained in this way can be used for the symmetry reduction of any system of differential equations which are invariant under $AC(1,3)$.

2 Maximal subalgebras of the conformal algebra

We denote by $\operatorname{Ad} L$ the group of inner automorphisms of the Lie algebra L. Unless otherwise stated, conjugacy of subalgebras of L means conjugacy with respect to the group $\operatorname{Ad} L$. We consider $\operatorname{Ad} L_1$ as a subgroup of $\operatorname{Ad} L_2$ whenever L_1 is a subalgebra of L_2. The connected identity component of a Lie group H is denoted by H_1.

Let $\mathbb{R}_{1,n}$ $(n \geq 2)$, be Minkowski space with metric $g_{\alpha\beta}$, where $(g_{\alpha\beta}) = \operatorname{diag}[1,-1,\ldots,-1]$ and $\alpha,\beta = 0,1,\ldots,n$. The transformation defined by the equations

$$x_\alpha = x_\alpha(y_0, y_1, \ldots, y_n), \quad \alpha = 0,1,\ldots,n$$

of a domain $U \subset \mathbb{R}_{1,n}$ into $\mathbb{R}_{1,n}$, is said to be conformal if

$$\frac{\partial x_\mu}{\partial y^\alpha}\frac{\partial x_\nu}{\partial y^\beta} g^{\mu\nu} = \lambda(x) g_{\alpha\beta},$$

where $\lambda(x) \neq 0$ and $x = (x_0, x_1, \ldots, x_n)$. The conformal transformations of $\mathbb{R}_{1,n}$ form a Lie group, the conformal group $C(1,n)$. The Lie algebra $AC(1,n)$ of the group $C(1,n)$ has as its basis the generators of pseudorotations $J_{\alpha\beta}$, the translations P_α, the nonlinear conformal translations K_α, and the dilatations D, where $\alpha,\beta = 0,1,\ldots,n$. These generators satisfy the following commutation relations:

$$\begin{aligned}
&[J_{\alpha\beta}, J_{\gamma\delta}] = g_{\alpha\delta}J_{\beta\gamma} + g_{\beta\gamma}J_{\alpha\delta} - g_{\alpha\gamma}J_{\beta\delta} - g_{\beta\delta}J_{\alpha\gamma},\\
&[P_\alpha, J_{\beta\gamma}] = g_{\alpha\beta}P_\gamma - g_{\alpha\gamma}P_\beta, \quad [P_\alpha, P_\beta] = 0, \quad [K_\alpha, J_{\beta\gamma}] = g_{\alpha\beta}K_\gamma - g_{\alpha\gamma}K_\beta,\\
&[K_\alpha, K_\beta] = 0, \quad [D, P_\alpha] = P_\alpha, \quad [D, K_\alpha] = -K_\alpha, \quad [D, J_{\alpha\beta}] = 0,\\
&[K_\alpha, P_\beta] = 2(g_{\alpha\beta}D - J_{\alpha\beta}).
\end{aligned} \tag{1}$$

The pseudo-orthogonal group $O(2,n+1)$ is the multiplicative group of all $(n+3)\times(n+3)$ real matrices C satisfying $C^t E_{2,n+1} C = E_{2,n+1}$, where $E_{2,n+1} = \operatorname{diag}[1,1,-1,\ldots,-1]$. We denote by I_{ab} the $(n+3)\times(n+3)$ matrix whose entries are zero except for 1 in the (a,b) position, with $a,b = 1,2,\ldots,n+3$. The Lie algebra $AO(2,n+1)$ of $O(2,n+1)$ has as its basis the following operators:

$$\begin{aligned}
&\Omega_{12} = I_{12} - I_{21}, \quad \Omega_{ab} = -I_{ab} + I_{ba} \ \ (a < b;\ a,b = 3,\ldots,n+3),\\
&\Omega_{ia} = -I_{ia} - I_{ai} \ \ (i = 1,2;\ a = 3,\ldots,n+3),
\end{aligned}$$

which satisfy the commutation relations

$$[\Omega_{ab},\Omega_{cd}]=\rho_{ad}\Omega_{bc}+\rho_{bc}\Omega_{ad}-\rho_{ac}\Omega_{bd}-\rho_{bd}\Omega_{ac}\quad (a,b,c,d=1,2,\dots,n+3),$$

where $(\rho_{ab})=E_{2,n+1}$. Let us denote by $\mathbb{R}_{2,n+1}$ the pseudo-Euclidean space of $n+3$ dimensions with metric ρ_{ab}. The matrices of the group $O(2,n+1)$ and the algebra $AO(2,n+1)$ will be identified with operators acting on the left on $\mathbb{R}_{2,n+1}$. Then, with this convention, $O(2,n+1)$ is the group of isometries of $\mathbb{R}_{2,n+1}$.

It is known (see for instance Ref. [9]) that there is a homomorphism $\Psi: O(2,n+1)\to C(1,n)$ with kernel $\{\pm E_{n+3}\}$, where $\{E_{n+3}\}$ is the unit $(n+3)\times(n+3)$ matrix. Thus we are able to identify $O(2,n+1)$ with $C(1,n)$. This homomorphism of groups induces an isomorphism f of the corresponding Lie algebras, $f: AO((2,n+1)\to AC(1,n)$, which is given by

$$\begin{aligned}&f(\Omega_{\alpha+2,\beta+2})=J_{\alpha\beta},\quad f(\Omega_{1,\alpha+2}-\Omega_{\alpha+2,n+3})=P_\alpha,\\&f(\Omega_{1,\alpha+2}+\Omega_{\alpha+2,n+3})=K_\alpha,\quad f(\Omega_{1,n+3})=-D\quad(\alpha,\beta=0,1,\dots,n).\end{aligned}$$

We shall in this article identify the two algebras, using this isomorphism, so that we can write the previous equations as

$$\begin{aligned}&\Omega_{\alpha+2,\beta+2}=J_{\alpha\beta},\quad \Omega_{1,\alpha+2}-\Omega_{\alpha+2,n+3}=P_\alpha,\\&\Omega_{1,\alpha+2}+\Omega_{\alpha+2,n+3}=K_\alpha,\quad \Omega_{1,n+3}=-D\quad(\alpha<\beta;\ \alpha,\beta=0,1,\dots,n).\end{aligned}$$

We shall use the matrix realization of the conformal algebra.

Each matrix C which belongs to the identity component $O_1(2,n+1)$ of the group $O(2,n+1)$ is a product of matrices which are rotations in the x_1x_2 and x_ax_b planes ($a<b$; $a,b=3,\dots,n+3$) and hyperbolic rotations in the x_ix_a planes ($i=1,2$; $a=3,\dots,n+3$). Thus each such matrix C can be given as a finite product of matrices of the form $\exp X$, where $X\in AO(2,n+1)$. From this, it follows that each inner automorphism of the algebra $AO(2,n+1)$ is a mapping

$$\varphi_C: Y\to CYC^{-1}, \tag{2}$$

where $Y\in AO(2,n+1)$ and $C\in O_1(2,n+1)$, and conversely each mapping of this type is an inner automorphism of the algebra $AO(2,n+1)$.

In the process of our investigation mappings of the above type (2) will occur for certain matrices $C\in O(2,n+1)$, so we call these types of mappings $O(2,n+1)$-automorphisms of the algebra $AO(2,n+1)$ corresponding to the matrix C.

If G is the group of $O(2,n+1)$-automorphisms of the algebra $AO(2,n+1)$, and H is the subgroup of G consisting of its inner automorphisms, then H is normal in G and $[G:H]\le 4$. Representatives of the cosets of G/H different from the identity will be

$$\begin{aligned}&C_1=\mathrm{diag}\,[-1,1,\dots,1,-1],\quad C_2=\mathrm{diag}\,[1,1,-1,1\dots,1],\\&C_3=\mathrm{diag}\,[-1,1,-1,1,\dots,1,-1],\end{aligned}\tag{3}$$

or

$$\begin{aligned}&C_1=\mathrm{diag}\,[-1,1,\dots,1,-1,1],\quad C_2=\mathrm{diag}\,[1,1,-1,1\dots,1],\\&C_3=\mathrm{diag}\,[1,-1,-1,1,\dots,1,-1,1].\end{aligned}\tag{4}$$

Given a subspace V of $\mathbb{R}_{2,n+1}$, there is a maximal subalgebra of $AO(2,n+1)$ which leaves V invariant. We call this algebra the normalizer in $AO(2,n+1)$ of the subspace V.

Let $Q_1,\ldots,Q_{n+3}$ be a system of unit vectors in $\mathbb{R}_{2,n+1}$. Then the normalizer in $AO(2,n+1)$ of the isotropic subspace $\langle Q_1+Q_{n+3}\rangle$ is the extended Poincaré algebra

$$A\tilde{P}(1,n)=\langle P_0,P_1,\ldots,P_n\rangle \uplus (AO(1,n)\oplus\langle D\rangle),$$

where $\uplus$ denotes semidirect sum, and $\oplus$ denotes direct sum of algebras; $AO(1,n)=\langle J_{\alpha,\beta}:\alpha,\beta=0,1,\ldots,n\rangle$. The normalizer in $AO(2,n+1)$ of the completely isotropic subspace $\langle Q_1+Q_{n+3},Q_2+Q_{n+2}\rangle$ is the ful1 Galilei algebra

$$AG_4(n-1)=\langle M,P_1,\ldots,P_{n-1},G_1,\ldots,G_{n-1}\rangle \uplus (AO(n-1)\oplus\langle R,S,T\rangle\oplus\langle Z\rangle),$$

where

$$\begin{gathered}M=P_0+P_n,\quad G_a=J_{0a}-J_{an}\quad (a=1,\ldots,n-1),\quad R=-(J_{0n}+D),\\ S=\frac{1}{2}(K_0+K_n),\quad T=\frac{1}{2}(P_0-P_n),\quad Z=J_{0n}-D.\end{gathered}$$

The generators of the algebra $AG_4(n-1)$ satisfy the following commutation relations:

$$\begin{gathered}[J_{ab},J_{cd}]=g_{ad}J_{bc}+g_{bc}J_{ad}-g_{ac}J_{bd}-g_{bd}J_{ac},\quad [G_a,J_{bc}]=g_{ab}G_c-g_{ac}G_b,\\ [P_a,J_{bc}]=g_{ab}P_c-g_{ac}P_b,\quad [G_a,G_b]=0,\quad [P_a,G_b]=\delta_{ab}M,\quad [G_a,M]=0,\\ [P_a,M]=0,\quad [J_{ab},M]=0,\quad [R,S]=2S,\quad [R,T]=-2T,\quad [T,S]=R,\\ [Z,R]=[Z,S]=[Z,T]=[Z,J_{ab}]=0,\quad [R,G_a]=G_a,\quad [R,P_a]=-P_a,\\ [R,M]=0,\quad [R,J_{ab}]=0,\quad [S,G_a]=0,\quad [S,P_a]=-G_a,\quad [S,M]=0,\\ [S,J_{ab}]=0,\quad [T,G_a]=P_a,\quad [T,P_a]=0,\quad [T,M]=0,\quad [T,J_{ab}]=0,\\ [Z,G_a]=-G_a,\quad [Z,P_a]=-P_a,\quad [Z,M]=-2M,\end{gathered}$$

with $a,b,c,d=1,\ldots,n-1$.

From these commutation relations we find that

$$\langle R,S,T\rangle=ASL(2,\mathbb{R}),\quad \langle R,S,T\rangle\oplus\langle Z\rangle=AGL(2,\mathbb{R}),$$

where $\mathbb{R}$ denotes the field of real numbers.

Let F be a reducible subalgebra of $AO(2,n+1)$. That is, there exists in $\mathbb{R}_{2,n+1}$ a nontrivial subspace W which is invariant under F. If W is isotropic, then there exists a totally isotropic subspace $W_0\subset W$ which is invariant under F. Since $\dim W_0$ is 1 or 2, then, by Witt's theorem [14] there exists an isometry $C\in O(2,n+1)$ such that CW_0 is either $\langle Q_1+Q_{n+3}\rangle$ or $\langle Q_1+Q_{n+3},Q_2+Q_{n+2}\rangle$. Taking into account that the matrices (3) do not change these subspaces and represent all the components of the group $O(2,n+1)$ different from the identity component $O_1(2,n+1)$, then we may assume that the above C lies in $O_1(2,n+1)$, the identity component. Thus there exists an inner automorphism φ of the algebra $AO(2,n+1)$ such that either $\varphi(F)\subset A\tilde{P}(1,n)$ or $\varphi(F)\subset AG_4(n-1)$.

If W is a nondegenerate subspace, then, by Witt's theorem, it is isometric with one of the following subspaces: $\mathbb{R}_{1,k}$ $(k\geq 2)$, $\mathbb{R}_{2,k}$ $(k\geq 1)$, $\mathbb{R}_k$ $(k\geq 1)$. Each of the isometrics (3) leaves invariant each of these subspaces, so that we may assume that the isometry which maps W onto one of these subspaces belongs to $O_1(2,n+1)$. From this, it follows that a subalgebra F is conjugate under the group of inner automorphisms of the algebra $AO(2,n+1)$ to a subalgebra of one of the following algebras:

(1) $AO'(1,k) \oplus AO''(1,n-k+1)$,
where $AO'(1,k) = \langle \Omega_{ab} : a,b = 1,3,\ldots,k+2\rangle$ and
$AO''(1,n-k+1) = \langle \Omega_{ab} : a,b = 2,k+3,\ldots,n+3\rangle$ with $n \geq 3$
and $k = 2,\ldots,[(n+1)/2]$;

(2) $AO(2,k) \oplus AO(n-k+1)$, where
$AO(n-k+1) = \langle \Omega_{ab} : a,b = k+3,\ldots,n+3\rangle$ with $k = 0,1,\ldots,n$.

In order to classify the subalgebras of these direct sums it is necessary to know the irreducible subalgebras of algebras of the type $AO(1,m)$ $(m \geq 2)$ and $AO(2,m)$ $(m \geq 3)$. It has been shown in Ref. [15] that $AO(1,m)$ has no irreducible subalgebras different from $AO(1,m)$. In Refs. [16] and [17] it has been shown that every semisimple irreducible subalgebra of $AO(2,m)$ $(m \geq 3)$ can be mapped by an automorphism of this algebra onto one of the following algebras:

(1) $AO(2,m)$;
(2) $ASU(1,(m/2)]$ when m is even;
(3) $\langle \Omega_{12} + \sqrt{3}\Omega_{13} + \Omega_{25}, -\Omega_{15} + \Omega_{24} - \sqrt{3}\Omega_{23}, \Omega_{12} - 2\Omega_{45}\rangle$ when $m = 3$.

