Symmetry of equations of nonlinear quantum mechanics

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The paper is devoted to description of nonlocal symmetries of linear and nonlinear equations of quantum mechanics and to symmetry classification of nonlinear multi-dimensional equations, compatible with Galilei relativity principle.t

The plan of the talk

- Discovery of the Schrödinger equation
- Derivation (uniqueness) of the Schrödinger equation
- High order equation of the Schrödinger type
- Nonlocal symmetry of the Schrödinger equation [2, 7]
- High-order evolution equations. Dependence of mass on velocity in the nonlocal Galilei-invariant theory [6, 19]
- Galilei relativity principle and nonlinear Schrödinger-type equations [2, 5, 10–17]
- Nonlocal symmetry of the linear Schrödinger–de Broglie–Klein–Gordon–Fock–Kudar–de Donder–Van Dunger [3, 18, 8]
- Nonlocal Galilei symmetry of a relativistic equation [8, 9]
- Nonlocal Galilei symmetry of the Dirac equation [7]
- Galilei symmetry of a relativistic equation.

1 Brief comment on discovery of the Schrödinger equation of motion in quantum mechanics

First I would like to remind that 70 years ago Erwin Schrödinger discovered motion equations and thus created the mathematical foundation for the quantum mechanics. On 21 June, 1926 E. Schrödinger submitted the paper “Quantisierung als Eigenwertprobleme” to the journal “Annalen der Physik” (1926, Vol. 81, 109–139, [1]) where he suggested the equation

$$\begin{aligned} &S\Psi = 0, \quad S = p_0 - \frac{p_a^2}{2m} - V(t,x), \\ &p_0 = i\hbar\frac{\partial}{\partial t}, \quad p_a = -i\hbar\frac{\partial}{\partial x_a}, \quad a = 1,2,3, \end{aligned} \tag{1}$$

where $\Psi = \Psi(x_0 = t, \vec{x})$ is a complex-valued wave function, V is a potential.

Proceedings on the XXI International Colloquium on Group Theoretical Methods in Physics, Group21 “Physical Applications and Mathematical Aspects of Geometry, Groups, and Algebras” (July 15–20, 1996, Goslar, Germany), Editors: H.-H. Doebner, W. Scherer, P. Natterman, Singapore, Word Scientific, 1997, V.1, P. 439–446.

This paper was the last of the series of four papers with the same title where the quantization problem in the atom physics was solved.

Can we say that E. Schrödinger had derived his equation?

Acquaintance with the original paper by E. Schrödinger gives us an ultimate answer to this question. E. Schrödinger had not derived this equation. The equation (1) was written without accurate substantiation. Moreover E. Schrödinger believed that the correct motion equations in the quantum mechanics should be fourth-order equations for the real function, and not the equation (1) for the complex function. E. Schrödinger considered the equation (1) as some auxiliary (interim) equation which enables to simplify calculations.

His previous papers were based on the equations

$$\Delta\Psi - \frac{2(E-V)}{E^2}\frac{\partial^2\Psi}{\partial t^2} = 0, \tag{2}$$

$$\Delta\Psi + \frac{8\pi^2}{\hbar^2}(E-V)\Psi = 0, \tag{3}$$

where Ψ is a real function, E is energy.

When the potential V does not depend on time, Schrödinger derives from (2), (3) the fourth-order wave equation

$$\left(\Delta - \frac{8\pi^2}{\hbar^2}V\right)^2\Psi + \frac{16\pi^2}{\hbar^2}\frac{\partial^2\Psi}{\partial t^2} = 0, \tag{4}$$

where Ψ is a real function.

Schrödinger write about the equation (4): "... the equation (4) is the unique and general wave equation for the field scalar Ψ .... the wave equation (4) contains the dispersion law and can serve as a foundation for the theory of conservative system which I had developed. Its generalization for the case of time-dependent potential demands some caution ... an attempt to generalize the equation (4) for non-conservative systems encounters the difficulty arising because of the term $\frac{\partial V}{\partial t}$. Therefore in the following I will go the other way which is simpler from the point of view of calculations. I consider this way to be the most correct in principle."

