Solutions of the relativistic nonlinear wave equation by solutions of the nonlinear Schrödinger equation

P. BASARAB-HORWATH, W.I. FUSHCHYCH, L.F. BARANNYK

Using an ansatz for nonlinear complex wave equations obtained by using Lie point symmetries, we show how to construct new solutions of the relativistic nonlinear wave equation from those of a nonlinear Schrödinger equation with the same nonlinearity. This ansatz reduces the number of space-time variables by one, and is not related to a contraction. We give some examples of other types of hyperbolic equations admitting solutions based on nonlinear Schrödinger equations.

1 Introduction

That nonlinear equations should play a role in quantum theory is not a new idea. This idea was propagated by de Broglie, Iwanenko and Heisenberg [1–3]. Nonlinear wave mechanics was taken up again by Bialynicki–Birula and Mycielski [4]. This theme has also been of interest more recently [5], and much work on exact solutions and modelling of nonlinear equations in quantum theory has also been done [12, 21, 22, 6].

In this article we consider a new aspect of some types of nonlinear relativistic equations, and we obtain a connection between solutions of nonlinear Schrödinger equations and our nonlinear relativistic equations. Our starting point is the nonlinear hyperbolic wave equation

$$\Box\Psi + \lambda F(|\Psi|)\Psi = 0, \tag{1}$$

where

$$\Box = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2},$$

with

$$x_\mu = g_{\mu\nu}x^\nu,\ \ \mu,\nu = 0,\dots,3, \quad g_{\mu\nu} = g^{\mu\nu} = \operatorname{diag}(1,-1,-1,-1), \quad |\Psi| = (\Psi\bar{\Psi})^{1/2},$$

and $\Psi = \Psi(x_0, x_1, x_2, x_3)$ is a complex function, $\bar{\Psi}$ being the complex conjugate of Ψ, and we use summation over repeated indices (here and in the rest of the paper). Using Lie point symmetries, exact solutions have been obtained for different choices of the nonlinearity F [7–12]. In this paper we obtain a new class of solutions to (1) by using the symmetries of (1) to establish a connection between (1) and the nonlinear Schrödinger equation

$$i\frac{\partial v}{\partial \tau} = -\Delta v + \lambda F(|v|)v. \tag{2}$$

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Equation (2) is invariant under point transformations generated by the Galilei group. Therefore it seems at first surprising that a Poincaré-invariant equation should be connected with a Galilei-invariant one. It is, however, known that the Poincaré algebra contains the Galilei algebra [20], and the conformal algebra contains the Schrödinger algebra [13–16]. The invariance of a restricted class of solutions of the generalized Bhabha equations (invariant under the 1+4 Poincaré group) with respect to the Galilei group was remarked upon in [20]. However, it is important to note that equation (1) is *not* invariant under the Galilei group.

The novelty of our result is that we use a hitherto unexploited symmetry of (1) to construct an ansatz (called the *Galilei* or *parabolic ansatz*) reducing (1) to (2), for arbitrary nonlinearities in the right-hand side of (1). Thus, we show how nonlinear equations themselves give rise to this connection. The ansatz we construct is shown to work in other cases where the nonlinearity contains derivatives. This is explained by the fact that the equations in question admit the same symmetry operator which is crucial to the construction of the ansatz. Furthermore, we do not establish the connection in terms of contractions, as is done in [13, 14].

The article is organized as follows: first, we give a symmetry classification of equation (1) and show how to construct the ansatz connecting (1) to (2). We also give the symmetry classification of (2), exhibiting the parallel with the symmetry classification of (1). We list the subalgebra classification of the symmetry algebra of (2), together with the corresponding ansatzes and reduced equations, in the appendix. Because of the types of nonlinearity, we are able to solve only some of the reduced equations, in Section 3. In Section 4, we give some examples of other equations for which our ansatz works, and give solutions of the relativistic equations which are related to solitons of the corresponding (using our reduction) Schrödinger equations in 1+1 space-time dimensions. We do not list exact solutions based on the heat equation: these can be obtained by using the results of [19].

2 Symmetry and Galilei ansatz for equation (1)

2.1. Symmetry classification. For the sake of completeness, we give the symmetry classification of equations of type (1) in the following result.

Theorem 1. *The Lie point symmetry algebra of equation (1) has basis vector fields as follows:*

(*i*) *when* $F(|\Psi|) = \mathrm{const}\,|\Psi|^2$*:*

$$\partial_\mu, \quad J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu, \quad K_\mu = 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu - 2x_\mu(\Psi\partial_\Psi + \bar\Psi\partial_{\bar\Psi}),$$
$$D = x^\nu\partial_\nu - (\Psi\partial_\Psi + \bar\Psi\partial_{\bar\Psi}), \quad M = i(\Psi\partial_\Psi - \bar\Psi\partial_{\bar\Psi}),$$

where $x^2 = x_\mu x^\mu$ *and* $\partial_\mu = \partial/\partial x^\mu$, $\partial_\Psi = \partial/\partial\Psi$*;*

(*ii*) *when* $F(|\Psi|) = \mathrm{const}\,|\Psi|^k$, $k \neq 0, 2$*:*

$$\partial_\mu, \quad J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu,$$
$$D_{(k)} = x^\nu\partial_\nu - \frac{2}{k}(\Psi\partial_\Psi + \bar\Psi\partial_{\bar\Psi}), \quad M = i(\Psi\partial_\Psi - \bar\Psi\partial_{\bar\Psi});$$

(*iii*) *when* $F(|\Psi|) = \mathrm{const}\,|\Psi|^k$ *for any* k*, but* $\dot F \neq 0$*:*

$$\partial_\mu, \quad J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu, \quad M = i(\Psi\partial_\Psi - \bar\Psi\partial_{\bar\Psi});$$

(iv) when $F(|\Psi|) = \text{const} \neq 0$*:*

$$\partial_\mu, \quad J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu, \quad M = i(\Psi\partial_\Psi - \bar{\Psi}\partial_{\bar{\Psi}}),$$
$$L = \Psi\partial_\Psi + \bar{\Psi}\partial_{\bar{\Psi}}, \quad L_1 = i(\bar{\Psi}\partial_\Psi - \Psi\partial_{\bar{\Psi}}), \quad L_2 = \bar{\Psi}\partial_\Psi + \Psi\partial_{\bar{\Psi}}, \quad B\partial_\Psi,$$

where B *is an arbitrary solution of* $\Box\Psi = \text{const}\,\Psi$*;*

(v) when $F(|\Psi|) = 0$*:*

$$\partial_\mu, \quad J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu, \quad K_\mu = 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu - 2x_\mu(\Psi\partial_\Psi + \bar{\Psi}\partial_{\bar{\Psi}}),$$
$$D = x^\mu\partial_\mu, \quad M = i(\Psi\partial_\Psi - \bar{\Psi}\partial_{\bar{\Psi}}), \quad L = \Psi\partial_\Psi + \bar{\Psi}\partial_{\bar{\Psi}},$$
$$L_1 = i(\bar{\Psi}\partial_\Psi - \Psi\partial_{\bar{\Psi}}), \quad L_2 = \bar{\Psi}\partial_\Psi + \Psi\partial_{\bar{\Psi}}, \quad B\partial_\Psi,$$

where B *is an arbitrary solution of* $\Box\Psi = 0$.