It follows then that when $m > 3$ is odd, the algebra $AO(2,m)$ has no irreducible subalgebras other than $AO(2,m)$. If $m = 2k$ and $k \geq 2$, then, up to inner automorphisms, $AO(2,m)$ has two nontrivial maximal irreducible subalgebras: $ASU(l,k) \oplus \langle Y\rangle$, and $ASU(l,k)' \oplus \langle Y'\rangle$, where

$$Y = \mathrm{diag}\,[J,\ldots,J], \quad Y' = \mathrm{diag}\,[J,-J,J\ldots,J]$$

with

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We note that a subalgebra L of $AG_4(n-1)$ is conjugate under $\mathrm{Ad}\,AO(2,n+1)$ with a subalgebra the algebra $A\tilde{P}(1,n)$ if and only if the projection of L onto $AGL(2,\mathbb{R}) = \langle R,S,T\rangle \oplus \langle Z\rangle$ is conjugate under $\mathrm{Ad}\,AGL(2,\mathbb{R})$ with a subalgebra of the algebra $\langle R,T,Z\rangle$.

3 Conjugacy under Ad $AP(1,n)$ of subalgebras of the Poincaré algebra $AP(1,n)$

The Poincaré group $P(1,n)$ is the multiplicative group of matrices

$$\begin{pmatrix} \Delta & Y \\ 0 & 1 \end{pmatrix},$$

where $\Delta \in O(1,n)$ and $Y \in \mathbb{R}_{n+1}$. Let I'_{ab}, $a,b = 0,1,\ldots,n+1$ be the $(n+2)\times(n+2)$ matrix whose entries are all zero except for the ab-entry, which is unity. Then a basis for $AP(1,n)$ is given by the matrices

$$J_{0a} = -I'_{0a} - I'_{0a}, \quad J_{ab} = -I'_{ab} + I'_{ba}, \quad P_0 = I'_{0,n+1}, \quad P_a = I'_{a,n+1},$$

with $a < b$; $a,b = 1,\ldots,n$. These basis elements obey the commutation relations (1). It is sometimes useful in calculations to identify elements of $AO(1,n)$ with matrices of the form

$$X = \begin{pmatrix} 0 & \beta_{01} & \beta_{02} & \cdots & \beta_{0n} \\ \beta_{01} & 0 & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{02} & -\beta_{12} & 0 & \cdots & \beta_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \beta_{0n} & -\beta_{1n} & -\beta_{2n} & \cdots & 0 \end{pmatrix}$$

and elements of the space $U = \langle P_0,\ldots,P_n\rangle$ are represented by $n+1$-dimensional columns Y. In this case, we take

$$P_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, P_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

and with this notation it is easy to see that $[X,Y] = XY$. We endow the space U with the metric of the pseudo-Euclidean space $\mathbb{R}_{1,n}$, so that the inner product of two vectors

$$\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

is $x_0y_0 - x_1y_1 - \cdots - x_ny_n$. The projection of $AP(1,n)$ onto $AO(1,n)$ is denoted by $\hat{\varepsilon}$. We also note that $AO(n)$, contained in $AO(1,n)$, is generated by J_{ab} ($a < b$; $a,b = 1,\ldots,n$).

Let B be a Lie subalgebra of the algebra $AO(1,n)$ which has no invariant isotropic subspaces in $\mathbb{R}_{1,n}$. Then B is conjugate under $\operatorname{Ad} AO(1,n)$ to a subalgebra of $AO(n)$ or to $AO(1,k) \oplus C$, where $k \geq 2$ and C is a subalgebra of the orthogonal algebra $AO'(n-k)$ generated by the matrices J_{ab} ($a,b = k+1,\ldots,n$). In the first case, B is not conjugate to any subalgebra of $AO(n-1)$.

Proposition 1. *Let B be a subalgebra of $AO(n)$ which is not conjugate to a subalgebra of $AO(n-1)$. If L is a subalgebra of $AP(1,n)$ and $\hat{\varepsilon}(L) = B$, then L is conjugate to an algebra $W \uplus C$, where W is a subalgebra of $\langle P_1,\ldots,P_n\rangle$, and C is a subalgebra of $B \oplus \langle P_0\rangle$. Two subalgebras $W_1 \uplus C_1$ and $W_2 \uplus C_2$ of this type are conjugate to each other under* $\operatorname{Ad} AP(1,n)$ *if and only if they are conjugate under* $\operatorname{Ad} AO(n)$.

Proof. The algebra B is a completely reducible algebra of linear transformations of the space U and annuls only the subspace $\langle P_0\rangle$ (other than the null subspace itself). Thus, by Theorem 1.5.3 [9], the algebra L is conjugate to an algebra of the form

$W \uplus C$ where $W \subset \langle P_1, \ldots P_n \rangle$ and $C \subset B \oplus \langle P_0 \rangle$. Now let $W_1 \uplus C_1$, and $W_2 \uplus C_2$ be of this form, conjugate under $\mathrm{Ad}\, AP(1,n)$. Then there exists a matrix $\Gamma \in P_1(1,n)$ such that $\varphi_\Gamma(W_1 \uplus C_1) = W_2 \uplus C_2$, and from this it follows that $\varphi_\Lambda(B_1) = B_2$ for some $\Lambda \in O_1(1,n)$. Let $V = \langle P_1, \ldots, P_n \rangle$. Since $[B_1, V] = V$, then $[B_2, \varphi_\Lambda(V)] = \varphi_\Lambda(V)$ and $\varphi_\Lambda(V) = V$. Thus we can assume that $\Lambda = \mathrm{diag}\,[1, \Lambda_1]$ where $\Lambda_1 \in SO(n)$, so that the given algebras are conjugate under $\mathrm{Ad}\, AO(n)$. The converse is obvious.

Proposition 2. *Let $B = AO(1,k) \oplus C$, where $k \geq 2$ and $C \subset AO'(n-k)$. If L is a subalgebra of $AP(1,n)$ and $\hat{\varepsilon}(L) = B$ then L is conjugate to $L_1 \oplus L_2$ where $L_1 = AO(1,k)$ or $L_1 = AP(1,k)$, and L_2 is a subalgebra of the Euclidean algebra $AE'(n-k)$ with basis P_a, J_{ab} $(a, b = k+1, \ldots, n)$. Two subalgebras of this form, $L_1 \oplus L_2$ and $L_1' \oplus L_2'$ are conjugate under $\mathrm{Ad}\, AP(1,n)$ if and only if $L_1 = L_1'$ and L_2 is conjugate to L_2' under the group of $E'(n-k)$-automorphisms.*

Proof. The proof is as in the proof of Proposition 1.

Lemma 1. *If $C \in O(1,n)$ and $C(P_0 + P_n) = \lambda(P_0 + P_n)$ then $\lambda \neq 0$ and*

$$C = \begin{pmatrix} \dfrac{1 + \lambda^2(1 + \boldsymbol{v}^2)}{2\lambda} & \lambda \boldsymbol{v}^t B & \dfrac{-1 + \lambda^2(1 - \boldsymbol{v}^2)}{2\lambda} \\ \boldsymbol{v} & B & -\boldsymbol{v} \\ \dfrac{-1 + \lambda^2(1 + \boldsymbol{v}^2)}{2\lambda} & \lambda \boldsymbol{v}^t B & \dfrac{1 + \lambda^2(1 - \boldsymbol{v}^2)}{2\lambda} \end{pmatrix}, \tag{5}$$

where $B \in B(n-1)$, $\boldsymbol{v}$ is an $(n-1)$-dimensional column vector, $\boldsymbol{v}^2$ is the scalar square of $\boldsymbol{v}$ and $\boldsymbol{v}^t$ is the transpose of $\boldsymbol{v}$. Conversely, every matrix C of this form satisfies $C(P_0 + P_n) = \lambda(P_0 + P_n)$.

Proof. Proof is by direct calculation.

Lemma 2. *Let $C \in O(1,n)$ have the form (5), with $\lambda > 0$. Then*

$$C = \mathrm{diag}\,[1, B, 1] \exp[(-\ln \lambda) J_{0n}] \exp(-\beta_1 G_1 - \cdots - \beta_{n-1} G_{n-1}),$$

where $G_a = J_{0a} - J_{an}$ and

$$\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{n-1} \end{pmatrix} = B^{-1} \boldsymbol{v}.$$

Proof. Direct calculation gives us

$$\exp(-\theta J_{0n}) = \begin{pmatrix} \cosh\theta & 0 & \sinh\theta \\ 0 & E_{n-1} & 0 \\ \sinh\theta & 0 & \cosh\theta \end{pmatrix}$$

and

$$\exp(-\beta_1 G_1 - \cdots - \beta_{n-1} G_{n-1}) = \begin{pmatrix} 1 + \dfrac{\boldsymbol{b}^2}{2} & \boldsymbol{b}^t & \dfrac{\boldsymbol{b}^2}{2} \\ \boldsymbol{b} & E_{n-1} & -\boldsymbol{b} \\ \dfrac{\boldsymbol{b}^2}{2} & \boldsymbol{b}^t & 1 - \dfrac{\boldsymbol{b}^2}{2} \end{pmatrix},$$

where $\boldsymbol{b} = (\beta_1, \ldots, \beta_{n-1})^t$. On putting $\lambda \exp\theta$ we have

$$\cosh\theta = \frac{\lambda^2+1}{2\lambda}, \quad \sinh\theta = \frac{\lambda^2-1}{2\lambda}.$$

Since we have

$$\begin{pmatrix} \dfrac{\lambda^2+1}{2\lambda} & 0 & \dfrac{\lambda^2-1}{2\lambda} \\ 0 & E_{n-1} & 0 \\ \dfrac{\lambda^2-1}{2\lambda} & 0 & \dfrac{\lambda^2+1}{2\lambda} \end{pmatrix} \begin{pmatrix} 1+\dfrac{\boldsymbol{b}^2}{2} & \boldsymbol{b}^t & \dfrac{\boldsymbol{b}^2}{2} \\ \boldsymbol{b} & E_{n-1} & -\boldsymbol{b} \\ \dfrac{\boldsymbol{b}^2}{2} & \boldsymbol{b}^t & 1-\dfrac{\boldsymbol{b}^2}{2} \end{pmatrix} =$$

$$= \begin{pmatrix} \dfrac{1+\lambda^2(1+\boldsymbol{b}^2)}{2\lambda} & \lambda\boldsymbol{b}^t & \dfrac{-1+\lambda^2(1-\boldsymbol{b}^2)}{2\lambda} \\ \boldsymbol{b} & E_{n-1} & -\boldsymbol{b} \\ \dfrac{-1+\lambda^2(1+\boldsymbol{b}^2)}{2\lambda} & \lambda\boldsymbol{b}^t & \dfrac{1+\lambda^2(1-\boldsymbol{b}^2)}{2\lambda} \end{pmatrix},$$

then

$$\exp(-\theta J_{0n})\exp(-\beta_1 G_1 - \cdots - \beta_{n-1}G_{n-1}) = \mathrm{diag}\,[1, \beta^{-1}, 1]C$$

from which it follows directly that

$$C = \mathrm{diag}\,[1, B, 1]\exp[(-\ln\lambda)J_{0n}]\exp(-\beta_1 G_1 - \cdots - \beta_{n-1}G_{n-1})$$

and the lemma is proved.

The set of F of matrices of the form (5) with $\lambda > 0$ is a group under multiplication. The mapping

$$C \to \begin{pmatrix} \lambda B & \lambda\boldsymbol{v} \\ 0 & 1 \end{pmatrix}$$

is an isomorphism of the group F onto the extended Euclidean group $\tilde{E}(n-1)$. Thus we shall mean the group F when talking of the extended Euclidean group, and the connected identity component $\tilde{E}_1(n-1)$ will be identified with the group of matrices of the form (5) with $\lambda > 0$ and $B \sin SO(n-1)$. From Lemma 2 it follows that the Lie algebra AF of the group F is generated by the basis elements J_{ab}, G_a, J_{0n} ($a < b$; $a, b = 1, \ldots, n-1$).

Lemma 3. *If $C \in O_1(1,n)$ and $C(P_0 + P_n) = \lambda(P_0 + P_n)$ then $\lambda > 0$ and $B \in SO(n-1)$ in (5).*

Proof. Since

$$\frac{1+\lambda^2(1+\boldsymbol{v}^2)}{2\lambda} > 0,$$

then we have $\lambda > 0$. From Lemma 2, $\mathrm{diag}\,[1, B, 1] \in O_1(1,n)$, so that $\det B > 0$. Thus $B \in SO(n-1)$ and the lemma is proved.

Lemma 4. *If $C \in O(1,n)$ and $\pm C \notin \tilde{E}(n-1)$ then $C = \pm A_1 C' A_2$ where $A_1, A_2 \in \tilde{E}(n-1)$ and $C' = \mathrm{diag}\,[1, \ldots, 1, -1]$.*

Proof. We can choose a matrix $\Lambda \in O(n-1)$ so that $\Lambda C(P_0+P_n) = \alpha P_0+\beta P_1+\gamma P_n$ where $\alpha^2-\beta^2-\gamma^2=0$. If $\beta \neq 0$ then $\alpha-\gamma \neq 0$. Let $\theta = \beta/(\alpha-\gamma)$. Then,

$$\exp(\theta G_1)(\alpha P_0+\beta P_1+\gamma P_n) = \frac{\alpha-\gamma}{2}(P_0-P_n)$$

and so there exists a matrix $\Gamma \in \tilde{E}(n-1)$ such that $\Gamma C(P_0+P_n) = \lambda(P_0+P_n)$ or $\Gamma C(P_0+P_n) = \lambda(P_0-P_n)$. In the first case, $\pm\Gamma C \in \tilde{E}(n-1)$, so that then we have $\pm C \in \tilde{E}(n-1)$, which is impossible. In the second case, $C'\Gamma C(P_0+P_n) = \lambda(P_0+P_n)$. For $\lambda > 0$ we find $C'\Gamma C \in \tilde{E}(n-1)$. Put $C'\Gamma C = A_2$, $\Gamma = A_1^{-1}$. Then $C = A_1 C' A_2$. If $\lambda < 0$ then we put $-C'\Gamma C = A_2$, in which case $C = -A_1 C' A_2$, and the lemma is proved.