Further Schrödinger writes down the equation (1) for the complex function Ψ. Just in this place of the paper [1] Schrödinger makes a step of genius (and non-logical), writing the equation (1) for a complex function.

As to the equation (1) Schrödinger writes: "There is certainly some difficulty in application of complex wave functions. If they are necessary in principle, and not only as a way to simplify calculations then it means that in principle two functions exist which only together can give the description of the state of the system ... The fact that in the pair of equations (1) we have only a substitute, which is extremal convenient at least for calculations. The real wave equation most certainly must be a fourth-order equation. Though I have not succeeded to find such equation for a non-conservative system $\left(\frac{\partial V}{\partial t} \neq 0\right)$."

We can make following conclusions from the above:

Conclusion 1. *In 1926 Schrödinger thought that the correct equation in quantum mechanics has to be a fourth-order equation. For the case when the potential does not depend on time this equation has the form (4).*

Conclusion 2. *In June, 1926 Schrödinger considered that the equation (1) which is first order in time and second order in space variables for the complex function is interim (not principal), which is to be used only to simplify calculations.*

Conclusion 3. *Schrödinger considered that in the case when the potential V depends on time, the motion equation has to be also of the fourth order for the given function. He could not derive such equation.*

Now we can undoubtedly say that E. Schrödinger did a mistake in respect of importance (fundamental role) of the equations (1), (4). Really, the equation (1) is a principal equation of the quantum mechanics, and the equation (4) cannot be a motion equation as it is not compatible with the Galilei relativity principle.

This statement follows from the symmetry analysis of the equations (1) and (4) [2, 4]:

the equation (1) is invariant with respect to the Galilei group;

the equation (4) is not invariant with respect to the Galilei group.

With respect to the above we shall answer the following questions below:

1. Which linear equations of second, fourth, n-th order are compatible with the Galilei relativity principle?

2. Does linear equations which are first-order in time variable, fourth order in space variables and are compatible with the Galilei relativity principle exist?

Theorem 1 [4] [Fushchych, 1987]. *The Euclid algebra $AE_1(1,3)$ is the maximal invariance algebra of the equation (4) $(V=0)$.*

We have the following corollaries of the adduced theorems.

Corollary 1. *The equation (4) is not compatible with the Galilei relativity principle. This means that (4) cannot be considered as an equation of particle motion in quantum mechanics.*

2 Derivation of the Schrödinger equation and higher order equations

Let us derive Schrödinger equation out of the requirement of invariance of an equation with respect to the Galilei transformations and to the group of space and time translations.

In [6] is proposed the following generalisation $(V=0)$ of the Schrödinger equation (1)

$$\begin{gathered}\left(\lambda_1 S+\lambda_2 S^2+\cdots+\lambda_n S^n\right)\Psi=\lambda\Psi,\\ S^2=\left(p_0-\frac{p_a^2}{2m}\right)^2,\ldots,S^n=\left(p_0-\frac{p_a^2}{2m}\right)^n,\end{gathered}\tag{5}$$

where $\lambda,\lambda_1,\lambda_2,\ldots,\lambda_n$ are arbitrary parameters.

The equation (5) is compatible with the Galilei relativity principle and is invariant with respect to the Galilei algebra $AG(1,3)$, but it is not invariant with respect to the scale operator D and projective operator Π ($\lambda_1\neq 0$, $\lambda_2\neq 0$).

The complete information on the symmetry of the equation (5) is given by the following theorem.

Theorem 2 [19] [Fushchych and Symenoh, 1997]. *There is only one equation among linear arbitrary order equations which is invariant with respect to the algebra $AG(1,3)$, and that is the equation (5). In the case when $\lambda = \lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 0$, the equation (5) is invariant with respect to the algebra $AG_2(1,3)$.*

Thus the class of linear Galilei-invariant equations of arbitrary order is rather narrow and reduced to the equation (5). All other Galilei-invariant equations are locally invariant to the equation (5).