The first case, $F(|\Psi|) = |\Psi|^2$, gives us the extended conformal algebra, the second case gives the extended Poincaré algebra. In all five cases (which exhaust all possible nonlinearities of the given type), the symmetry algebra contains the subalgebra $\langle P_\mu, J_{\mu\nu}\rangle$, which is the Poincaré algebra, and the operator $M = i(\Psi\partial_\Psi - \bar{\Psi}\partial_{\bar{\Psi}})$. It is this operator which we combine with the generators of space-time translations ∂_μ in order to build an ansatz which reduces equation (1) to a nonlinear Schrödinger equation. This gives a reduction of a hyperbolic equation to a parabolic equation, and for this reason we call it a *parabolic symmetry* of the nonlinear wave equation. In this fashion we are able to construct new solutions of (1), even making a contact with the Zakharov–Shabat soliton solution [18] when $F(|\Psi|) = |\Psi|^2$. The appearance of the parabolic symmetry M is a feature of the fact that Ψ is a complex-valued function and of the type of nonlinearity we consider. In our previous article [19] we considered a similar reduction of a linear equation (corresponding to $F = \text{const}$) to the heat equation using the operator $u\partial_u$ which is the counterpart of the other parabolic symmetry operator L. Using M, we improve upon our result in that we are able to include nonlinearities and still obtain a reduction to a parabolic equation. If we were to use L instead, then we would reduce (1) to the heat equation with a complex function. This, however, may be done only in the cases $F = \text{const} \neq 0$ and $F = 0$, as it is only then that L appears as a symmetry. On writing $\Psi = ue^{iw}$, one finds that $L = u\partial_u$ whereas $M = \partial_w$. Therefore, equations admitting the symmetry M involve only the derivatives of the phase.

In [17] we investigated equation (1) from a slightly different point of view: taking the phase-amplitude representation of Ψ, we used results about the compatibility of the system

$$\Box v = F_1(v), \quad \partial^\mu v\partial_\mu v = F_2(v),$$

to obtain new solutions of non-Lie type (that is, not obtainable by reduction by Lie symmetries). The same approach can be taken for the nonlinear Schrödinger equation, and the methods of [17] can also be combined with those of this article.

2.2. The Galilei ansatz and reduction to the Schrödinger equation. Equation (1) is invariant under ∂_μ and M, and therefore under any constant linear combination of them:

$$\varepsilon^\mu\partial_\mu + kM. \tag{3}$$

The operator (3) gives rise to the invariant surface conditions

$$\varepsilon^\mu \partial_\mu \Psi = ik\Psi, \quad \varepsilon^\mu \partial_\mu \bar{\Psi} = -ik\bar{\Psi}$$

for Ψ and $\bar{\Psi}$, where ε^μ and k are real constants. These conditions give us the Lagrangian system

$$\frac{dx_\mu}{\varepsilon_\mu} = \frac{d\Psi}{ik\Psi} = \frac{d\bar{\Psi}}{-ik\bar{\Psi}}. \tag{4}$$

It is straightforward to show that (4) is equivalent to

$$\frac{d(cx)}{c\varepsilon} = \frac{d\Psi}{ik\Psi} = \frac{d\bar{\Psi}}{-ik\bar{\Psi}} \tag{5}$$

for any constant four-vector c, where $cx = c^\mu x_\mu$, $c\varepsilon = c^\mu \varepsilon_\mu$. Then choose ε light-like, so that $\varepsilon^2 = 0$ and, further, choose α, β, δ so that

$$\alpha^2 = \beta^2 = -1, \quad \delta^2 = 0, \quad \alpha\beta = \alpha\delta = \alpha\varepsilon = \beta\delta = \beta\varepsilon = 0, \quad \delta\varepsilon = 1.$$

That is, α, β, δ, ε is a hybrid 2+2 basis of Minkowski space consisting of two space-like vectors (α, β) and two light-like vectors (δ, ε). Then put c in (5) successively equal to α, β, δ, ε, and we obtain the Lagrangian system

$$\frac{d(\alpha x)}{0} = \frac{d(\beta x)}{0} == \frac{d(\varepsilon x)}{0} = \frac{d(\delta x)}{1} = \frac{d\Psi}{ik\Psi} = \frac{d\bar{\Psi}}{-ik\bar{\Psi}}. \tag{6}$$

The system (6) then integrates to give

$$\Psi = e^{ik(\delta x)} v(\varepsilon x, \alpha x, \beta x), \quad \bar{\Psi} = e^{-ik(\delta x)} \bar{v}(\varepsilon x, \alpha x, \beta x), \tag{7}$$

where v is a smooth function. Substituting equations (7) as ansatzes in (1), we obtain (after some elementary manipulation) the equation

$$i\frac{\partial v}{\partial t} = \frac{1}{2k}\Delta v - \frac{\lambda}{2k} F(|v|)v,$$

where we have used the notation $t = \varepsilon x$, $y_1 = \alpha x$, $y_2 = \beta x$ and $\Delta = \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}$. For convenience, we choose $k = -\frac{1}{2}$, and we then have the nonlinear Schrödinger equation in 2+1 space-time dimensions

$$i\frac{\partial v}{\partial t} = -\Delta v + \lambda F(|v|)v. \tag{8}$$

This is a well-studied equation, at least in 1+1 space-time dimensions, exhibiting soliton solutions and being completely integrable (possessing infinitely many commuting flows) for $F(|v|) = |v|^2$ (see [18]). It has been studied in other dimensions in [20–23, 27] in terms of symmetries and conditional symmetries.