Lemma 5. *If $C \in O_1(1,n)$ and $C \notin \tilde{E}_1(n-1)$, then $C = D_1 Q D_2$, where $D_1, D_2 \in \tilde{E}_1(n-1)$, and $Q = \operatorname{diag}[1,-1,1,\dots,1,-1]$.*

Proof. If $\pm C \in \tilde{E}(n-1)$, then $C(P_0+P_n) = \gamma(P_0+P_n)$. By Lemma 3, $\gamma > 0$ and $C \in \tilde{E}_1(n-1)$, which contradicts the assumption. Thus, $\pm C \notin \tilde{E}(n-1)$. By Lemma 4, $C = \pm A_1 C' A_2$. From this it follows that $C = D_1 \Gamma D_2$, where $D_1, D_2 \in \tilde{E}_1(n-1)$, and F is one of the matrices $\pm C'$, $\pm Q$. However, $\Gamma \in O_1(1,n)$, since $\Gamma = D_1^{-1} C D_2^{-1}$, find from this it follows that $\Gamma = Q$. The Lemma is proved.

Direct calculation shows that the normalizer of the space $\langle P_0+P_n\rangle$ in $AO(1,n)$ is generated by the matrices G_a, J_{ab}, J_{0n} $(a,b=1,\dots,n-1)$, which satisfy the commutation relations

$$[G_a, J_{bc}] = g_{ab}G_c - g_{ac}G_b, \quad [G_a, G_b] = 0, \quad [G_a, J_{0n}] = G_a.$$

This means that the normalizer of the space $\langle P_0+P_n\rangle$ in the algebra $AO(1,n)$ is the extended Euclidean algebra

$$A\tilde{E}(n-1) = \langle G_1,\dots,G_{n-1}\rangle \uplus (AO(n-1) \oplus \langle J_{0n}\rangle)$$

in an $(n-1)$-dimensional space, where the generators of translations are $G_1,\dots,G_{n-1}$ and the generator of dilatations is the matrix J_{0n}.

Let K be a subalgebra of $AP(1,n)$ such that its projection onto $AO(1,n)$ has an invariant isotropic subspace in Minkowski space $\mathbb{R}_{1,n}$. The subalgebra K is conjugate under $\operatorname{Ad} AP(1,n)$ with a subalgebra of the algebra $\mathcal{A} = AG_1(n-1) \uplus \langle J_{0n}\rangle$ where $AG_1(n-1)$ is the usual Galilei algebra with basis M, T, P_a, G_a, J_{ab} $(a,b=1,\dots,n-1)$, and $M = P_0+P_n$, $T = \frac{1}{2}(P_0-P_n)$.

Proposition 3. *Let L_1 and L_2 be subalgebras of $\mathcal{A}$, with L_1 not conjugate under $\operatorname{Ad}\mathcal{A}$ to any subalgebra having zero projection onto $\langle G_1,\dots,G_{n-1}\rangle$. If $\varphi(L_1) = L_2$ for some $\varphi \in \operatorname{Ad} AP(1,n)$, then there exists an inner automorphism ψ of the algebra $\mathcal{A}$ with $\psi(L_1) = L_2$.*

Proof. Since $\operatorname{Ad}\mathcal{A}$ contains automorphisms which correspond to matrices of the form

$$\exp\left(\sum_{\gamma=}^{n} a_\gamma P_\gamma\right) \tag{6}$$

and since $P(1,n)$ is a semidirect product of the group of matrices of the form (6) and the group $O(1,n)$ of matrices of the form $\operatorname{diag}[\Delta, 1]$, then we may assume that

$\varphi = \varphi_C$ with $C \in O_1(1,n)$. If $C \notin \tilde{E}_1(n-1)$, then by Lemma 5, $C = D_1 Q D_2$. In that case we find that

$$(D_1QD_2)\hat{\varepsilon}(L_1)(D_2^{-1}QD_1^{-1}) = \hat{\varepsilon}(L_2),$$

whence

$$Q(D_2\hat{\varepsilon}(L_1)D_2^{-1})Q = D_1^{-1}\hat{\varepsilon}(L_2)D_1. \tag{7}$$

However,

$$QG_aQ = Q(J_{0a} - J_{an})Q = \begin{cases} J_{0a} + J_{an}, & \text{when } a \neq 1, \\ -(J_{01} + J_{1n}), & \text{when } a = 1. \end{cases}$$

This means that $QG_aQ \notin \mathcal{A}$. Because of this, the left-hand side of (7) does not belong to $\mathcal{A}$, whereas the right-hand side of (7) is a subalgebra of $\mathcal{A}$. This then implies that we must have $C \in \tilde{E}_1(n-1)$ and thus we have $\psi(L_1) = L_2$ for some $\psi \in \operatorname{Ad}\mathcal{A}$.

Proposition 4. *Let $\tilde{\mathcal{A}}$ be a Lie algebra with basis P_0, P_a, P_n, J_{ab}, J_{0n} $(a,b = 1,\ldots,n-1)$ and let L_1, L_2 be subalgebras of $\tilde{\mathcal{A}}$ such that at least one of them has a nonzero projection onto $\langle J_{0n}\rangle$. If $\varphi(L_1) = L_2$ for some $\varphi \in \operatorname{Ad}AP(1,n)$, then there exists an inner automorphism $\psi \in \tilde{\mathcal{A}}$ so that either $\psi(L_1) = L_2$ or $\psi(L_1) = \varphi_Q(L_2)$ where $Q = \operatorname{diag}[1,-1,1,\ldots,1,-1]$.*

Proof. As in the proof of Proposition 3, we may assume that $\varphi = \varphi_C$ where $C \in O_1(1,n)$. We shall also assume that the projection of L_1 onto $\langle J_{0n}\rangle$ is nonzero. If $C \in \tilde{E}_1(n-1)$ and $C \notin \tilde{O}_1(n-1)$ then the projection of the algebra $\varphi(L_1)$ onto $\langle G_1,\ldots,G_{n-1}\rangle$ is nonzero, and hence the projection of L_2 onto $\langle G_1,\ldots,G_{n-1}\rangle$ is nonzero, which contradicts the assumptions of the proposition. Thus, if $C \in \tilde{E}_1(n-1)$ then $\varphi \in \operatorname{Ad}\tilde{\mathcal{A}}$.

Let $C \notin \tilde{E}_1(n-1)$. By Lemma 5, $C = D_1QD_2$ where $D_1, D_2 \in \tilde{E}_1(n-1)$. Then $\varphi(L_1) = L_2$ can be written as

$$\varphi_Q(\sigma_{D_2}(L_1)] = \varphi_{D_1^{-1}}(L_2).$$

If $D_2 \notin \tilde{O}_1(n-1)$ then the projection of $\varphi_{D_2}(L_1)$ onto $\langle G_1,\ldots,G_{n-1}\rangle$ is nonzero and hence $\varphi_Q[\varphi_{D_2}(L_1)]$ does not belong to $\mathcal{A}$. But then $\varphi_{D_1^{-1}}(L_2)$ is also not in $\mathcal{A}$. This is a contradiction. Thus $D_1, D_2 \in \tilde{O}_1(n-1)$. From this it follows that $\varphi_Q(\psi(L_1)) = L_2$ where $\psi = \varphi_D$ is an inner automorphism of the algebra $\tilde{\mathcal{A}}$. This proves the proposition.

Proposition 5. *Suppose $2 \le m \le n-1$. Let F be a subalgebra of the algebra $AO(m)$ which is not conjugate under $\operatorname{Ad}AO(m)$ to a subalgebra of $AO(m-1)$, and let L be a subalgebra of $\langle P_0, P_1,\ldots,P_n\rangle \uplus F$ such that $\hat{\varepsilon}(L) = F$. Then L is conjugate to an algebra $W \uplus K$, where W is a subalgebra of $\langle P_1,\ldots,P_m\rangle$ and K is a subalgebra of $F \oplus \langle P_0, P_{m+1},\ldots,P_n\rangle$. Two subalgebras $W_1 \uplus K_1$ and $W_2 \uplus K_2$ of this type are conjugate under $\operatorname{Ad}AP(1,n)$ if and only if there exists an automorphism $\psi \in \operatorname{Ad}AO(m) \times \operatorname{Ad}AO(1,n-m)$ such that $\psi(W_1 \uplus K_1) = W_2 \uplus K_2$ or $\psi(W_1 \uplus K_1) = Q(W_2 \uplus K_2)Q$ where*

$$AO(1,n-m) = \langle J_{\alpha\beta} : \alpha,\beta = 0, m+1,\ldots,n\rangle$$

and $Q = \operatorname{diag}[1,-1,1,\ldots,1,-1]$.

4 Conjugacy of subalgebras of the extended Poincaré algebra $A\tilde{P}(1,n)$ under Ad $AC(1,n)$

Lemma 6. *If $C \in O(2,n+1)$ and $C(Q_1 + Q_{n+3}) = \lambda(Q_1 + Q_{n+3})$ then $\lambda \neq 0$ and*

$$C = \begin{pmatrix} \dfrac{1+\lambda^2(1-\boldsymbol{v}^2)}{2\lambda} & -\lambda \boldsymbol{v}^t E_{1,n} B & \dfrac{-1+\lambda^2(1+\boldsymbol{v}^2)}{2\lambda} \\ \boldsymbol{v} & B & -\boldsymbol{v} \\ \dfrac{-1+\lambda^2(1-\boldsymbol{v}^2)}{2\lambda} & -\lambda \boldsymbol{v}^t E_{1,n} B & \dfrac{1+\lambda^2(1+\boldsymbol{v}^2)}{2\lambda} \end{pmatrix}, \qquad (8)$$

where $B \in O(1,n)$, $E_{1,n} = \operatorname{diag}[1,-1,\dots,-1]$, $\boldsymbol{v}$ is an $(n+1)\times 1$ matrix and $\boldsymbol{v}^2$ is its scalar square in $\mathbb{R}_{1,n}$. Conversely, every matrix C of the form (8) satisfies the condition $C(Q_1 + Q_{n+3}) = \lambda(Q_1 + Q_{n+3})$.

Proof. Direct calculation.

Lemma 7. *Let $C \in O(2,n+1)$ have the form (8), with $\lambda > 0$. Then*

$$C = \operatorname{diag}[1,B,1]\exp[(\ln\lambda)D]\exp(-\beta_0 P_0 - \beta_1 P_1 - \cdots - \beta_n P_n),$$

where

$$\begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = B^{-1}\boldsymbol{v}.$$

Proof. The proof of Lemma 7 is similar to that of Lemma 2.

The mapping

$$f : C \to \begin{pmatrix} \lambda B & \lambda \boldsymbol{v} \\ 0 & 1 \end{pmatrix}$$

is a homomorphism of the group of matrices (8) onto the extended Poincaré group $\tilde{P}(1,n)$. The kernel of this homomorphism is the group of order two, $\{-E_{n+3}, E_{n+3}\}$. Let us denote by H the set of matrices of the form (8) with $\lambda > 0$. Then f is an isomorphism of H onto $\tilde{P}(1,n)$. For this reason we shall, in the remainder of this article, mean the group H when referring to $\tilde{P}(1,n)$. Its Lie algebra is the extended Poincaré algebra $A\tilde{P}(1,n)$ given in Section 2.

Lemma 8. *Let $C \in O_1(2,n+1)$ and let it be of the form (8) with $\lambda > 0$. Then $B \in B_1(1,n)$.*

Remark 1. Note that when $\lambda < 0$ it is possible that B does not belong to $O_1(2,n+1)$.

Lemma 9. *If $C \in O_1(2,n+1)$ and $\pm C \notin \tilde{P}(1,n)$ then either $C = \pm A_1 Q A_2$ or $C = A_1 F(\theta) A_2$, where $A_1, A_2 \in \tilde{P}(1,n)$, $Q = \operatorname{diag}[1,\dots,1-1]$ and $F(\theta) = \exp[(\theta/2)(K_0 + P_0 + K_n - P_n)]$.*

Proof. There exists a matrix $\Lambda\tilde{P}(1,n)$ such that

$$\Lambda C(Q_1 + Q_{n+3}) = \alpha_1 Q_1 + \alpha_2 Q_2 + \alpha_3 Q_{n+2} + \alpha_4 Q_{n+3},$$

where $\alpha_1^2+\alpha_2^2-\alpha_3^2-\alpha_4^2=0$ and $\alpha_2\alpha_3\geq 0$. If $\alpha_1\neq\alpha_4$ then, as in the proof of Lemma 4, we obtain that

$$\exp(\beta_0P_0+\beta_nP_n)\Lambda C(Q_1+Q_{n+3})=\gamma(Q_1\pm Q_{n+3})$$

for some real numbers β_0, β_n, γ. From this it follows that

$$\Gamma\exp(\beta_0P_0+\beta_nP_n)\Lambda C(Q_1+Q_{n+3})=\lambda(Q_1+Q_{n+3}),$$

where $\lambda>0$ and $\Gamma=\pm E_{n+3}$ or $\Gamma=\pm Q$. By Lemma 6 and Lemma 7, we obtain

$$\Gamma\exp(\beta_0P_0+\beta_nP_n)\Lambda C=\tilde\Lambda,\quad \tilde\Lambda\in\tilde P(1,n).$$

Since $\pm C\notin\tilde P(1,n)$, then $\Gamma=\pm Q$, and so $C=\pm A_1QA_2$, where $A_1=\Lambda^{-1}\exp(-\beta_0P_0 -\beta_nP_n)$, $A_2=\tilde\Lambda$.

If $\alpha_1=\alpha_4$, then also $\alpha_2=\alpha_3$. It is easy to verify that

$$F(\theta)\Lambda C(Q_1+Q_{n+3})=(\alpha_1\cos\theta+\alpha_2\sin\theta)(Q_1+Q_{n+3})+ \\ +(\alpha_2\cos\theta-\alpha_1\sin\theta)(Q_2+Q_{n+2}).$$

If $\alpha_1=0$ then we put $\theta=(\pi/2)$, when $\alpha_2>0$ and $\theta=-(\pi/2)$, when $\alpha_2<0$. If $\alpha_1\neq 0$ then we let $\alpha_2\cos\theta-\alpha_1\sin\theta=0$. In that case,

$$\tan\theta=\frac{\alpha_2}{\alpha_1},\quad \alpha_1\cos\theta+\alpha_2\sin\theta=\alpha_1\cos\theta(1+\tan^2\theta).$$

We choose the value of θ so that $\alpha_1\cos\theta>0$. With this choice of θ we have

$$F(\theta)\Lambda C(Q_1+Q_{n+3})=\lambda(Q_1+Q_{n+3}),$$

where $\lambda>0$. But then, as a result of Lemma 6 and Lemma 7, $F(\theta)\Lambda C=\tilde\Lambda$, $\tilde\Lambda\in\tilde P(1,n)$, and so $C=A_1F(-\theta)A_2$, where $A_1=\Lambda^{-1}$, $A_2=\tilde\Lambda$. The result is proved.