3 Nonlocal Galilei symmetry of the relativistic pseudodifferential wave equation

Let us consider a pseudodifferential equation

$$p_0 u = Eu, \quad E \equiv (p_a^2 + m^2)^{1/2}, \quad u = u(x_0, \vec{x}). \tag{6}$$

We may consider the equation (6) as a "square root of the wave operator" for a scalar complex function u.

We can check by direct calculation that the equation (6) is invariant with respect to the standard representation of the Poincaré algebra and not invariant with respect to the standard representation of the Galilei algebra.

Theorem 3 [8] [Fushchych, 1977]. *The equation (6) is invariant with respect to the 11-dimensional Galilei algebra with the following basis operators:*

$$\begin{gathered} P_0^{(2)} = \frac{p^2}{2m} = -\frac{\Delta}{2m}, \quad P_a^{(2)} = p_a = -\frac{\partial}{\partial x_a}, \quad J_{ab}^{(2)} = x_a p_b - x_b p_a \equiv J_{ab}, \\ G_a^{(2)} = t\widetilde{p}_a - m x_a, \quad \widetilde{p}_a \equiv \frac{m}{E} p_a, \quad E = (p_a^2 + m^2)^{1/2}. \end{gathered} \tag{7}$$

The proof of the theorem is reduced to checking the invariance condition

$$[p_0 - E, Q_l]u = 0, \tag{8}$$

where Q_l is any operator from the set (7).

The operators (7) satisfy the commutation relations of the Galilei algebra.

$G_a^{(2)}$ are pseudodifferential operators which generate, as distinct from the standard operators G_a, nonlocal transformations.

So the set of solutions of the motion equation (6) for a scalar particle (field) with positive energy has a nonlocal Galilei symmetry, whose Lie algebra is given by the operators (7).

4 Nonlocal Galilei symmetry of the Dirac equation

It is well-known that the Dirac equation

$$p_0 \Psi = (\gamma_0 \gamma_a p_a + \gamma_0 \gamma_4 m)\Psi = H(p)\Psi \tag{9}$$

is invariant with respect to the Poincaré algebra with the basis operators (see [2])

$$P_0 = i\frac{\partial}{\partial x_0}, \quad P_k = -i\frac{\partial}{\partial x_k}, \quad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}, \quad S_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]. \tag{10}$$

The Dirac equation, as it was established in our papers (see references in [2]) has wide nonlocal symmetry.

In this paragraph we shall establish nonlocal Galilei symmetry of the Dirac equation. For this purpose, using the method described in [2], by means of the integral operator

$$W = \frac{1}{\sqrt{2}}\left(1+\gamma_0\frac{H}{E}\right), \quad E = (p_a^2+m^2)^{1/2}, \quad H = \gamma_0\gamma_a p_a + \gamma_0\gamma_4 m \tag{11}$$

we transform the system of four connected first-order differential equations to the system of non-connected pseudodifferential equations

$$i\frac{\partial\Phi}{\partial t} = \gamma_0 E\Phi, \quad \gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{12}$$

$$\Phi = W\Psi, \quad \gamma_0 E = WHW^{-1}. \tag{13}$$

Having found additional symmetry of the equation (12), we simultaneously establish symmetry of the Dirac equation (9).

Theorem 4 [8] [Fushchych, 1977]. *The equation (12) is invariant with respect to the 11-dimensional Galilei algebra with the following basis operators:*

$$\begin{aligned} &P_0^{(3)} = \frac{\vec{p}^2}{2m}, \quad P_a^{(3)} = p_a = -\frac{\partial}{\partial x_a}, \quad I, \\ &J_{ab}^{(3)} = x_a p_b - x_b p_a + S_{ab}, \quad G_a^{(3)} = t\widetilde{p_a} - m x_a, \qquad \widetilde{p_a} \equiv \gamma_0 \frac{m}{E} p_a. \end{aligned} \tag{14}$$

The operators (14) satisfy the commutation relations of the Galilei algebra $AG(1,3)$.