The Cauchy problem for equation (8) is well-posed for $t > 0$, and (8) has solutions which are singular for $t = 0$. This leads to similar problems for the wave equation when $\varepsilon x = 0$, which is a characteristic ($\varepsilon^2 = 0$), and so the initial-value problem of (8) is related to the initial-value problem of (1) on a characteristic, known as Goursat's problem. For the linear equation, this has been studied in [28].

It is an interesting question as to what quantum-mechanical implications (8) has for (1), but we shall not pursue this in the present article.

We emphasise that the connection between the hyperbolic equation (1) and the Schrödinger equation (8) is obtained by an ansatz which reduces the number of space-time dimensions by one; it is not a contraction as in [13].

2.3. Symmetries of the Schrödinger equation (8). The symmetry algebra of equation (8) is given by the following result: its classification according to the type of nonlinearity is in a direct correspondence to that of the symmetry algebra of equation (1).

Theorem 2. *Equation (8) has maximal point symmetry algebra (with the given vector fields as basis) depending on the nonlinearity $F(|v|)$:*

(i) $AG_2(1,2)$*, when* $F(|v|) = \mathrm{const}\,|v|^2$*:*

$$T = \partial_t, \quad P_a = -\partial_a, \quad J_{12} = x_1\partial_{x_2} - x_2\partial_{x_1},$$
$$G_a = t\partial_a + \frac{1}{2}ix_a(v\partial_v - \bar{v}\partial_{\bar{v}}), \quad D_2 = 2t\partial_t + x_a\partial_a - (v\partial_v + \bar{v}\partial_{\bar{v}}),$$
$$S = t^2\partial_t + tx_a\partial_a + \frac{1}{4}ix_ax_a(v\partial_v - \bar{v}\partial_{\bar{v}}) - t(v\partial_v + \bar{v}\partial_{\bar{v}}),$$
$$M = -\frac{1}{2}i(v\partial_v - \bar{v}\partial_{\bar{v}});$$

(ii) $AG_1(1,2)$*, when* $F(|v|) = \mathrm{const}\,|v|^k$*,* $k \neq 0, 2$*:*

$$T = \partial_t, \quad P_a = -\partial_a, \quad J_{12} = x_1\partial_{x_2} - x_2\partial_{x_1}, \quad G_a = t\partial_a + \frac{1}{2}ix_a(v\partial_v - \bar{v}\partial_{\bar{v}}),$$
$$D_2 = 2t\partial_t + x_a\partial_a - \tfrac{2}{k}(v\partial_v + \bar{v}\partial_{\bar{v}}), \quad M = -\frac{1}{2}i(v\partial_v - \bar{v}\partial_{\bar{v}});$$

(iii) $AG(1,2)$*, when* $F(|v|) \neq \mathrm{const}\,|v|^k$*, for any* k *but* $\dot{F} \neq 0$*:*

$$T = \partial_t, \quad P_a = \partial_a, \quad J_{12} = x_1\partial_{x_2} - x_2\partial_{x_1},$$
$$G_a = t\partial_a + \frac{1}{2}ix_a(v\partial_v - \bar{v}\partial_{\bar{v}}), \quad M = -\frac{1}{2}i(v\partial_v - \bar{v}\partial_{\bar{v}});$$

(iv) $AG_2(1,2) \oplus \langle B\rangle$*, when* $F = 0$*, where* $\langle B\rangle$ *infinite space of arbitrary solutions of the free Schrödinger equation:*

$$T = \partial_t, \quad P_a = \partial_a, \quad J_{12} = x_1\partial_{x_2} - x_2\partial_{x_1}, \quad G_a = t\partial_a + \frac{1}{2}ix_a(v\partial_v - \bar{v}\partial_{\bar{v}}),$$
$$S = t^2\partial_t + tx_a\partial_a + \frac{1}{4}ix_ax_a(v\partial_v - \bar{v}\partial_{\bar{v}}) - t(v\partial_v + \bar{v}\partial_{\bar{v}}),$$
$$M = -\frac{1}{2}i(v\partial_v - \bar{v}\partial_{\bar{v}}), \quad D = 2t\partial_t + x_a\partial_a, \quad L = v\partial_v + \bar{v}\partial_{\bar{v}}, \quad B\partial_v,$$

where B *is an arbitrary solution of the free Schrödinger equation.*

The algebra in Theorem 2i is the Schrödinger algebra [14], which is a subalgebra of the conformal algebra. This is reflected in the fact that the nonlinearity in Theorem 2i is the same as in Theorem 1i, for which the wave equation (1) is invariant under the conformal group. Note that Theorems 2iv, v correspond to Theorem 1v, since for equation (8) the case $F = \mathrm{const} \neq 0$ can be gauged to the case (iv) on putting $\hat{v} = e^{it\lambda F}v$, and then $\hat{v}$ satisfies the free (no potential) Schrödinger equation. The above is an exhaustive list of the types of symmetries for all the different types of nonlinearities. Again, in each of the four cases, we find the operator $M = i(v\partial_v - \bar{v}\partial_{\bar{v}})$, and we can use this in a similar way to the reduction of the wave equation, in order to reduce (8) to the corresponding Schrödinger equation in 1+1 space-time dimensions; this time with the same nonlinearity and 'coupling' constant λ. Thus we can think of the linear and nonlinear Schrödinger equations as part of a chain of successive reductions, beginning with a nonlinear (hyperbolic) wave equation in $n+1$ space-time dimensions, as in (1).

Theorem 2 now allows us to classify the reductions of equation (8), according to the type of nonlinearity. If we exclude the case $F = 0$, then there are only three types of algebras: $AG(1,2) = \langle T, P_a, G_a, J_{12}, M\rangle$, $AG_1(1,2) = \langle T, P_a, G_a, J_{12}, M, D\rangle$, and $AG_2(1,2) = \langle T, P_a, G_a, J_{12}, M, D, S\rangle$. These are the maximal symmetry algebras of the equations:

$$i\frac{\partial v}{\partial t} = -\Delta v + \lambda F(|v|)v, \quad \text{with} \quad F(|v|) \neq |v|^k, \quad \dot{F} \neq 0, \tag{9}$$

$$i\frac{\partial v}{\partial t} = -\Delta v + \lambda |v|^k v, \quad k \neq 0, 2, \tag{10}$$

$$i\frac{\partial v}{\partial t} = -\Delta v + \lambda |v|^2 v, \tag{11}$$

respectively. The Lie algebra $AG_2(1,2)$ was considered in [19]. It is the semi-direct sum

$$ASL(2,\mathbb{R}) \oplus AO(2) \dotplus \langle M, P_a, G_a\rangle,$$

where $ASL(2,\mathbb{R})$ is the Lie algebra of the group $SL(2,\mathbb{R})$, and $AO(2)$ is the Lie algebra of the group $O(2)$. The other two algebras are subalgebras of $AG_2(1,2)$.