Lemma 10. *Let L_1 and L_2 be subalgebras of $A\tilde P(1,n)$ which are not conjugate under $A\tilde P(1,n)$ to subalgebras of $A\tilde O(1,n)=AO(1,n)\oplus\langle D\rangle$. Then L_1, L_2 are conjugate under* $\mathrm{Ad}\,AC(1,n)$ *if and only if they are conjugate under* $\mathrm{Ad}\,A\tilde P(1,n)$ *or if one of the following conditions holds:*

(1) *n is an odd number and there exists an automorphism $\psi\in\mathrm{Ad}\,A\tilde P(1,n)$ with $\psi(L_1)=C_2L_2C_2^{-1}$ (see Eq. (3) for notation);*

(2) *there exist automorphisms $\psi_1,\psi_2\in A\tilde P(1,n)$ with*

$$\psi_1(L_1)=F(\theta)[\psi_2(L_2)]F(-\theta).$$

Proof. Let $CL_1C^{-1}=L_2$ for some $C\in O_1(2,n+1)$. By Lemma 9, we may assume that $\pm C\in\tilde P(1,n)$ or that C is one of the matrices $\pm A_1QA_2$, $A_1F(\theta)A_2$ (we use the notation of Lemma 9) . If $C\in\tilde P(1,n)$ then, by Lemma 8, C belongs to the identity component of the group $\tilde P(1,n)$ and thus φ_C is an inner automorphism of the algebra $A\tilde P(1,n)$. Now suppose $-C\in\tilde P(1,n)$. Then by Lemma 7, $C=-\mathrm{diag}\,[1,B,1]$, where $B\in O(1,n)$ and $\Delta\in\tilde P_1(1,n)$. Thus we may assume that $C=-\mathrm{diag}\,[1,B,1]$. From this it follows that $B\in O_1(1,n)$ for odd n and we have

$$\mathrm{diag}\,[1,1,-1,1,\dots,1,1]B\in O_1(1,n)$$

For even n this means that the algebras L_1, L_2 are conjugate to each other under $\operatorname{Ad} A\tilde{P}(1,n)$ or that there exists an automorphism $\psi \in \operatorname{Ad} A\tilde{P}(1,n)$ such that $\psi(L_1) = C_2 L_2 C_2^{-1}$.

Let $C = \pm A_1 Q A_2$. Then $C = \Gamma_1 \Delta \Gamma_2$ with $\Gamma_1, \Gamma_2 \in \tilde{P}(1,n)$ and $\Delta = \pm\operatorname{diag}[1, \varepsilon_1, 1, \ldots, 1, \varepsilon_2, -1]$ with $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$. Clearly, $\Delta \in O_1(2, n+1)$. When $C = A_1 Q_2$ we have $\varepsilon_1 = 1$, $\varepsilon_2 = -1$ and when $C = -A_1 Q A_2$, $\varepsilon_1 = 1$, $\varepsilon_2 = (-1)^n$. Since

$$\Delta P_n \Delta^{-1} = \pm K_n, \quad \Delta P_\alpha \Delta^{-1} = \pm K_\alpha$$

with $\alpha < n$, then from $\Gamma_1^{-1} L_2 \Gamma_1 = \Delta(\Gamma_2 L_1 \Gamma_2^{-1})\Delta^{-1}$ it follows that the algebra $\Gamma_1^{-1} L_2 \Gamma_1$ has a nonzero projection onto $\langle K_0, K_1, \ldots, K_n\rangle$, which is impossible. Thus the matrix C is different from $\pm A_1 Q A_2$.

Now let $C = A_1 F(\theta) A_2$. If Γ is one of the matrices (4), then $\Gamma F(\theta)\Gamma^{-1} = F(\pm\theta)$, so that

$$C = A_1' F(\theta) A_2' \Delta,$$

where $A_1', A_2' \in \tilde{P}(1,n)$ and $\Delta = E$ or Δ is one of the matrices (4). Since Δ can be represented as a product of matrices in $O_1(2,n)$, then the last case is impossible, and we have proved the Lemma.

Theorem 1. *Let L_1 and L_2 be subalgebras of $A\tilde{P}(1,n)$ which are not conjugate under $A\tilde{P}(1,n)$ to subalgebras of $A\tilde{O}(1,n)$ and such that their projections onto $AO(1,n)$ have no invariant isotropic subspace in $\mathbb{R}_{1,n}$. The subalgebras L_1 and L_2 are conjugate under $\operatorname{Ad} AC(1,n)$ if and only if they are conjugate under $\operatorname{Ad} A\tilde{P}(1,n)$ or when there exists an automorphism $\psi \in \operatorname{Ad} A\tilde{P}(1,n)$ such that $\psi(L_1) = C_2 L_2 C_2^{-1}$, where $C_2 = \operatorname{diag}[1, 1, -1, 1, \ldots, 1]$.*

Proof. By Lemma 10 we may assume that $\psi_1(L_1) = F(\theta)[\psi_2(L_2)]F(-\theta)$ for some $\psi_1, \psi_2 \in A\tilde{P}(1,n)$. Under the given assumptions, the projection of $\psi_2(L_2)$ onto $AO(1,n)$ contains an element of the form

$$X = \sum_{b=1}^{n-1} (\alpha_b J_{0b} + \gamma_b J_{bn}) + \sum_{b,c=1}^{n-1} \sigma_{bc} J_{bc},$$

where $\alpha_q \neq -\gamma_q$ for some q $(1 \le q \le n-1)$. Since

$$F(\theta) J_{0q} F(-\theta) = J_{0q}\cos\theta + \frac{1}{2}(K_q + P_q)\sin\theta$$

and

$$F(\theta) J_{qn} F(-\theta) = J_{nq}\cos\theta + \frac{1}{2}(K_q - P_q)\sin\theta$$

we have that $F(\theta) X F(-\theta)$ contains the term

$$F(\theta)[\alpha_q J_{0q} + \gamma_q J_{qn}]F(-\theta) = (\alpha_q J_{0q} + \gamma_q J_{qn})\cos\theta + \\ + \frac{1}{2}[\alpha_q(K_q + P_q) + \gamma_q(K_q - P_q)]\sin\theta$$

and from this it follows that $(\alpha_q + \gamma_q)\sin\theta = 0$ so that $\sin\theta = 0$. But then $\theta = m\pi$. When $m = 2d$ we have $F(\theta) = E_{n+3}$. When $m = 2d+1$ then $F(\theta) = \operatorname{diag}[-1, -1, E_{n-1}, -1, -1]$. However,

$$F(\theta)[\psi_2(L_2)]F(-\theta) = (-F(\theta))[\psi_2(L_2)](-F(-\theta))$$

from which it follows that we may assume that $\psi_1(L_1) = C[\psi_2(L_2)]C^{-1}$ where $C = \operatorname{diag}[1, 1, -E_{n-1}, 1, 1]$. If n is odd, then φ_C is an inner automorphism of $A\tilde{P}(1, n)$. If n is even, then $\varphi_{C_2}\varphi_C$ is an inner automorphism of the algebra $A\tilde{P}(1, n)$. In the first case, $\psi_3(L_1) = L_2$ where $\psi_3 = \psi_2^{-1}\varphi_C^{-1}\psi_1$ is an inner automorphism of the algebra $A\tilde{P}(1, n)$. In the second case, $\psi(L_1) = \varphi_{C_2}(L_2)$ for some $\psi \in \operatorname{Ad} A\tilde{P}(1, n)$. The theorem is proved.

Theorem 2. *Let L_1 and L_2 be subalgebras of $A\tilde{O}(1, n)$ having no invariant isotropic subspaces in $\mathbb{R}_{1,n}$. The subalgebras L_1, L_2 are conjugate under $\operatorname{Ad} AC(1, n)$ if and only if they are conjugate under $\operatorname{Ad} A\tilde{O}(1, n)$ or when there exists an automorphism $\psi \in \operatorname{Ad} A\tilde{O}(1, n)$ such that $\psi(L_1) = CL_2C^{-1}$ where C is one of the $(n+3)\times(n+3)$ matrices*

$$\operatorname{diag}[1, 1, -1, 1, \ldots, 1], \quad \operatorname{diag}[1, \ldots, 1, -1], \quad \operatorname{diag}[1, \ldots, 1, -1, -1].$$

We note that $A\tilde{O}(1, n) \subset AO(2, n+1)$ and that the matrix C is $(n+3)\times(n+3)$.

5 Subalgebras of the full Galilei algebra

Lemma 11. *Let $C \in O(2, n+1)$ and $W = \langle Q_1 + Q_{n+3}, Q_2 + Q_{n+2}\rangle$. If $CW = W$, then*

$$\begin{aligned} C = {} & \exp[\theta(S+T)] \operatorname{diag}[1, \varepsilon, K, \varepsilon, 1] \exp(\alpha R + \beta Z) \times \\ & \times \exp\left(\sum_{i=1}^{n-1} \gamma_i G_i\right)\left(\delta M + \lambda T + \sum_{i=1}^{n-1} \mu_i P_i\right), \end{aligned} \tag{9}$$

where $\varepsilon = \pm 1$, $K \in O(n-1)$.

Proof. We have

$$C(Q_1 + Q_{n+3}) = \alpha_1(Q_1 + Q_{n+3}) + \alpha_2(Q_2 + Q_{n+2})$$

and so

$$\begin{aligned} F(-\theta)C(Q_1 + Q_{n+3}) = {} & (\alpha_1 \cos\theta - \alpha_2 \sin\theta)(Q_1 + Q_{n+3}) + \\ & + (\alpha_2 \cos\theta + \alpha_1 \sin\theta)(Q_2 + Q_{n+2}). \end{aligned}$$

If $\alpha_1 = 0$ then we put $\theta = (3\pi/2)$ when $\alpha_2 > 0$ and $\theta = (\pi/2)$ when $\alpha_2 < 0$. If $\alpha_1 \neq 0$ then we put $\alpha_1 \sin\theta + \alpha_2 \cos\theta = 0$ and then $\tan\theta = -\alpha_2/\alpha_1$ and $\alpha_1 \cos\theta - \alpha_2 \sin\theta = \alpha_1 \cos\theta(1 + \tan^2\theta)$. We choose θ so that $\alpha_1 \cos\theta > 0$. For this choice of θ we have $F(-\theta)C(Q_1 + Q_{n+3}) = \xi(Q_1 + Q_{n+3})$, where $\xi > 0$. Using Lemma 7, we obtain

$$F(-\theta)C = A = \operatorname{diag}[1, B, 1] \exp([\ln\xi]D) \exp\left(-\sum_{i=0}^{n} \beta_i P_i\right) \in \tilde{P}(1, n),$$

where $B \in O(1, n)$. Then $C = F(\theta)A$. The matrix A has the form (8). Direct calculation gives

$$A(Q_2 + Q_{n+2}) = \alpha(Q_1 + Q_{n+3}) + \beta Q_2 + \gamma Q_{n+2} + \sum_{i=3}^{n+1} \delta_i Q_i.$$

From this it follows that

$$F(\theta)A(Q_2+Q_{n+2}) = (\alpha\cos\theta+\beta\sin\theta)Q_1 + (-\alpha\sin\theta+\beta\cos\theta)Q_2 + \\ + (\gamma\cos\theta-\alpha\sin\theta)Q_{n+2} + (\gamma\sin\theta+\alpha\cos\theta)Q_{n+3} + \sum_{i=3}^{n+1}\delta_i Q_i.$$

Now we have $F(\theta)A(Q_2+Q_{n+2}) \in W$, from which we have

$$\alpha\cos\theta+\beta\sin\theta = \gamma\sin\theta+\alpha\cos\theta, \quad -\alpha\sin\theta+\beta\cos\theta = \gamma\cos\theta-\alpha\sin\theta$$

and so we conclude that $\beta=\gamma$ and $\delta_j=0$, $j=3,\ldots,n+1$. But in that case we have

$$\mathrm{diag}\,[1,B,1](Q_2+Q_{n+2}) = \beta(Q_2+Q_{n+2}).$$

By Lemma 2, we have

$$\pm B = \mathrm{diag}\,[1,K,1]\exp[(-\ln|\beta|)J_{0n}]\exp\left(\sum_{i=1}^{n-1}\gamma_i G_i\right),$$

where $K\in O(n-1)$. We note that

$$K_0+P_0-K_n-P_n = 2(S+T), \quad J_{0n} = \frac{1}{2}(Z-R), \quad D = -\frac{1}{2}(Z+R),$$
$$P_0 = \frac{1}{2}(M+2T), \quad P_n = \frac{1}{2}(M-2T), \quad [D,G_a]=0, \quad [D,J_{0n}]=0.$$

The lemma is proved.

Lemma 12. *Let $C\in O_1(2,n+1)$ and $W=\langle Q_1+Q_{n+3}, Q_2+Q_{n+2}\rangle$. If $CW=W$ then the matrix C has the form (9) with $\varepsilon=1$ and $K\in SO(n-1)$.*

Proof. From the conditions of Lemma 1 1 and the fact that we ask for $C\in O_1(2,n+1)$, it follows that $\mathrm{diag}\,[1,\varepsilon,K,\varepsilon,1]\in O_1(2,n+1)$. It follows now that $\varepsilon>0$ and that

$$\begin{vmatrix} K & 0 \\ 0 & \varepsilon \end{vmatrix} > 0$$

and thus we have $\varepsilon=1$ and $|K|>0$, whence $K\in SO(n-1)$. This proves the lemma.

The matrices of the form (9) with $\varepsilon=1$ and $K\in SO(n-1)$ form a group under multiplication, which we denote by $G_4(n-1)$ since its Lie algebra is the full Galilei algebra $AG_4(n-1)$. It is easy to see that $G_4(n-1)\subset O_1(2,n+1)$.

Lemma 13. *If $C\in O_1(2,n+1)$ but $C\notin G_4(n-1)$, then $C=A_1\Gamma A_2$, where $A_1,A_2\in G_4(n-1)$ and Γ is one of the matrices*

$$\Gamma_1 = \mathrm{diag}\,[1,\ldots,1,-1], \quad \Gamma_2 = \mathrm{diag}\,[1,1,-1,1,\ldots,1,-1,1]. \tag{10}$$

Proof. Let

$$C(Q_1+Q_{n+3}) = \sum_{i=1}^{n+3}\alpha_i Q_i, \quad \alpha_1^2+\alpha_2^2-\alpha_3^3-\cdots-\alpha_{n+3}^2 = 0.$$

There exists a matrix $\Lambda=\mathrm{diag}\,[1,1,\Delta,1,1]$ with $\Delta\in SO(n-1)$ such that $\Lambda C(Q_1+Q_{n+3})$ does not contain $Q_4,\ldots,Q_{n+1}$. Hence we may assume $\alpha_1^2+\alpha_2^2-\alpha_{n+2}^2-\alpha_{n+3}^2=0$.