To prove the theorem is necessary to make sure that the invariance condition

$$[p_0 - \gamma_0 E, Q_l]\Psi = 0 \tag{15}$$

is satisfied for any operator Q_l from the set (14).

$G_a^{(3)}$ are integral operators which generate nonlocal transformations, which do not coincide with the standard Galilei transformations.

Thus the equation (12), and also the Dirac equation (9), has the nonlocal symmetry, which is given by the operators (14). The explicit form of the operators (14) for the equation (9) is calculated by means of the formula

$$\widetilde{Q}_l = W^{-1} Q_l W. \tag{16}$$

1. Schrödinger E., Quantisierung als Eigenwertprobleme, *Annalen der Physik*, 1926, **81**, 109.
2. Fushchych W., Shtelen W., Serov N., Symmetry analysis and exact solutions of eEquations of nonlinear mathematical physics, Dordrecht, Kluwer Academic Publishers, 1993.
3. Fushchych W., Nikitin A., Symmetries of equations of quantum mechanics, New York, Allerton Press, 1994.
4. Fushchych W., How to extend symmetry of differential equations?, in Symmetry and Solutions of Nonlinear Mathematical Physics, Kyiv, Inst. of Math., 1987.

5. Fushchych W., Cherniha R., The Galilei relativictic principle and nonlinear partial differential equations, *J. Phys. A: Math. and Gen.*, 1987, **18**, 3491.
6. Fushchych W., Symmetry in problems of mathematical physics, in Algebraic-Theortic Studies in Mathematical Physics, Kyiv, Inst. of Math., Kyiv, 1981.
7. Fushchych W., Seheda Yu., On a new invariance algebra for the free Schrödinger equation, *Doklady Acad. Sci. USSR*, 1977, **232**, № 4, 800.
8. Fushchych W., Group properties of equations of quantum mechanics, in Asymptotic problems in theory of nonlinear oscilations, Kyiv, Naukova Dumka, 1977, 238.
9. Fushchych W., Nikitin A., Group invariance for quasirelativistic equation of motion, *Doklady Acad. Sci. USSR*, 1978, **238**, № 1, 46.
10. Fushchych W., Serov M., On some exact solutions of the three-dimensional non-linear Schrödinger equation, *J. Phys. A: Math. and Gen.*, 1987, **20**, L929.
11. Fushchych W., Boyko V., Continuity equation in nonlinear quantum mechanics and the Galilei relativity principle, *J. Nonlinear Math. Phys.*, 1997, **4**, 124–128.
12. Schiller R., Quasi-classical transformation theory, *Phys. Rev.*, 1962, **125**, 1109.
13. Rosen N., *American J. Phys.*, 1965, **32**, 597.
14. Guerra F., Pusterla M., Nonlinear Klein–Gordon equation carrying a nondispersive solitonlike singularity, *Lett. Nuovo Cimento*, 1982, **35**, 256.
15. Guéret Ph., Vigier J.-P., Relativistic wave equation with quantum potential nonlinearity, *Lett. Nuovo Cimento*, 1983, **38**, 125.
16. Doebner H.-D., Goldin G.A., Properties of nonlinear Schrödinger equations associated with diffeomorphism group representations, *J. Phys. A: Math. Gen.*, 1994, **27**, 1771.
17. Fushchych W., Cherniha R., Chopyk V., On unique symmetry of two nonlinear generalizations of the Schrödinger equation, *J. Nonlinear Math. Phys.*, 1996, **3**, 296.
18. Fushchych W., On additional invariance of the Klein–Gordon–Fock equation, *Doklady Acad. Sci. USSR*, 1976, **230**, № 3, 570.
19. Fushchych W., Symenoh Z., Higher-order equations of motion in quantum mechanics and Galilean relativity, *J. Phys. A: Math. Gen.*, 1997, **30**, L131.