3 Some exact solutions

In this section we obtain some exact solutions of the wave equation using results from the tables in the appendix. The other reduced equations are difficult to solve, so we leave them for future consideration, remarking only that they give exact solutions of equation (1) when we use the ansatz in equation (7).

First, we take the case of the subalgebra $\langle P_2, T + 2\alpha M\rangle$ from Table 1 in the appendix, with $F(|\phi|) = |\phi|^n$ and $n > 0$. The reduced equation is then

$$\ddot{\phi} + a\phi = \lambda |\phi|^n \phi.$$

On putting

$$\phi(\omega) = \rho(\omega) e^{i\theta(\omega)}$$

into this equation, with ρ, θ being real functions and $\rho > 0$, we obtain

$$\ddot{\rho} + a\rho - \rho\dot{\theta}^2 = \lambda\rho^{n+1}, \tag{12}$$

$$\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta} = 0. \tag{13}$$

Equation (13) readily integrates to give us

$$\dot{\theta} = \frac{A}{\rho^2}, \tag{14}$$

where A is a constant of integration. Put now equation (14) into equation (12) and we find

$$\ddot{\rho} + a\rho - \frac{A^2}{\rho^3} = \lambda\rho^{n+1},$$

which is the Ermakov–Pinney [31] equation when $\lambda = 0$. Multiplying this equation by $2\dot{\rho}$ and integrating, we obtain

$$\dot{\rho}^2 + a\rho^2 + \frac{A^2}{\rho^2} = \frac{2\lambda}{n+2}\rho^{n+2} + C, \tag{15}$$

where C is another constant of integration. We now consider three cases of equation (15).

Case 1. $A = 0$, $C = 0$, $a \neq 0$. Since $A = 0$ here, we have $\theta = \text{const}$, and (15) becomes

$$\dot{\rho}^2 = \frac{2\lambda}{n+2}\rho^{n+2} - a\rho^2,$$

from which we deduce

$$\int \frac{d\rho}{\sqrt{\frac{2\lambda}{n+2}\rho^{n+2} - a\rho^2}} = \pm\omega + C_1.$$

On writing $u = -\rho^{-n/2}$, this integral reduces to

$$\int \frac{du}{\sqrt{\frac{2\lambda}{n+2} - au^2}} = -\frac{n}{2}(\pm\omega + C_1).$$

For $\lambda > 0$, $a < 0$ we obtain (after some calculation)

$$u^2 = \frac{\lambda}{a(n+2)}\left[1 - \cosh\left(n\sqrt{-a}(C_1 \pm \omega)\right)\right]$$

or

$$\rho = \sqrt[n]{\frac{a(n+2)}{\lambda}\frac{1}{1 - \cosh\left(n\sqrt{-a}(C_1 \pm \omega)\right)}}.$$

Finally, noting that we have $\omega = y_1 = \alpha x$, in the notation of Section 2.3, we find that

$$\Psi = e^{-i(a(\varepsilon x) + (\delta x)/2)}\sqrt[n]{\frac{a(n+2)}{\lambda}\frac{1}{1 - \cosh\left(n\sqrt{-a}(C_1 \pm \alpha x)\right)}}$$

is a solution of

$$\Box\Psi = -\lambda|\Psi|^n\Psi,$$

when $\lambda > 0$, $a < 0$. If we take $\lambda > 0$, $a > 0$, then we obtain, with similar calculations, that

$$\Psi = e^{-i(a(\varepsilon x)+(\delta x)/2)} \sqrt[n]{\frac{a(n+2)}{\lambda} \frac{1}{1-\cos\left(n\sqrt{a}(C_1 \pm \alpha x)\right)}}$$

is a solution of

$$\Box\Psi = -\lambda|\Psi|^n\Psi.$$

Case 2. $A = 0$, $n = 2$, $a \neq 0$. In this case we also have $\theta = \text{const}$, and (15) becomes

$$\dot{\rho}^2 + a\rho^2 - \frac{1}{2}\lambda\rho^4 = C. \qquad (16)$$

Equation (16) can be solved using Jacobian elliptic functions. For the definitions, we refer to [29]. Following [30], we take a, λ and C as functions of a real parameter κ, with $|\kappa| < 1$, and using the generic notation $E(\omega, \kappa)$ for solutions of (16), we have the following table of exact solutions:

$E(\omega,\kappa)$	$a(\kappa)$	$\lambda(\kappa)$	$C(\kappa)$
sn	$1+\kappa^2$	$2\kappa^2$	1
cn	$1-2\kappa^2$	$-2\kappa^2$	$1-\kappa^2$
dn	κ^2	-2	κ^2-1
ns = 1/sn	$1+\kappa^2$	2	κ^2
nc = 1/cn	$1-2\kappa^2$	$2(1-\kappa^2)$	$-\kappa^2$
nd =1/dn	κ^2-2	$2(\kappa^2-1)$	-1
sc = sn/cn	κ^2-2	$2(1-\kappa^2)$	1
sd = sn/dn	$1-2\kappa^2$	$2\kappa^2(\kappa^2-1)$	1
cs = cn/sn	κ^2-2	2	$1-\kappa^2$
cd = cn/dn	$1+\kappa^2$	$2\kappa^2$	1
ds = dn/sn	$1-2\kappa^2$	2	$\kappa^2(\kappa^2-1)$
dc = dn/cn	$1+\kappa^2$	2	κ^2

Using this table and the notation of Section 2.3, we find that

$$\Psi = e^{-i(a(\kappa)(\varepsilon x)+(\delta x)/2)} E(\alpha x, \kappa)$$

is an exact solution of

$$\Box\Psi = -\lambda(\kappa)|\Psi|^2\Psi,$$

where $a(\kappa)$ and $\lambda(\kappa)$ are the appropriate functions of the parameter κ, as given in the above table. This gives us elliptic solutions of a nonlinear relativistic wave equation. We note that solutions of nonlinear wave equations in terms of elliptic functions were obtained by Petiau [35]. The solutions we present here are for a different nonlinearity.