Since

$$S+T=\frac{1}{2}(K_0+P_0+K_n-P_n)=\Omega_{12}+\Omega_{n+2,n+3},$$

then, up to a factor $\exp[\theta(S+T)]$, we may suppose that $\alpha_1\neq 0$, $\alpha_2=0$. If $\alpha_1^2=\alpha_{n+3}^2$ then $\alpha_3=0$, $\alpha_{n+2}=0$. Assume $\alpha_1\neq\alpha_{n+3}$. As in the proof of Lemma 4, we find that

$$\begin{aligned}&\exp(\beta_1P_1+\beta_2P_2)(\alpha_1Q_1+\alpha_3Q_3+\alpha_{n+2}Q_{n+2}+\alpha_{n+3}Q_{n+3})=\\&\quad=\alpha_1'Q_1+\alpha_{n+3}'Q_{n+3},\end{aligned}$$

where $\alpha_1'^2-\alpha_{n+3}'^2=0$. Thus there exists a matrix $A_1\in G_4(n-1)$ such that

$$\begin{aligned}&A_1^{-1}C(Q_1+Q_{n+3})=\gamma(Q_1\pm Q_{n+3}),\\&A_1^{-1}C(Q_2+Q_{n+2})=\delta_1Q_1+\delta_2Q_2+\delta_3Q_3+\delta_4Q_{n+2}+\delta_5Q_{n+3}.\end{aligned}\tag{11}$$

Since the pseudo-orthogonal transformations preserve the scalar product, it follows that the right-hand sides in (11) are also orthogonal, which implies that $\gamma(\delta_1\mp\delta_5)=0$ so that $\delta_5=\pm\delta_1$. If $\delta_2\neq\delta_4$ then multiplying the left- and right-hand sides in (11) by $\exp(\theta G_1)$ does not change the right-hand side of the first equality, and allows us to eliminate δ_3 by transforming it into 0. If $\delta_2=\delta_4$, then one easily deduces that $\delta_3=0$. Thus we may assume that $\delta_3=0$. But then we have $\delta_4=\pm\delta_2$ because $\delta_5=\pm\delta_1$ and $\delta_1^2+\delta_2^2-\delta_4^2-\delta_5^2=0$.

Let $W=\langle Q_1+Q_{n+3},Q_2+Q_{n+2}\rangle$. The above reasoning implies that for some matrix $A_1\in G_4(n-1)$ we have $\Gamma A_1^{-1}CW=W$ where Γ is one of the matrices (10). The fact that $\Gamma A_1^{-1}C\in O_1(2,n+1)$ implies, using Lemma 12, $\Gamma A_1^{-1}C=A_2\in G_4(n-1)$. Thus $C=A_1\Gamma A_2$ and the lemma is proved.

Lemma 14. *The subalgebras L_1 and L_2 of $AG_4(n-1)$ are conjugate under* $\mathrm{Ad}AC(1,n)$ *if and only if they are conjugate under* $\mathrm{Ad}\,AG(n-1)$ *or if there exist automorphisms ψ_1, ψ_2 in* $\mathrm{Ad}\,AG_4(n-1)$ *with $\psi_1(L_1)=\Gamma[\psi_2(L_2)]\Gamma^{-1}$, where Γ is one of the matrices (10).*

Proof. The result follows immediately from Lemma 13.

In the following table we give the action on the full Galilei algebra $AG_4(n-1)$ of the automorphisms where

$$C_4=\exp\left(\frac{\pi}{2}(S+T)\right),\quad C_5=\exp(\pi(S+T))$$

(see (3) and (10) for the notation).

Theorem 3. *Let L_1 and L_2 be subalgebras of $AG_4(n-1)$ which are not conjugate under* $\mathrm{Ad}\,AG_4(n-1)$ *with subalgebras of*

$$\langle M,T,P_1,\ldots,P_{n-1}\rangle\uplus(AO(n-1)\oplus\langle D,J_{0n}\rangle$$

and

$$AO(n-1)\oplus\langle S+T,Z\rangle.$$

Then the subalgebras L_1 and L_2 are conjugate under $\mathrm{Ad}\,AC(1,n)$ *if and only if they are conjugate under* $\mathrm{Ad}\,AG_4(n-1)$.

Table 1. Action of automorphisms on elements of $AG_4(n-1)$ for $n \geq 2$.

Element of $AG_4(n-1)$	φ_{Γ_1}	φ_{Γ_2}	φ_{C_1}	φ_{C_4}	φ_{C_5}	Restrictions
P_1	K_1	$-P_1$	$-P_1$	$-G_1$	$-P_1$	
P_a	K_a	P_a	$-P_a$	$-G_a$	$-P_a$	$a = 2, \ldots, n-1$
M	$K_0 - K_n$	$2T$	$-M$	M	M	
G_1	$J_{01} + J_{1n}$	$-(J_{01} + J_{1n})$	G_1	P_1	$-G_1$	
G_a	$J_{0a} + J_{an}$	$J_{0a} + J_{an}$	G_a	P_a	$-G_a$	$a = 2, \ldots, n-1$
J_{1a}	J_{1a}	$-J_{1a}$	J_{1a}	J_{1a}	J_{1a}	$a = 2, \ldots, n-1$
J_{ab}	J_{ab}	J_{ab}	J_{ab}	J_{ab}	J_{ab}	$a, b = 2, \ldots, n-1$
R	$-R$	Z	R	$-R$	R	
S	T	$\frac{1}{2}(K_0 - K_n)$	$-S$	T	S	
T	S	$\frac{1}{2}M$	$-T$	S	T	
Z	$-Z$	R	Z	Z	Z	

Proof. If the subalgebras L_1 and L_2 are conjugate under $\operatorname{Ad} AG_4(n-1)$ then they are conjugate under $\operatorname{Ad} AC(1,n)$. Now suppose that they are conjugate under $\operatorname{Ad} AC(1,n)$. In order to prove their conjugacy under $\operatorname{Ad} AG_4(n-1)$ it is sufficient (by Lemma 14) to show that for an arbitrary $\psi \in \operatorname{Ad} AG_4(n-1)$ and for each matrix Γ of the form (10), the subalgebra $\Gamma\psi(L_1)\Gamma^{-1}$ either equals $\psi(L_1)$ or is not contained in $AG_4(n-1)$, for then the only possibility is that they are conjugate under $\operatorname{Ad} AG_4(n-1)$.

If the projection of $\psi(L_1)$ onto $\langle G_1, \ldots, G_{n-1}\rangle$ is nonzero, then, using Table 1, the subalgebra $\Gamma\psi(L_1)\Gamma^{-1}$ contains an element Y whose projection for some a, $1 \leq a \leq n-1$ onto $\langle J_{0a}, J_{an}\rangle$ is of the form $\lambda(J_{0a}+J_{an})$ with $\lambda \neq 0$. If $\Gamma\psi(L_1)\Gamma^{-1} \subset AG_4(n-1)$, then the projection of Y onto $\langle J_{0a}, J_{an}\rangle$ would have the form $\mu(J_{0a}-J_{an})$ which would imply $\lambda = \mu = -\mu = 0$, an obvious contradiction.

Now let the projection of $\psi(L_1)$ onto $\langle G_1, \ldots, G_{n-1}\rangle$ be zero. Denote by $\tau\psi(L_1)$ the projection of $\psi(L_1)$ onto $\langle R, S, T\rangle$. If $\tau\psi(L_1) = \langle R, S, T\rangle$, then $\langle R, S, T\rangle \subset \psi(L_1)$. From this it follows that $\Gamma_2\psi(L_1)\Gamma_2^{-1}$ is not a subset of $AG_4(n-1)$. If we assume that $\Gamma_1\psi(L_1)\Gamma_1^{-1} \subset AG_4(n-1)$, we obtain, from Table 1, that the projection of $\psi(L_1)$ onto $\langle P_1, \ldots, P_n, M\rangle$ is zero, and consequently we have either $\psi(L_1) = \langle R, S, T\rangle$ or $\psi(L_1) = \langle R, S, T\rangle \oplus \langle Z\rangle$. In this case, $\Gamma_1\psi(L_1)\Gamma_1^{-1} = \psi(L_1)$. If $\tau\psi(L_1) = \langle R + \alpha S, T + \beta S\rangle$, with $\alpha \neq 0$, then $\Gamma_2\psi(L_1)\Gamma_2^{-1}$ is not contained in $AG_4(n-1)$. If we had $\Gamma_1\psi(L_1)\Gamma_1^{-1} \subset AG_4(n-1)$, then the projection of $\psi(L_1)$ onto $\langle P_1, \ldots, P_n, M\rangle$ would be zero. But then $\psi(L_1)$ would be conjugate under $\operatorname{Ad} AG_4(n-1)$ with a subalgebra of $AO(n-1) \oplus \langle R, T, Z\rangle$, which contradicts the assumptions of the theorem. The theorem is proved.

Theorem 4. *Let L_1 and L_2 be subalgebras of the algebra*

$$L = \langle M, T, P_1, \ldots, P_{n-1}\rangle \uplus (AO(n-1) \oplus \langle D, J_{0n}\rangle)$$

having nonzero projection on $\langle J_{0n}\rangle$ and $\langle D\rangle$ and are not conjugate under $\operatorname{Ad} L$ *with subalgebras of the algebra $\langle M, T\rangle \uplus (AO(n-1) \oplus \langle D, J_{0n}\rangle)$. Then L_1 and L_2 are conjugate under* $\operatorname{Ad} AC(1,n)$ *if and only if they are conjugate under* $\operatorname{Ad} L$ *or if there exists an automorphism $\psi \in \operatorname{Ad} L$ such that $\psi(L_1) = \Lambda L_2 \Lambda^{-1}$ where Λ is one of the matrices Γ_2, C_5, $\Gamma_2 C_5$ (see Table 1).*

Proof. If $\psi \in \operatorname{Ad} AG_4(n-1)$, then $\psi = \varphi_C$ where C is a matrix of the form (9). By theorem IV.3.4 of Ref. [9], the subalgebra L_1 is, up to an automorphism of $\operatorname{Ad} AG_4(n-$

1), one of the following algebras:

(1) $(U_1 + U_2 + U_3) \uplus F$, where $U_1 \subset \langle M \rangle$, $U_2 \subset \langle T \rangle$, $U_3 \subset \langle P_1, \dots, P_{n-1} \rangle$ and $F \subset AO(n-1) \oplus \langle D, J_{0n} \rangle$;

(2) $(U_1 + U_2) \uplus F$, where $U_1 \subset \langle T \rangle$, $U_2 \subset \langle P_1, \dots, P_{n-1} \rangle$ and F is a subalgebra of $AO(n-1) \oplus \langle R, M \rangle$;

(3) $(U_1 + U_2) \uplus F$, where $U_1 \subset \langle M \rangle$, $U_2 \subset \langle P_1, \dots, P_{n-1} \rangle$ and F is a subalgebra of $AO(n-1) \oplus \langle Z, T \rangle$.

By assumption, the projection of L_1 onto $\langle P_1, \dots, P_{n-1} \rangle$ is nonzero.

If $\psi(L_1) = L_2$, then in formula (9) $\theta = 0$ or $\theta = \pi$ because for other values of θ the projection of $\psi(L_1)$ onto $\langle G_1, \dots, G_{n-1} \rangle$ is nonzero. For this reason, $\gamma_1 = \cdots = \gamma_{n-1} = 0$ and so $\psi \in \mathrm{Ad}\, L$ or $\varphi_{C_5}\psi \in \mathrm{Ad}\, L$. Let there be automorphisms $\psi_1, \psi_2 \in \mathrm{Ad}\, AG_4(n-1)$ with $\Gamma\psi_1(L_1)\Gamma = \psi_2(L_2)$ where Γ is one of the matrices (10). If $\mathrm{Ad}\, L$ did not contain ψ_1 and $\varphi_{C_5}\psi_1$, then the projection of $\psi_1(L_1)$ on $\langle G_1, \dots, G_{n-1} \rangle$ would be nonzero, and so, by Table 1, $\psi_2(L_2)$ would not be in $AG_4(n-1)$. Thus ψ_j or $\varphi_{C_5}\psi_j$ belongs to $\mathrm{Ad}\, L$ for each $j = 1, 2$. For $\Gamma = \Gamma_1$ the projection of $\Gamma\psi_1(L_1)\Gamma$ onto $\langle K_1, \dots, K_{n-1} \rangle$ is nonzero, so we have $\Gamma = \Gamma_2$. In this case $\Gamma\psi_2(L_2)\Gamma = \psi_2'(\Gamma L_2 \Gamma)$. Using Lemma 14, the theorem is proved.

In a similar way, one proves the following results.

Theorem 5. *Let B be a subalgebra of the algebra*

$$N = \langle M, P_1, \dots, P_{n-1} \rangle \uplus (AO(n-1) \oplus \langle D, T \rangle)$$

and let B have nonzero projection onto $\langle D \rangle$. Then B is conjugate under $\mathrm{Ad}\, AC(1,n)$ to the algebra

$$F = (W_1 \oplus W_2) \uplus E, \tag{12}$$

where E is a subalgebra of the algebra $AO(n-1) \oplus \langle D \rangle$, $W_1 \subset \langle P_1, \dots, P_{n-1} \rangle$ and W_2 is one of the algebras 0, $\langle P_0 \rangle$, $\langle P_n \rangle$, $\langle P_n \rangle$, $\langle P_0, P_n \rangle$. If $W_2 = \langle P_n \rangle$, or $W_2 = \langle P_0, P_n \rangle$ then the subalgebra $W_1 \uplus E$ is not conjugate under $\mathrm{Ad}\, AO(n-1)$ with any subalgebra of $\langle P_1, \dots, P_{n-2} \rangle \uplus (AO(n-2) \oplus \langle D \rangle)$. Subalgebras F_1, F_2 of the type (12) of the algebra N with nonzero projection onto $\langle D \rangle$, which are not conjugate under $\mathrm{Ad}\, N$ to subalgebras of $\langle M, T \rangle \uplus (AO(n-1) \oplus \langle D \rangle)$, will be conjugate under $AC(1,n)$ if and only if they are conjugate under $\mathrm{Ad}\, L$ or when there exists an automorphism $\psi \in \mathrm{Ad}\, L$ with $\psi(F_1) = \Gamma_2 F_2 \Gamma_2^{-1}$ (see (10)), where $L = AO(n-1)$ (we consider $\mathrm{Ad}\, AO(n-1)$ to be a subgroup of $\mathrm{Ad}\, AC(1,n)$).