Case 3. $n = 2$, $a = 0$. If we put $n = 2$ and $a = 0$ in (15), we obtain the equation

$$\dot{\rho}^2 + \frac{A^2}{\rho^2} = \frac{\lambda}{2}\rho^4 + C.$$

On multiplying this equation by ρ^2, and putting $z = \rho^2$, we obtain the following equation for z:

$$\dot{z}^2 = \frac{\lambda}{2}\left[4z^3 + \frac{8C}{\lambda}z - \frac{8A^2}{\lambda}\right],$$

which gives us the solution

$$z = \wp\left(\sqrt{\frac{1}{2}}\lambda\omega\right),$$

where $\wp(\xi)$ is the Weierstrass elliptic function (see [29]), provided that $27A^4 + 8C^3/\lambda \neq 0$ (the equation $(d\xi/ds)^2 = 4\xi^3 - g_2\xi - g_3$ has $\wp(s)$ as solution provided $g_2^3 - 27g_3 \neq 0$). From this it is straightforward to deduce that

$$\Psi = \sqrt{\wp\left(\sqrt{\frac{1}{2}}\lambda(\alpha x)\right)} \exp\left[-\left(\frac{\delta x}{2} + \frac{2A}{\lambda}\int^{\sqrt{\frac{\lambda}{2}}(\alpha x)} \frac{d\sigma}{\wp(\sigma)}\right)\right]$$

is a solution of

$$\Box\Psi = -\lambda|\Psi|^2\Psi.$$

Next we turn to the case $\langle G_1 + aP_1, G_2\rangle$ in Table 1. The reduced equation is

$$\dot{\phi} + \frac{1}{2}\left(\frac{1}{\omega - a} + \frac{1}{\omega}\right)\phi = -i\lambda F(|\phi|)\phi.$$

Using the amplitude-phase representation $\phi = \rho e^{i\theta}$ in this equation, as before, we find the following system:

$$\dot{\rho} + \frac{1}{2}\left(\frac{1}{\omega - a} + \frac{1}{\omega}\right)\rho = 0, \tag{17}$$

$$\dot{\theta} = -\lambda F(\rho). \tag{18}$$

Equation (17) integrates immediately to give

$$\rho = \frac{C}{\sqrt{\omega(\omega - a)}},$$

where C is a constant of integration. Using this, (18) now yields

$$\theta = -\lambda\int F\left(\frac{C}{\sqrt{\omega(\omega - a)}}\right)d\omega + C_1.$$

Combining this with the corresponding ansatz for the solution v of (8), and using the notation of Section 2.3, we obtain that

$$\Psi = \frac{C}{\sqrt{(\varepsilon x)^2 - a(\varepsilon x)}} \times$$
$$\times \exp\left[-i\left(\lambda\int^{\varepsilon x} F\left(\frac{C}{\sqrt{\xi(\xi - a)}}\right)d\xi + \frac{\delta x}{2} + \frac{(\alpha x)^2 + (\beta x)^2}{4\varepsilon x}\right)\right]$$

is an exact solution of

$$\Box\Psi = -\lambda F(|\Psi|)\Psi,$$

and when $F(\xi) = \xi^n$, with $n \geq 2$, we have

$$\Psi = \frac{C}{\sqrt{(\varepsilon x)^2 - a(\varepsilon x)}} \times \\ \times \exp\left[-i\left(-\lambda\frac{C^n}{(n-1)[(\varepsilon x)^2 - a(\varepsilon x)]^{(n-1)/2}} + \frac{\delta x}{2} + \frac{(\alpha x)^2 + (\beta x)^2}{4\varepsilon x}\right)\right]$$

as an exact solution.

4 Special solutions of some nonlinear complex wave equations

In this section we give some particular solutions of some multi-dimensional hyperbolic ('relativistic') equations which can be reduced to Schrödinger equations with our ansatz (7). In some cases, the nonlinear Schrödinger equation involved admits a soliton solution in 1+1 space-time.

First we take the hyperbolic equation

$$\Box\Psi = \lambda|\Psi|^n\Psi.$$

The ansatz (7) (with $k = -1/2$) reduces this to

$$iv_t + \Delta v + \lambda|v|^n v = 0,$$

as we have already noted. It is a simple matter to verify that for $\lambda = \boldsymbol{a}^2 b^2 \frac{2}{n}\left(\frac{2}{n} + 1\right)$ we have

$$v = \frac{\exp(4i\boldsymbol{a}^2 b^2 t/n^2)}{\cosh^{2/n}(b\boldsymbol{a}\cdot\boldsymbol{y})}$$

as a solution. Here $\boldsymbol{a} = (a_1, a_2)$, $\boldsymbol{y} = (y_1, y_2)$, where $\boldsymbol{a} = (a_1, a_2)$ is an arbitrary vector and b an arbitrary real number. Applying the Galilean boosts (which are symmetries of the above nonlinear Schrödinger equation)

$$G_a = t\partial_a + \frac{1}{2}ix_a(v\partial_v - \bar{v}\partial_{\bar{v}}) \tag{19}$$

(where $a = 1, 2$) to this solution, we obtain the solution

$$v = \frac{\exp[i(4\boldsymbol{a}^2 b^2 t/n^2 + \boldsymbol{V}\cdot\boldsymbol{y}/2 - \boldsymbol{V}^2 t/4)]}{\cosh^{2/n}(b\boldsymbol{a}\cdot(\boldsymbol{y} - \boldsymbol{V}t))},$$

where $\boldsymbol{V} = (V_1, V_2)$ is an arbitrary vector. For $n = 2$ and in 1+1 space-time, we have

$$v = \frac{\exp[i(a^2 b^2 t + Vy/2 - V^2 t/4)]}{\cosh(ab(y - Vt))},$$

which is the Zakharov–Shabat soliton. Finally, using (7), we obtain

$$\Psi = \frac{\exp[i(-\delta x/2 + 4\boldsymbol{a}^2 b^2(\varepsilon x)/n^2 + (V_1(\alpha x) + V_2(\beta x))/2 - \boldsymbol{V}^2(\varepsilon x)/4)]}{\cosh^{2/n}(b[a_1(\alpha x - V_1 t) + a_2(\beta x - V_2(\varepsilon x))])}$$

as a solution of

$$\Box\Psi = \boldsymbol{a}^2 b^2 \frac{2}{n}\left(\frac{2}{n}+1\right)|\Psi|^n\Psi$$

in 1+3 space-time.

There are some other hyperbolic equations which can be reduced to nonlinear Schrödinger equations, but with nonlinearities involving derivatives. The hyperbolic equations of the form

$$\Box\Psi = \lambda F(|\Psi|, |\Psi|_\mu |\Psi|_\mu)\Psi \tag{20}$$

can also be reduced to nonlinear Schrödinger equations with derivative nonlinearities, using the same ansatz (7) (which is not surprising as the same symmetry operator is responsible for the ansatz). Indeed, ansatz (7) with $k = -1/2$ gives us

$$iv_t + \Delta v + \lambda F(|v|, -|v|_a|v|_a)v = 0, \tag{21}$$

where $|v|_a|v|_a = |v|^2_{y_1} + |v|^2_{y_2}$. Equations of the type (21) were discussed in [21] from a group-theoretical point of view. One of this type of Schrödinger equations is

$$iv_t + \Delta v = 2\frac{|v|_a|v|_a}{|v|^2}v, \tag{22}$$

with $\lambda = -2$ and $F(|v|, |v|_a|v|_a) = \frac{|v|_a|v|_a}{|v|^2}$. Equation (22) admits the two solutions:

$$v = A\frac{\exp(-i\boldsymbol{a}^2 t)}{\cosh(\boldsymbol{a}\cdot\boldsymbol{y})}, \quad v = A\frac{\exp(-i\boldsymbol{a}^2 t)}{\sinh(\boldsymbol{a}\cdot\boldsymbol{y})},$$

where $\boldsymbol{a} = (a_1, a_2)$ is an arbitrary vector and A is an arbitrary number. Applying the Galilei boosts (19) (they are symmetries of (22)) to these solutions, we find

$$v = A\frac{\exp[i(\boldsymbol{V}\cdot\boldsymbol{y}/2 - t\boldsymbol{V}^2/4 - t\boldsymbol{a}^2)]}{\cosh(\boldsymbol{a}\cdot\boldsymbol{y} - \boldsymbol{a}\cdot\boldsymbol{V}t)},$$

and

$$v = A\frac{\exp[i(\boldsymbol{V}\cdot\boldsymbol{y}/2 - t\boldsymbol{V}^2/4 - t\boldsymbol{a}^2)]}{\sinh(\boldsymbol{a}\cdot\boldsymbol{y} - \boldsymbol{a}\cdot\boldsymbol{V}t)},$$

as solutions of (22), with $\boldsymbol{V} = (V_1, V_2)$ an arbitrary vector. From this we find that the hyperbolic equation

$$\Box\Psi = -\frac{2|\Psi|_\mu|\Psi|_\mu}{|\Psi|^2}\Psi$$

admits the solutions

$$\Psi = A\frac{\exp[i(V_1(\alpha x)/2 + V_2(\beta x)/2 - \boldsymbol{V}^2(\varepsilon x)/4 - \delta x/2 - \boldsymbol{a}^2(\varepsilon x))]}{\cosh(a_1(\alpha x) + a_2(\beta x) - (\boldsymbol{a}\cdot\boldsymbol{V})(\varepsilon x))}$$

and

$$\Psi = A\frac{\exp[i(V_1(\alpha x)/2 + V_2(\beta x)/2 - \boldsymbol{V}^2(\varepsilon x)/4 - \delta x/2 - \boldsymbol{a}^2(\varepsilon x))]}{\sinh(a_1(\alpha x) + a_2(\beta x) - (\boldsymbol{a}\cdot\boldsymbol{V})(\varepsilon x))}.$$

Note that we have only used two Galilean boosts to obtain these two-parameter families of solutions. We can introduce more parameters by using the other symmetries of the hyperbolic equation and the corresponding Schrödinger equations.

A third example is the hyperbolic equation

$$\Box\Psi = 2p|\Psi|^2\Psi - C\frac{\Psi_\mu\Psi_\mu}{\Psi} \tag{23}$$

with $C \neq 1$. Using the ansatz

$$\Psi = e^{-i(\delta x)/2(1+C)}v(\alpha x, \beta x, \varepsilon x)$$

is straightforward to show that (23) reduces to the equation

$$iv_t + \Delta v + 2p|v|^2 v = -C\frac{v_a v_a}{v}. \tag{24}$$

In 1+1 space-time, equation (24) is the Malomed–Stenflo equation [32] in plasma physics which admits solitons. Equation (24) admits the solution

$$v = A\,\mathrm{sech}\,(\boldsymbol{n}\cdot\boldsymbol{y})\exp(i(C+1)\boldsymbol{n}^2 t)$$

(which in 1+1 dimensions is the Malomed–Stenflo soliton), where $A^2 = \boldsymbol{n}^2(C+2)/2p$ and $\boldsymbol{n} = (n_1, n_2)$ is an arbitrary vector. We can now act on this solution with the Galilean boosts

$$G_a = t\partial_a + \frac{iy_a}{2(1+C)}(v\partial_v - \bar{v}\partial_{\bar{v}}),$$

which are symmetries of (24), and we obtain

$$v = A\,\mathrm{sech}\,(\boldsymbol{n}\cdot\boldsymbol{y} - \boldsymbol{n}\cdot\boldsymbol{V}t)\exp\left[i\left((C+1)\boldsymbol{n}^2 t + \frac{\boldsymbol{V}\cdot\boldsymbol{y}}{2(1+C)} - \frac{\boldsymbol{V}^2 t}{4(1+C)}\right)\right]$$

as a two-parameter family of solutions of (24). We are then able to construct the following solution of (23):

$$\Psi = A\frac{\exp\left[i\left((1+C)\boldsymbol{n}^2(\varepsilon x) - \frac{\delta x}{2(1+C)} + \frac{V_1(\alpha x)}{2(1+C)} + \frac{V_2(\beta x)}{2(1+C)} - \frac{\boldsymbol{V}^2(\varepsilon x)}{4(1+C)}\right)\right]}{\cosh(n_1(\alpha x) + n_2(\beta x) - (\boldsymbol{n}\cdot\boldsymbol{V})(\varepsilon x))}.$$

5 Conclusions

These are just some examples of hyperbolic equations which reduce down to nonlinear Schrödinger equations. There are of course more. For instance, the hyperbolic equation

$$\Box\Psi = \frac{\Box|\Psi|}{|\Psi|}\Psi - \lambda\Psi, \tag{25}$$

which arises in the context of de Broglie's double solution [33, 1], reduces, with our ansatz, to

$$i\partial_t v = -\Delta v + \frac{\Delta|v|}{|v|}v + \lambda v; \tag{26}$$

an equation which was considered by Guerra and Pusterla [34] in the context of a nonlinear Schrödinger equation. The terms $\Box|\Psi|/|\Psi|$ and $\Delta|v|/|v|$ are called the quantum potentials [1]. Both equations (25) and (26) are conformally invariant, (25) being invariant under the conformal algebra $AC(1,n+2)$, and (26) under $AC(1,n+1)$ in $n+1$ space-time dimensions (see [40]). These remarkable symmetry properties are due to the quantum potential term. They share this symmetry with a wide class of other equations [36, 37].