Theorem 6. *Let B be a subalgebra of the algebra*

$$N = \langle M, P_1, \dots, P_{n-1} \rangle \uplus (AO(n-1) \oplus \langle J_{0n}, T \rangle)$$

and let B have nonzero projection onto $\langle J_{0n} \rangle$. Then B is conjugate under $\mathrm{Ad}\, AC(1,n)$ with the algebra

$$F = W \uplus E, \tag{13}$$

where E is a subalgebra of the algebra $\langle P_1, \dots, P_{n-1} \rangle \uplus (AO(n-1) \oplus \langle J_{0n} \rangle)$ and W is one of the algebras 0, $\langle M \rangle$, $\langle P_0, P_n \rangle$. Let $L = N \uplus \langle D \rangle$. Subalgebras F_1, F_2 of the *type (13) of the algebra N which are not conjugate under* $\mathrm{Ad}\,N$ *with subalgebras of the algebra* $\langle M\rangle \uplus (AO(n-1)\oplus\langle J_{0n},T\rangle)$*, will be conjugate under* $\mathrm{Ad}\,AC(1,n)$ *if and only if they are conjugate under* $\mathrm{Ad}\,L$ *or if there exists an automorphism* $\psi\in\mathrm{Ad}\,L$ *with* $\psi(F_1)=\Lambda F_2\Lambda^{-1}$ *where* Λ *is one of the matrices* Γ_2, C_5, $\Gamma_2 C_5$ *(see Table 1).*

Theorem 7. *Let* L_1, L_2 *be subalgebras of the algebra* $L=\langle M,S+T,Z\rangle\oplus AO(n-1)$ *which have nonzero projection onto* $\langle S+T\rangle$*. The algebras* L_1 *and* L_2 *are conjugate under* $\mathrm{Ad}\,AC(1,n)$ *if and only if they are conjugate under* $\mathrm{Ad}\,L$ *or if there exists an automorphism* $\psi\in\mathrm{Ad}\,L$ *such that* $\psi(L_1)=\Gamma_1 L_2\Gamma_1^{-1}$ *(see Table 1).*

6 Subalgebras of $AC(1,3)$

We recall that in this article the conformal algebra $AC(1,3)$ is realized as the pseudo-orthogonal algebra $AO(2,4)$. It turns out that it is convenient to divide the subalgebras of $AO(2,4)$ into seven classes:

(1) subalgebras not having invariant isotropic subspaces in $\mathbb{R}_{2,4}$;

(2) subalgebras conjugate to subalgebras of $AG_1(2)$;

(3) subalgebras conjugate to subalgebras of $AG_1(2)\uplus\langle J_{03}\rangle$ and having nonzero projection onto $\langle J_{03}\rangle$;

(4) subalgebras conjugate to subalgebras of $AP(1,3)$ but not conjugate to subalgebras of $AG_1(2)\uplus\langle J_{03}\rangle$;

(5) subalgebras conjugate to subalgebras of $AG_1(2)\uplus\langle J_{03},D\rangle$ but not conjugate to subalgebras of $AG_1(2)\uplus\langle J_{03}\rangle$;

(6) subalgebras conjugate to subalgebras of $AP(1,3)$ but not conjugate to subalgebras of $AG_1(2)\uplus\langle J_{03},D\rangle$;

(7) subalgebras conjugate to subalgebras of $AG_4(2)$ but not conjugate to subalgebras of $AP(1,3)$.

Since subalgebras conjugate under $\mathrm{Ad}\,AC(1,3)$ are identified, we omit mentioning conjugacy when referring to classes. So, for instance, we shall consider the second class as consisting of subalgebras of $AG_1(2)$. In order to have a better survey of subalgebras it is convenient to split the classes into subclasses corresponding to certain properties of the projections of the subalgebras of a class onto the homogeneous part of the algebra.

The division of the set of subalgebras of $AC(1,3)$ into the classes (1)–(7) allows us easily to construct the set of subalgebras of each of the algebras $AG_1(2)$, $AP(1,3)$, $A\tilde{P}(1,3)$, $AG_4(2)$. Up to conjugacy under $\mathrm{Ad}\,AC(1,3)$ we have

(a) the set of subalgebras of $AG_1(2)$ coincides with class (2);

(b) the set of subalgebras of $AP(1,3)$ is the union of classes (2), (3) and (4);

(c) the set of subalgebras of $A\tilde{P}(1,3)$ coincides with the union of classes (2)–(6);

(d) the set of subalgebras of $AG_4(2)$ is the union of classes (2), (3), (5), and (7).

We use the notation $F:U_1,\dots,U_m$ for $U_1\uplus F,\dots,U_m\uplus F$.

A. Subalgebras not possessing invariant isotropic subspaces in $\mathbb{R}_{2,4}$

This class is divided into subclasses by the existence for the subalgebras of invariant irreducible subspaces of a particular kind in the space $\mathbb{R}_{2,4}$.

1. Irreducible subalgebras of $AO(2,4)$

$AC(1,3)$;

$$ASU(1,2) = \langle P_0 + K_0 + 2J_{12}, P_0 + K_0 + K_3 - P_3, P_1 + K_1 + 2J_{02}, P_3 + K_3 + K_0 - P_0, K_2 - P_2 + 2J_{13}, P_2 + K_2 - 2J_{01}, D + J_{03}, K_1 - P_1 - 2J_{23}\rangle;$$

$$ASU'(1,2) = \langle P_0 + K_0 - 2J_{12}, P_0 + K_0 + K_3 - P_3, P_1 + K_1 - 2J_{02}, P_3 + K_3 + K_0 - P_0, K_2 - P_2 - 2J_{13}, P_2 + K_2 + 2J_{01}, D + J_{03}, K_1 - P_1 + 2J_{23}\rangle;$$

$ASU(1,2) \oplus \langle P_0 + K_0 - 2J_{12} - K_3 + P_3\rangle$;

$ASU'(1,2) \oplus \langle P_0 + K_0 + 2J_{12} - K_3 + P_3\rangle$;

$\langle P_0 + K_0 - 2J_{12} - K_3 + P_3\rangle \oplus \langle P_1 + K_1 + 2J_{02}, P_3 + K_3 + K_0 - P_0, K_2 - P_2 + 2J_{13}\rangle$;

$\langle P_0 + K_0 + 2J_{12} - K_3 + P_3\rangle \oplus \langle P_1 + K_1 - 2J_{02}, P_3 + K_3 + K_0 - P_0, K_2 - P_2 - 2J_{13}\rangle$.

2. Irreducible subalgebras $AO(1,4)$

$AC(3)$.

3. Irreducible subalgebras of $AO(2,3)$

$AC(1,2)$;

$\langle P_2 + K_2 + \sqrt{3}(P_1 + K_1) + K_0 - P_0, D + J_{02} - \sqrt{3}J_{01}, P_0 + K_0 - 2(K_2 - P_2)\rangle$;

$\langle P_2 + K_2 - \sqrt{3}(P_1 + K_1) + K_0 - P_0, D + J_{02} + \sqrt{3}J_{01}, P_0 + K_0 - 2(K_2 - P_2)\rangle$.

4. Subalgebras of $AO(2,2) \oplus AO(2)$ with irreducible projection onto $AO(2,2)$

$\langle J_{01} - D, K_0 - P_0 - P_1 - K_1, P_0 + K_0 - K_1 + P_1\rangle \oplus$
$\oplus \langle P_0 + K_0 + K_1 - P_1\rangle \oplus F$, where $F = 0$ or $F = \langle J_{23}\rangle$;

$\langle J_{01} + D, K_0 - P_0 + P_1 + K_1, P_0 + K_0 + K_1 - P_1\rangle \oplus$
$\oplus \langle P_0 + K_0 - K_1 + P_1\rangle \oplus F$, where $F = 0$ or $F = \langle J_{23}\rangle$;

$AC(1,1)$, $AC(1,1) \oplus \langle J_{23}\rangle$, where $AC(1,1) = \langle P_0, P_1, K_0, K_1, J_{01}, D\rangle$;

$\langle J_{01} - D, K_0 - P_0 - P_1 - K_1, P_0 + K_0 - K_1 + P_1\rangle \oplus$
$\oplus \langle P_0 + K_0 + K_1 - P_1 + \alpha J_{23}\rangle$ $(\alpha \neq 0)$;

$\langle J_{01} + D, K_0 - P_0 + P_1 + K_1, P_0 + K_0 + K_1 - P_1\rangle \oplus$
$\oplus \langle P_0 + K_0 - K_1 + P_1 + \alpha J_{23}\rangle$ $(\alpha \neq 0)$.

5. Subalgebras of the type $AO(2,1) \oplus F$ with $F \subset AO(3)$

$AC(1) \oplus L$, where $AC(1) = \langle D, P_0, K_0\rangle$,
and L is one of the algebras: 0, $\langle J_{12}\rangle$, $\langle J_{12}, J_{13}, J_{23}\rangle$.

6. Subalgebras of $AO(2) \oplus AO(4)$ having an irreducible projection

$\langle P_0 + K_0\rangle$; $\langle P_0 + K_0\rangle \oplus \langle 2J_{12} + \alpha(K_3 - P_3)\rangle$ $(|\alpha| \le 1)$;
$\langle P_0 + K_0\rangle \oplus \langle J_{12}, K_3 - P_3\rangle$; $\langle P_0 + K_0\rangle \oplus \langle J_{12} + J_{13}, J_{23}\rangle$;
$\langle P_0 + K_0\rangle \oplus \langle 2J_{12} + \varepsilon(K_3 - P_3), 2J_{13} - \varepsilon(K_2 - P_2),$
$2J_{23} + \varepsilon(K_1 - P_1)\rangle$ $(\varepsilon = \pm 1)$;
$\langle P_0 + K_0\rangle \oplus \langle 2J_{12} + \varepsilon(K_3 - P_3), 2J_{13} - \varepsilon(K_2 - P_2), 2J_{23} + \varepsilon(K_1 - P_1)\rangle \oplus$
$\oplus \langle 2J_{12} - \varepsilon(K_3 - P_3)\rangle$ $(\varepsilon = \pm 1)$;
$\langle P_0 + K_0\rangle \oplus \langle K_1 - P_1, K_2 - P_2, K_3 - P_3, J_{12}, J_{13}, J_{23}\rangle$;
$\langle P_0 + K_0 + 2\alpha J_{12}\rangle$ $(\alpha \ne 0,\ |\alpha| \ne 1)$;
$\langle P_0 + K_0 + 2\alpha J_{12} + \beta(K_3 - P_3)\rangle$ $(\alpha \ne 0,\ |\alpha| \ne 1,\ \beta \ge \alpha,\ \beta \ne 1)$;
$\langle 2J_{12} + \alpha(P_0 + K_0), K_3 - P_3 + \beta(P_0 + K_0)\rangle$
$(\alpha \ne 0,\ \beta \ge 0,$ with $|\alpha| \ne 1$ when $\beta = 0)$;
$\langle \alpha(P_0 + K_0) + 2\varepsilon J_{12} - K_3 + P_3\rangle \oplus \langle 2\varepsilon J_{12} + K_3 - P_3, 2\varepsilon J_{13} - K_2 + P_2,$
$2\varepsilon J_{23} + K_1 - P_1\rangle$ $(\alpha \ge 0)$;
$\langle 2\varepsilon J_{12} + K_3 - P_3, 2\varepsilon J_{13} - K_2 + P_2, 2\varepsilon J_{23} + K_1 - P_1\rangle$ $(\varepsilon = \pm 1)$;
$\langle 2\varepsilon J_{12} + K_3 - P_3, 2\varepsilon J_{13} - K_2 + P_2, 2\varepsilon J_{23} + K_1 - P_1\rangle \oplus$
$\oplus \langle 2\varepsilon J_{12} - K_3 + P_3\rangle$ $(\varepsilon = \pm 1)$;
$\langle K_1 - P_1, K_2 - P_2, K_3 - P_3, J_{12}, J_{13}, J_{23}\rangle$.

7. Subalgebras of $AO(1,2) \oplus AO(1,2)$

$\langle P_1 + K_1, P_2 + K_2, J_{12}\rangle \oplus \langle K_0 - P_0, K_3 - P_3, J_{03}\rangle$;
$\langle P_1 + K_1 + 2\varepsilon J_{03}, P_2 + K_2 + K_0 - P_0, 2\varepsilon J_{12} + K_3 - P_3\rangle$ $(\varepsilon = \pm 1)$;
$\langle P_1 + K_1, P_2 + K_2, J_{12}\rangle \oplus \langle K_3 - P_3\rangle$.

B. Subalgebras of $AG_1(2)$

The classical Galilei algebra $AG_1(2)$ is the semidirect sum of a solvable ideal, generated by $\langle P_1, P_2, M, T\rangle$, and the Euclidean algebra $AE(2) = \langle G_1, G_2, J_{12}\rangle$. The projection of $AG_1(2)$ onto $AO(1,3)$ coincides with $AE(2)$, which has, up to inner automorphisms, the subalgebras 0, $\langle J_{12}\rangle$, $\langle G_1\rangle$, $\langle G_1, G_2\rangle$, $\langle G_1, G_2, J_{12}\rangle$. The first two subalgebras are completely reducible algebras of linear transformations of Minkowski space $\mathbb{R}_{1,3}$, whereas the others are not of this type. Thus we divide this class into two subclasses A and B.

1. Subalgebras with completely reducible projection onto $AO(1,3)$

0, $\langle P_0\rangle$, $\langle P_1\rangle$, $\langle M\rangle$, $\langle P_0, P_3\rangle$, $\langle M, P_1\rangle$, $\langle P_1, P_2\rangle$, $\langle M, P_1, P_2\rangle$, $\langle P_0, P_1, P_2\rangle$,
$\langle P_1, P_2, P_3\rangle$, $\langle P_0, P_1, P_2, P_3\rangle$;
$\langle J_{12}\rangle :$ 0, $\langle P_0\rangle$, $\langle P_3\rangle$, $\langle M\rangle$, $\langle P_0, P_3\rangle$, $\langle P_1, P_2\rangle$, $\langle P_0, P_1, P_2\rangle$, $\langle M, P_1, P_2\rangle$,
$\langle P_1, P_2, P_3\rangle$, $\langle P_0, P_1, P_2, P_3\rangle$;
$\langle J_{12} + P_0\rangle :$ 0, $\langle P_3\rangle$, $\langle P_1, P_2\rangle$, $\langle P_1, P_2, P_3\rangle$;
$\langle J_{12} \pm P_3\rangle :$ 0, $\langle P_0\rangle$, $\langle P_1, P_2\rangle$, $\langle P_0, P_1, P_2\rangle$;
$\langle J_{12} \pm 2T\rangle :$ 0, $\langle M\rangle$, $\langle P_1, P_2\rangle$, $\langle M, P_1, P_2\rangle$.

2. Subalgebras whose projection onto $AO(1,3)$ is not completely reducible

$\langle G_1\rangle:\ \langle P_2\rangle,\ \langle M,P_1\rangle,\ \langle M,P_2\rangle,\ \langle M,P_1+\alpha P2\rangle,\ \langle M,P_1,P_2\rangle,$
$\quad \langle P_0,P_1,P_3\rangle,\ \langle P_0,P_1,P_2,P_3\rangle\ (\alpha\neq 0);$
$\langle G_1\pm P_2\rangle:\ 0,\ \langle M\rangle,\ \langle M,P_1\rangle,\ \langle P_0,P_1,P_3\rangle;$
$\langle G_1+2T\rangle:\ 0,\ \langle P_2\rangle,\ \langle M\rangle,\ \langle M,P_1\rangle,\ \langle M,P_2\rangle,\ \langle M,P_1+\alpha P_2\rangle,$
$\quad \langle M,P_1,P_2\rangle\ (\alpha\neq 0);$
$\langle G_1,G_2\rangle:\ \langle M,P_1,P_2\rangle,\ \langle P_0,P_1,P_2,P_3\rangle;$
$\langle G_1+\varepsilon P_2,G_2-\varepsilon P_1,M\rangle,\ \langle G_1+\varepsilon P_2,G_2-\varepsilon P_1+\alpha P_2,M\rangle\ (\varepsilon=\pm 1,\ \alpha\neq 0);$
$\langle G_1+\alpha P_2,G_2+2T,M,P_1\rangle\ (\alpha\in\mathbb{R});$
$\langle G_1\pm P_2,G_2,M,P_1\rangle,\ \langle G_1,G_2+2T,M,P_1,P_2\rangle;$
$\langle G_1,G_2,J_{12}\rangle:\ \langle M,P_1,P_2\rangle,\ \langle P_0,P_1,P_2,P_3\rangle;$
$\langle G_1,G_2,J_{12}\pm 2T,M,P_1,P_2\rangle,\ \langle G_1+\varepsilon P_2,G_2-\varepsilon P_1,J_{12},M\rangle\ (\varepsilon=\pm 1).$

C. Subalgebras of $AG_1(2)\uplus\langle J_{03}\rangle$ with nonzero projection onto $\langle J_{03}\rangle$

We divide also this class into two subclasses which are distinguished by whether or not they have a completely reducible projection onto $AO(1,3)$.

1. Subalgebras with completely reducible projection onto $AO(1,3)$

$\langle J_{03}\rangle:\ 0,\ \langle P_1\rangle,\ \langle M\rangle,\ \langle P_0,P_3\rangle,\ \langle M,P_1\rangle,\ \langle P_1,P_2\rangle,\ \langle P_0,P_1,P_3\rangle,\ \langle M,P_1,P_2\rangle,$
$\quad \langle P_0,P_1,P_2,P_3\rangle;$
$\langle J_{03}+P_1\rangle:\ 0,\ \langle P_2\rangle,\ \langle M\rangle,\ \langle P_0,P_3\rangle,\ \langle M,P_2\rangle,\ \langle P_1,P_2,P_3\rangle;$
$\langle J_{12}+\alpha J_{03}\rangle:\ 0,\ \langle M\rangle,\ \langle P_0,P_3\rangle,\ \langle P_1,P_2\rangle,\ \langle M,P_1,P_2\rangle,$
$\quad \langle P_0,P_1,P_2,P_3\rangle,\ (\alpha\neq 0);$
$\langle J_{12},J_{03}\rangle:\ 0,\ \langle M\rangle,\ \langle P_0,P_3\rangle,\ \langle P_1,P_2\rangle,\ \langle M,P_1,P_2\rangle,\ \langle P_0,P_1,P_2,P_3\rangle.$

2. Subalgebras with projections onto $AO(1,3)$ which are not completely reducible

$\langle G_1,J_{03}\rangle:\ 0,\ \langle M\rangle,\ \langle P_2\rangle,\ \langle M,P_1\rangle,\ \langle M,P_2\rangle,\ \langle M,P_1+\alpha P_2\rangle,\ \langle M,P_1,P_2\rangle,$
$\quad \langle P_0,P_1,P_3\rangle,\ \langle P_0,P_1,P_2,P_3\rangle\ (\alpha\neq 0);$
$\langle G_1,J_{03}+P_2\rangle:\ 0,\ \langle M\rangle,\ \langle M,P_1\rangle,\ \langle M,P_1+\alpha P_2\rangle,\ \langle P_0,P_1,P_3\rangle,\ (\alpha\neq 0);$
$\langle G_1,J_{03}+P_1\rangle:\ \langle M\rangle,\ \langle M,P_2\rangle;$
$\langle G_1,J_{03}+P_1+\alpha P_2,M\rangle\ (\alpha\neq 0);$
$\langle G_1,G_2,J_{03}\rangle:\ 0,\ \langle M\rangle,\ \langle M,P_1\rangle,\ \langle M,P_1,P_2\rangle,\ \langle P_0,P_1,P_2,P_3\rangle;$
$\langle G_1,G_2,J_{03}+P_1,M\rangle,\quad \langle G_1,G_2,J_{03}+P_2,M,P_1\rangle;$
$\langle G_1,G_2,J_{12}+\alpha J_{03}\rangle:\ 0,\ \langle M\rangle,\ \langle M,P_1,P_2\rangle,\ \langle P_0,P_1,P_2,P_3\rangle\ (\alpha\neq 0);$
$\langle G_1,G_2,J_{12},J_{03}\rangle:\ 0,\ \langle M\rangle,\ \langle M,P_1,P_2\rangle,\ \langle P_0,P_1,P_2,P_3\rangle.$

D. Subalgebras of $AP(1,3)$ which are not conjugate to subalgebras of $AG_1(2) \uplus \langle J_{03}\rangle$

This class consists of those subalgebras of the Poincaré algebra $AP(1,3)$ whose projection onto $AO(1,3)$ do not possess isotropic invariant subspaces in $\mathbb{R}_{1,3}$. Since the projections are simple algebras, then each subalgebra of the fourth class splits. The full list of such algebras is

$AO(1,2):\ 0,\ \langle P_3\rangle,\ \langle P_0,P_1,P_2\rangle,\ \langle P_0,P_1,P_2,P_3\rangle;$

$AO(3):\ 0,\ \langle P_0\rangle,\ \langle P_1,P_2,P_3\rangle,\ \langle P_0,P_1,P_2,P_3\rangle;$

$AO(1,3):\ 0,\ \langle P_0,P_1,P_2,P_3\rangle.$

E. Subalgebras of $AG_1(2) \uplus \langle J_{03}, D\rangle$ which are not conjugate to subalgebras of $AG_1(2) \uplus \langle J_{03}\rangle$

Let K be a subalgebra of $AG_1(2) \uplus \langle J_{03}, D\rangle$ with nonzero projection onto $\langle D\rangle$, and let $\hat\theta$ be the projection of K onto $\langle J_{03}, D\rangle$. By Propositions IV.2.3 and IV.2.5 in Ref. [9], the algebra K, as a subalgebra of $A\tilde{P}(1,3)$, is split whenever $\hat\theta(K)$ is one of the subalgebras 1) $\langle D\rangle$; 2) $\langle \gamma D - J_{03}\rangle$ ($\gamma \neq \pm1, 0, 2$); 3) $\langle D, J_{03}\rangle$. This leads us to dividing this class of subalgebras into two subclasses of nonsplittable subalgebras K of $A\tilde{P}(1,3)$, denoted by D and E, for which the projection onto $\langle G_1, G_2\rangle$ is non-zero, and for which $\hat\theta(K)$ is $\langle J_{03} \pm D\rangle$ and $\langle J_{03} - 2D\rangle$ respectively. It is also useful to distinguish the subclass A of subalgebras having zero projection onto $\langle G_1, G_2\rangle$. The subalgebras in this subclass differ from the other subalgebras in that their projections onto $AO(1,3)$ are completely reducible algebras of linear transformations of Minkowski space $\mathbb{R}_{1,3}$. All the other subalgebras are split, and we divide them formally into subclasses B and C, depending on the dimension of their projection onto $\langle D, J_{03}\rangle$.

1. Subalgebras with zero projection on $\langle G_1, G_2\rangle$

$\langle D\rangle:\ \langle P_0\rangle,\ \langle P_0,P_3\rangle,\ \langle P_0,P_1,P_2\rangle,\ \langle P_1,P_2,P_3\rangle,\ \langle P_0,P_1,P_2,P_3\rangle;$

$\langle J_{12}+\alpha D\rangle:\ \langle P_0\rangle,\ \langle P_3\rangle:\ \langle P_0,P_3\rangle,\ \langle P_0,P_1,P_2\rangle,\ \langle P_1,P_2,P_3\rangle,$
$\quad \langle P_0,P_1,P_2,P_3\rangle\ (\alpha>0);$

$\langle J_{12}, D\rangle:\ \langle P_0\rangle,\ \langle P_3\rangle:\ \langle P_0,P_3\rangle,\ \langle P_0,P_1,P_2\rangle,\ \langle P_1,P_2,P_3\rangle,\ \langle P_0,P_1,P_2,P_3\rangle;$

$\langle J_{03}+\alpha D\rangle\ (0<\alpha\le 1);$

$\langle J_{03}+\alpha D, M\rangle\ (0<|\alpha|\le 1);$

$\langle J_{03}+\alpha D\rangle:\ \langle P_1\rangle,\ \langle P_0,P_3\rangle,\ \langle P_1,P_2\rangle,\ \langle P_0,P_1,P_3\rangle,\ \langle P_0,P_1,P_2,P_3\rangle\ (\alpha>0);$

$\langle J_{03}+\alpha D\rangle:\ \langle M,P_1\rangle,\ \langle M,P_1,P_2\rangle,\ (\alpha\neq 0);$

$\langle J_{03}-D\pm 2T\rangle:\ 0,\ \langle P_1\rangle,\ \langle M\rangle,\ \langle P_1,P_2\rangle,\ \langle M,P_1\rangle,\ \langle M,P_1,P_2\rangle;$

$\langle J_{03}, D\rangle:\ 0,\ \langle P_1\rangle,\ \langle M\rangle,\ \langle P_0,P_3\rangle,\ \langle P_1,P_2\rangle,\ \langle M,P_1\rangle,\ \langle M,P_1,P_2\rangle,$
$\quad \langle P_0,P_1,P_3\rangle,\ \langle P_0,P_1,P_2,P_3\rangle;$

$\langle \varepsilon J_{12}+\alpha J_{03}+\beta D\rangle\ (0<\alpha\le\beta,\ \varepsilon=\pm1);$

$\langle J_{12}+\alpha J_{03}+\beta D, M\rangle\ (0<|\alpha|\le|\beta|);$

$\langle \varepsilon J_{12}+\alpha J_{03}+\beta D\rangle:\ \langle P_0,P_3\rangle,\ \langle P_1,P_2\rangle,\ \langle P_0,P_1,P_2,P_3\rangle\ (\varepsilon=\pm1,\ \alpha,\beta>0);$

$\langle J_{12}+\alpha J_{03}+\beta D, M, P_1, P_2\rangle\ (\alpha\neq 0,\ \beta\neq 0);$

$\langle J_{12}+\alpha(J_{03}-D\pm 2T)\rangle:\ 0,\ \langle M\rangle, \langle P_1,P_2\rangle,\ \langle M,P_1,P_2\rangle\ (\alpha\neq 0);$

$\langle J_{12} + \alpha J_{03}, D\rangle :\ 0,\ \langle M\rangle,\ \langle P_1, P_2\rangle,\ \langle P_0, P_3\rangle,\ \langle M, P_1, P_2\rangle,$
$\langle P_0, P_1, P_2, P_3\rangle\ (\alpha \neq 0);$
$\langle J_{03} + \alpha D, J_{12} + \beta D\rangle :\ \langle P_0, P_3\rangle,\ \langle P_1, P_2\rangle,\ \langle M, P_1, P_2\rangle,$
$\langle P_0, P_1, P_2, P_3\rangle\ (\alpha^2 + \beta^2 \neq 0);$
$\langle J_{03} + \alpha D, J_{12} + \beta D\rangle :\ (|\alpha| \leq 1,\ \beta \geq 0,\ |\alpha| + \beta \neq 0);$
$\langle J_{03} + \alpha D, J_{12} + \beta D, M\rangle :\ (|\alpha| \leq 1,\ \beta \geq 0,\ |\alpha| + \beta \neq 0);$
$\langle J_{03} + \alpha D, J_{12} + \beta D, M, P_1, P_2\rangle :\ (\alpha, \beta \in \mathbb{R},\ \alpha^2 + \beta^2 \neq 0);$
$\langle J_{03} - D \pm 2T, J_{12} + 2\alpha T\rangle :\ 0,\ \langle M\rangle,\ \langle P_1, P_2\rangle,\ \langle M, P_1, P_2\rangle;$
$\langle J_{03} - D, J_{12} \pm T\rangle :\ 0,\ \langle M\rangle,\ \langle P_1, P_2\rangle,\ \langle M, P_1, P_2\rangle;$
$\langle J_{03}, J_{12}, D\rangle :\ 0,\ \langle M\rangle,\ \langle P_0, P_3\rangle,\ \langle P_1, P_2\rangle,\ \langle M, P_1, P_2\rangle,\ \langle P_0, P_1, P_2, P_3\rangle.$

2. Subalgebras with two-dimensional projection onto $\langle J_{03}, D\rangle$ and nonzero projection onto $\langle G_1, G_2\rangle$