Despite this connection, we are as yet unable to give a clear physical meaning to the reduction and the ansatz, other than the purely Lie-algebraic one. That we should expect some sort of physical interpretation is suggested by the use of complex hyperbolic equations by Grundland and Tuszynski in [10] in the context of superfluidity and liquid crystal theory.

It is also natural to ask if it is possible to obtain a nonlinear complex hyperbolic wave equation from a Schrödinger equation. It is, of course, not possible from an equation of the form

$$iv_t + \Delta v = F(|v|)v.$$

However, if we consider

$$iv_t + \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = F(|v|)v,$$

and put

$$v = e^{i(x+y)}w(x-2t, y+2t),$$

then we find that w satisfies the equation

$$\frac{\partial^2 w}{\partial \xi^2} - \frac{\partial^2 w}{\partial \eta^2} = F(|w|)w,$$

with $\xi = x - 2t$, $\eta = y + 2t$. It thus seems of interest to investigate equations of the type

$$i\frac{\partial \Psi}{\partial t} + \Box\Psi = F(|\Psi|)\Psi.$$

This type of equation is also of interest in quantum physics: the equation

$$i\frac{\partial \Psi}{\partial t} = \frac{1}{2m}(\Box - m^2)\Psi$$

(with interaction terms involving the electromagnetic potential) was used by Fock as an analogue of the Hamilton–Jacobi equation in quantum mechanics, where t was interpreted as the proper time (see [38] for more details on parametrized relativistic quantum theories). Feynman in [39] considered the equation

$$i\frac{\partial \Psi}{\partial t} = \frac{1}{2}(\partial_\mu - eA_\mu)(\partial^\mu - eA^\mu)\Psi.$$

Table 1. Reduction to ordinary differential equations. These ansazes and reduced equations are for equations (9), (10) and (11).

Subalgebra	Ansatz	ω	Reduced equation
$\langle P_2, T+2aM\rangle$ $(a\in\mathbb{R})$	$v=\exp(-i\alpha t)\phi(\omega)$	y_1	$\ddot{\phi}+a\phi=\lambda F(\vert\phi\vert)\phi$
$\langle J_{12}+2aM, T-2bM\rangle$ $(a,b\in\mathbb{R})$	$v=\exp\left[i\left(bt+a\arctan\dfrac{y_1}{y_2}\right)\right]\phi(\omega)$	$y_1^2+y_2^2$	$4\omega\ddot{\phi}+4\dot{\phi}-\left(b+\dfrac{a^2}{\omega}\right)\phi=\lambda F(\vert\phi\vert)\phi$
$\langle T+aG_1, P_2\rangle$ $(a>0)$	$v=\exp\left[i\left(-\dfrac{a^2t^3}{6}+\dfrac{ay_1t}{2}\right)\right]\phi(\omega)$	at^2-2y_1	$4\ddot{\phi}+\dfrac{a\omega}{4}\phi=\lambda F(\vert\phi\vert)\phi$
$\langle G_1, P_2\rangle$	$v=\exp\left(\dfrac{iy_1^2}{4t}\right)\phi(\omega)$	t	$4\dot{\phi}+\dfrac{1}{2\omega}\phi=-\lambda F(\vert\phi\vert)\phi$
$\langle G_1+aP_1, G_2\rangle$ $(a\in\mathbb{R})$	$v=\exp\left[i\left(\dfrac{y_1^2}{4(t-a)}+\dfrac{y_2^2}{4t}\right)\right]\phi(\omega)$	t	$\dot{\phi}+\dfrac{1}{2}\left(\dfrac{1}{\omega-a}+\dfrac{1}{\omega}\right)\phi=-\lambda F(\vert\phi\vert)\phi$
$\langle P_1, P_2\rangle$	$v=\phi(\omega)$	t	$i\dot{\phi}=\lambda F(\vert\phi\vert)\phi$

Table 2. Reduction by two-dimensional subalgebras of $AG_1(1,2)$. These ansazes and reduced equations are for equations (10) and (11).

Subalgebra	Ansatz	ω	Reduced equation
$\langle P_2, D+4aM\rangle$ $(a\in\mathbb{R})$	$v=t^{-(ia+1/k)}\phi(\omega)$	$\dfrac{y_1^2}{t}$	$4\omega\ddot{\phi}+(2-i\omega)\dot{\phi}+\left(a-\dfrac{i}{k}\right)=\lambda\vert\phi\vert^k\phi$
$\langle G_2, D+4aM\rangle$ $(a\in\mathbb{R})$	$v=t^{-(ia+1/k)}\exp\left(\dfrac{iy_2^2}{4t}\right)\phi(\omega)$	$\dfrac{y_1^2}{t}$	$4\omega\ddot{\phi}+(2-i\omega)\dot{\phi}+\left(a+\dfrac{i}{2}-\dfrac{i}{k}\right)=\lambda\vert\phi\vert^k\phi$
$\langle T, D+2aM\rangle$ $(a\in\mathbb{R})$	$v=y_1^{-(ia+2/k)}\phi(\omega)$	$\dfrac{y_1}{y_2}$	$(\omega^2+1)\ddot{\phi}+2\omega\left(ia+\dfrac{2}{k}+1\right)\dot{\phi}+$ $+\left(ia+\dfrac{2}{k}\right)\left(ia+\dfrac{2}{k}+1\right)\phi=\lambda\vert\phi\vert^k\phi$
$\langle J_{12}+2aM, D+4bM\rangle$ $(a\geq 0,\ b\in\mathbb{R})$	$v=t^{-(ib+1/k)}\exp\left(ia\arctan\dfrac{y_1}{y_2}\right)\phi(\omega)$	$\dfrac{y_1^2+y_2^2}{t}$	$4\omega\ddot{\phi}+(4-i\omega)\dot{\phi}-$ $-\left(\dfrac{a^2}{\omega}-b+\dfrac{i}{k}\right)\phi-\lambda\vert\phi\vert^k\phi=0$
$\langle T, J_{12}+\dfrac{a}{2}D+2abM\rangle$ $(a\geq 0,\ b\in\mathbb{R})$	$v=(y_1^2+y_2^2)^{-(ib+1/k)}\phi(\omega)$	$a\arctan\dfrac{y_1}{y_2}-(y_1^2+y_2^2)$	$(a^2+4)\ddot{\phi}-8\left(\dfrac{1}{k}+ib\right)\dot{\phi}+$ $+4\left(\dfrac{1}{k}+ib\right)^2\phi=\lambda\vert\phi\vert^k\phi$

Table 3. Reduction by two-dimensional subalgebras of $AG_2(1,2)$. These ansazes and reduced equations are for equation (11).