$\langle G_1, J_{03}, D\rangle :\ \langle P_2\rangle,\ \langle M, P_1\rangle,\ \langle M, P_2\rangle,\ \langle M, P_1 + \alpha P_2\rangle,\ \langle M, P_1, P_2\rangle,$
$\langle P_0, P_1, P_3\rangle,\ \langle P_0, P_1, P_2, P_3\rangle;$
$\langle G_1, G_2, J_{03}, D\rangle :\ \langle M, P_1, P_2\rangle,\ \langle P_0, P_1, P_2, P_3\rangle;$
$\langle G_1, G_2, J_{12} + \alpha J_{03}, D\rangle :\ \langle M, P_1, P_2\rangle,\ \langle P_0, P_1, P_2, P_3\rangle\ (\alpha \neq 0);$
$\langle G_1, G_2, J_{03} + \alpha D, J_{12} + \beta D, P_1, P_2\rangle\ (|\alpha| \leq 1,\ \beta \geq 0,\ |\alpha| + \beta \neq 0);$
$\langle G_1, G_2, J_{03} + \alpha D, J_{12} + \beta D, P_0, P_1, P_2, P_3\rangle\ (\alpha^2 + \beta^2 \neq 0);$
$\langle G_1, G_2, J_{03}, J_{12}, D\rangle :\ \langle M, P_1, P_2\rangle,\ \langle P_0, P_1, P_2, P_3\rangle.$

3. Split subalgebras with one-dimensional projection onto $\langle J_{03}, D\rangle$ and nonzero projection onto $\langle G_1, G_2\rangle$

$\langle G_1 + D\rangle :\ \langle P_0, P_1, P_3\rangle,\ \langle P_0, P_1, P_2, P_3\rangle;$
$\langle G_1, D\rangle :\ \langle P_0, P_1, P_3\rangle,\ \langle P_0, P_1, P_2, P_3\rangle;$
$\langle G_1 + D, G_2, P_0, P_1, P_2, P_3\rangle,\ \langle G_1, G_2, D, P_0, P_1, P_2, P_3\rangle;$
$\langle G_1, J_{03} + \alpha D\rangle :\ \langle P_2\rangle,\ \langle M, P_1\rangle,\ \langle M, P_2\rangle,\ \langle M, P_1 + \beta P_2\rangle$
$(|\alpha| \leq 1,\ \alpha \neq 0,\ \beta \neq 0);$
$\langle G_1, J_{03} + \alpha D\rangle :\ \langle M, P_1, P_2\rangle,\ \langle P_0, P_1, P_3\rangle,\ \langle P_0, P_1, P_2, P_3\rangle\ (\alpha \neq 0);$
$\langle G_1, G_2, J_{03} + \alpha D, M, P_1, P_2\rangle\ (0 < |\alpha| \leq 1);$
$\langle G_1, G_2, J_{03} + \alpha D, P_0, P_1, P_2, P_3\rangle\ (\alpha \neq 0);$
$\langle G_1, G_2, J_{12} + \alpha D, P_0, P_1, P_2, P_3\rangle\ (\alpha \neq 0);$
$\langle G_1, G_2, J_{12}, D, P_0, P_1, P_2, P_3\rangle;$
$\langle G_1, G_2, J_{12} + \alpha J_{03} + \beta D, M, P_1, P_2\rangle\ (0 < |\alpha| \leq |\beta|);$
$\langle G_1, G_2, J_{12} + \alpha J_{03} + \beta D, P_0, P_1, P_2, P_3\rangle\ (\beta \neq 0).$

4. Nonsplit subalgebras of $AG_1(2) \uplus \langle J_{03} \mp D\rangle$ with nonzero projection onto $\langle G_1, G_2\rangle$ and $\langle J_{03} \mp D\rangle$

$\langle J_{03} - D, G_1 \pm P_2\rangle :\ 0,\ \langle M\rangle,\ \langle M, P_1\rangle,\ \langle P_0, P_1, P_3\rangle;$
$\langle J_{03} - D \pm 2T, G_1 + \alpha P_2, M, P_1\rangle;$
$\langle J_{03} - D \pm 2T, G_1, M, P_1, P_2\rangle,\ \langle J_{03} - D + M, G_1, P_2\rangle;$

$\langle J_{03} - D, G_1 + \varepsilon P_2, G_2 - \varepsilon P_1 + \alpha P_2, M\rangle$ ($\varepsilon = \pm 1$, $\alpha \in \mathbb{R}$);
$\langle J_{03} - D, G_1 \pm P_2, G_2, M, P_1\rangle$, $\langle J_{03} - D \pm 2T, G_1, G_2, P_1, P_2, M\rangle$;
$\langle J_{12} + \alpha(J_{03} - D), G_1 + \varepsilon P_2, G_2 - \varepsilon P_1, M\rangle$ ($\varepsilon = \pm 1$, $\alpha \neq 0$);
$\langle J_{12} + \alpha(J_{03} - D \pm 2T), G_1, G_2, M, P_1, P_2\rangle$ ($\alpha \neq 0$);
$\langle J_{12} \pm 2T, J_{03} - D, G_1, G_2, M, P_1, P_2\rangle$;
$\langle J_{12} + 2\alpha T, J_{03} - D \pm 2T, G_1, G_2, M, P_1, P_2\rangle$ ($\alpha \in \mathbb{R}$);
$\langle J_{12}, J_{03} - D, G_1 + \varepsilon P_2, G_2 - \varepsilon P_1, M\rangle$ ($\varepsilon = \pm 1$).

5. Nonsplit subalgebras of $AG_1(2) \uplus \langle J_{03} - 2D\rangle$ with nonzero projection onto $\langle G_1, G_2\rangle$ and $\langle J_{03} - 2D\rangle$

$\langle J_{03} - 2D, G_1 + 2T\rangle :$ 0, $\langle M\rangle$, $\langle P_2\rangle$, $\langle M, P_1\rangle$, $\langle M, P_2\rangle$, $\langle M, P_1 + \alpha P_2\rangle$, $\langle M, P_1, P_2\rangle$ ($\alpha \neq 0$);
$\langle J_{03} - 2D, G_1, G_2 + 2T\rangle :$ $\langle M, P_1\rangle$, $\langle M, P_1, P_2\rangle$.

F. Subalgebras of $A\tilde{P}(1,3)$ not conjugate to subalgebras of $AP(1,3)$ and of $AG_1(2) \uplus \langle J_{03}, D\rangle$

This class consists of those subalgebras of $AP(1,3)$ whose projection onto $AO(1,3)$ do not have invariant isotropic subspaces in $\mathbb{R}_{1,3}$ and with a nonzero projection onto $\langle D\rangle$. We have

$AO(1,2) \oplus \langle D\rangle :$ 0, $\langle P_3\rangle$, $\langle P_0, P_1, P_2\rangle$, $\langle P_0, P_1, P_2, P_3\rangle$;
$AO(3) \oplus \langle D\rangle :$ 0, $\langle P_0\rangle$, $\langle P_1, P_2, P_3\rangle$, $\langle P_0, P_1, P_2, P_3\rangle$;
$AO(1,3) \oplus \langle D\rangle :$ 0, $\langle P_0, P_1, P_2, P_3\rangle$.

G. Subalgebras of $AG_4(2)$ which are not conjugate to subalgebras of $A\tilde{P}(1,3)$

Let K be a subalgebra of $AG_4(2)$ and $\tau(K)$ its projection onto $AGL(2,\mathbb{R})$. By Propositions V.2.1 and V.2.2 of Ref. [9], the algebra K belongs to this class if and only if $\tau(K)$ is conjugate to one of the following algebras: $\langle S + T\rangle$, $\langle S + T\rangle + \langle Z\rangle$ (subdirect sum), $ASL(2,\mathbb{R}) = \langle R, S, T\rangle$, $AGL(2,\mathbb{R}) = \langle R, S, T, Z\rangle$. Because of this, we divide this seventh class into three subclasses, each of which consists of subalgebras having a corresponding projection onto $AGL(2,\mathbb{R})$; those sub-algebras whose projections are either $ASL(2,\mathbb{R})$ or $AGL(2,\mathbb{R})$ are put into the same subclass.

1. Subalgebras whose projection onto $AGL(2,\mathbb{R})$ is $\langle S + T\rangle$

$\langle S + T\rangle :$ 0, $\langle M\rangle$, $\langle G_1, P_1, M\rangle$, $\langle G_1 - \alpha^{-1}P_2, G_2 + \alpha P_1, M\rangle$, $\langle G_1, G_2, P_1, P_2, M\rangle$ ($0 < |\alpha| \leq 1$);
$\langle S + T \pm M\rangle$, $\langle S + T + \alpha J_{12} \pm M\rangle$ ($\alpha \neq 0$);
$\langle S + T + \alpha J_{12}\rangle :$ 0, $\langle M\rangle$, $\langle G_1 + \varepsilon P_2, G_2 - \varepsilon P_1, M\rangle$, $\langle G_1, G_2, P_1, P_2, M\rangle$ ($\varepsilon = \pm 1, \alpha \neq 0$);
$\langle S + T + \varepsilon J_{12}\rangle :$ $\langle G_1 + \varepsilon P_2\rangle$, $\langle G_1 + \varepsilon P_2, M\rangle$, $\langle G_1 + \varepsilon P_2, G_1 - \varepsilon P_2, G_2 + \varepsilon P_1, M\rangle$ ($\varepsilon = \pm 1$);
$\langle S + T + \varepsilon J_{12} \pm M, G_1 + \varepsilon P_2\rangle$ ($\varepsilon = \pm 1$);

$\langle S+T+\varepsilon J_{12}+\varepsilon G_1+P_2\rangle:\ 0,\ \langle M\rangle,\ \langle G_2-\varepsilon P_1, M\rangle,$
$\quad\langle G_1-\varepsilon P_2, G_2+\varepsilon P_1, M\rangle,\ \langle G_2-\varepsilon P_1, G_1-\varepsilon P_2, G_2+\varepsilon P_1, M\rangle\ (\varepsilon=\pm 1);$
$\langle J_{12}, S+T\rangle:\ 0,\ \langle M\rangle,\ \langle G_1+\varepsilon P_2, G_2-\varepsilon P_1, M\rangle,$
$\quad\langle G_1, G_2, P_1, P_2, M\rangle\ (\varepsilon=\pm 1);$
$\langle J_{12}\pm M, S+T+\alpha M\rangle\ (\alpha\in\mathbb{R});$
$\langle J_{12}, S+T\pm M\rangle.$

2. Subalgebras whose projection onto $AGL(2,\mathbb{R})$ is the subdirect sum $\langle S+T\rangle+\langle Z\rangle$

$\langle S+T+\alpha Z\rangle:\ 0,\ \langle M\rangle,\ \langle G_1, P_1, M\rangle,\ \langle G_1-\beta^{-1}P_2, G_2+\beta P_1, M\rangle,$
$\quad\langle G_1, G_2, P_1, P_2, M\rangle\ (0<|\beta|\le 1,\ \alpha\ne 0);$
$\langle S+T, Z\rangle:\ 0,\ \langle M\rangle,\ \langle G_1, P_1, M\rangle,\ \langle G_1-\alpha^{-1}P_2, G_2+\alpha P_1, M\rangle,$
$\quad\langle G_1, G_2, P_1, P_2, M\rangle\ (0<|\alpha|\le 1);$
$\langle S+T+\alpha J_{12}+\beta Z\rangle:\ 0,\ \langle M\rangle,\ \langle G_1+\varepsilon P_2, G_2-\varepsilon P_1, M\rangle,$
$\quad\langle G_1, G_2, P_1, P_2, M\rangle\ (\varepsilon=\pm 1, \alpha \not- 0,\ \beta>0);$
$\langle S+T+\alpha J_{12}, Z\rangle:\ 0,\ \langle M\rangle,\ \langle G_1+\varepsilon P_2, G_2-\varepsilon P_1, M\rangle,$
$\quad\langle G_1, G_2, P_1, P_2, M\rangle\ (\varepsilon=\pm 1,\ \alpha\ne 0);$
$\langle S+T+\varepsilon J_{12}+\alpha Z\rangle:\ \langle G_1+\varepsilon P_2\rangle,\ \langle G_1+\varepsilon P_2, M\rangle,$
$\quad\langle G_1+\varepsilon P_2, G_1-\varepsilon P_2, G_2+\varepsilon P_1, M\rangle\ (\varepsilon=\pm 1,\ \alpha\ne 0);$
$\langle S+T+\varepsilon J_{12}, Z\rangle:\ \langle G_1+\varepsilon P_2\rangle,\ \langle G_1+\varepsilon P_2, M\rangle,$
$\quad\langle G_1+\varepsilon P_2, G_1-\varepsilon P_2, G_2+\varepsilon P_1, M\rangle\ (\varepsilon=\pm 1);$
$\langle J_{12}+\alpha Z, S+T+\beta Z\rangle:\ 0,\ \langle M\rangle,\ \langle G_1+\varepsilon P_2, G_2-\varepsilon P_1, M\rangle,$
$\quad\langle G_1, G_2, P_1, P_2, M\rangle\ (\varepsilon=\pm 1,\ |\alpha|+|\beta|\ne 0);$
$\langle J_{12}, S+T, Z\rangle:\ 0,\ \langle M\rangle,\ \langle G_1+\varepsilon P_2, G_2-\varepsilon P_1, M\rangle,$
$\quad\langle G_1, G_2, P_1, P_2, M\rangle\ (\varepsilon=\pm 1).$

3. Subalgebras whose projection onto $AGL(2,\mathbb{R})$ contains $ASL(2,\mathbb{R})$

$\langle R, S, T\rangle:\ 0,\ \langle M\rangle,\ \langle G_1, P_1, M\rangle,\ \langle G_1, G_2, P_1, P_2, M\rangle;$
$\langle J_{12}\rangle\oplus\langle R, S, T\rangle:\ 0,\ \langle M\rangle,\ \langle G_1, G_2, P_1, P_2, M\rangle;$
$\langle J_{12}\pm M\rangle\oplus\langle R, S, T\rangle;$
$\langle R, S, T, Z\rangle:\ 0,\ \langle M\rangle,\ \langle G_1, P_1, M\rangle,\ \langle G_1, G_2, P_1, P_2, M\rangle;$
$\langle R, S, T\rangle\oplus\langle J_{12}+\alpha Z\rangle:\ 0,\ \langle M\rangle,\ \langle G_1, G_2, P_1, P_2, M\rangle\ (\alpha\ne 0);$
$\langle R, S, T\rangle\oplus\langle J_{12}, Z\rangle:\ 0,\ \langle M\rangle,\ \langle G_1, G_2, P_1, P_2, M\rangle.$

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Professor Wilhelm Fuschych died on April 7, 1997, after a short illness. This is a great loss for his family, his many students, and for the scientific community. His many and deep contributions to the field of symmetry analysis of differential equations have made the Kyiv school of symmetries known throughout the world. We take this opportunity to express our deep sense of loss as well as our gratitude for all the encouragement in research that Wilhelm Fushchych gave during the years we knew him.

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