Subalgebra	Ansatz	ω	Reduced equation
$\langle J_{12}+S+T+2aM, G_1+P_2\rangle$ $(a\in\mathbb{R})$	$v=\frac{1}{\sqrt{t^2+1}}\exp\Big[i\Big(-a\arctan t+\frac{y_1^2}{4t}+\frac{t^2-1}{4t}\Big(\frac{y_1+ty_2}{t^2+1}\Big)^2\Big)\Big]\phi(\omega)$	$\frac{y_1+ay_2}{t^2+1}$	$\ddot\phi+(a-\omega^2)\phi=\lambda\vert\phi\vert^2\phi$
$\langle J_{12}+2aM, S+T+2bM\rangle$ $(a\geq 0,\ b\in\mathbb{R})$	$v=\frac{1}{\sqrt{t^2+1}}\exp\Big[i\Big(-b\arctan t+a\arctan\frac{y_1}{y_2}+\frac{t(y_1^2+y_2^2)}{4(t^2+1)}\Big)\Big]\phi(\omega)$	$\frac{y_1^2+y_2^2}{t^2+1}$	$4\ddot\phi+4\dot\phi+\Big(b-\frac{a^2}{\omega}-\frac{\omega}{4}\Big)\phi=\lambda\vert\phi\vert^2\phi$

Table 4. Reduction by one-dimensional subalgebras of $AG(1,2)$. These ansazes and reduced equations are for equations (9), (10) and (11).

Subalgebra	Ansatz	ω	Reduced equation
$\langle P_2\rangle$	$v=\phi(\omega_1,\omega_2)$	$\omega_1=t,\ \omega_2=y_1$	$i\phi_1+\phi_{22}=\lambda F(\vert\phi\vert)\phi$
$\langle G_2\rangle$	$v=\exp\left(\frac{iy_2^2}{4t}\right)\phi(\omega_1,\omega_2)$	$\omega_1=t,\ \omega_2=y_1$	$i\phi_1+\phi_{22}+\frac{i}{2\omega_1}\phi=\lambda F(\vert\phi\vert)\phi$
$\langle G_1+aP_2\rangle$ $(a>0)$	$v=\exp\left(\frac{iy_1^2}{4t}\right)\phi(\omega_1,\omega_2)$	$\omega_1=t,\ \omega_2=ay_1+ty_2$	$i\phi_1+(\omega_1^2+a^2)\phi_{22}+\frac{i\omega_2}{\omega_1}\phi_2+\frac{i}{2\omega_1}\phi=\lambda F(\vert\phi\vert)\phi$
$\langle T-2aM\rangle$ $(a\in\mathbb{R})$	$v=\exp(iat)\phi(\omega_1,\omega_2)$	$\omega_1=y_1,\ \omega_2=y_2$	$\phi_{11}+\phi_{22}-a\phi=\lambda F(\vert\phi\vert)\phi$
$\langle T+aG_1\rangle$ $(a>0)$	$v=\exp\left(-\frac{a^2t^3}{6}+\frac{aty_1}{2}\right)\phi(\omega_1,\omega_2)$	$\omega_1=at^2-2y_1,\ \omega_2=y_2$	$4\phi_{11}+\phi_{22}+\frac{a\omega_1}{4}\phi=\lambda F(\vert\phi\vert)\phi$
$\langle J_{12}+aT+2bM\rangle$ $(a>0,\ b\in\mathbb{R}$, or $a=0,\ b\geq 0$	$v=\exp(-ibt)\phi(\omega_1,\omega_2)$	$\omega_1=y_1^2+y_2^2$, $\omega_2=a\arctan\frac{y_1}{y_2}+t$	$4\omega_1\phi_{11}+\frac{a^2}{\omega_1}\phi_{22}+4\phi_1+i\phi_2+\phi=\lambda F(\vert\phi\vert)\phi$

Table 5. Reduction by one-dimensional subalgebras of $AG_1(1,2)$. These ansazes and reduced equations are for equations (10) and (11).

Subalgebra	Ansatz	ω	Reduced equation
$\langle D+4aM\rangle$ $(a\in\mathbb{R})$	$v=t^{-(ia+1/k)}\phi(\omega_1,\omega_2)$	$\omega_1=\dfrac{y_1^2}{t}$, $\omega_2=\dfrac{y_2^2}{t}$	$4\omega_1\phi_{11}+4\omega_2\phi_{22}+(2-i\omega_1)\phi_1+$ $+(2-i\omega_2)\phi_2-i\left(ia+\dfrac{1}{k}\right)\phi=\lambda\lvert\phi\rvert^k\phi$
$\langle J_{12}+\frac{1}{2}aD+2abM\rangle$ $(a\geq 0,\ b\in\mathbb{R})$	$v=t^{-(ib+1/k)}\phi(\omega_1,\omega_2)$	$\omega_1=\dfrac{y_1^2+y_2^2}{t}$, $\omega_2=a\arctan\dfrac{y_1}{y_2t}+t$	$4\omega_1\phi_{11}+\dfrac{a^2}{\omega_1}\phi_{22}+(4-i\omega_1)\phi_1+$ $+i\phi_2+\left(b-\dfrac{i}{k}\right)\phi=\lambda\lvert\phi\rvert^k\phi$

Table 6. Reduction by one-dimensional subalgebras of $AG_2(1,2)$. These ansazes and reduced equations are for equation (11).

Subalgebra	Ansatz	ω	Reduced equation
$\langle S+T+2aM\rangle$ $(a\in\mathbb{R})$	$v=\dfrac{1}{\sqrt{t^2+1}}\exp\left[i\left(-a\arctan t+\dfrac{t(y_1^2+y_2^2)}{4(t^2+1)}\right)\right]\phi(\omega_1,\omega_2)$	$\omega_1=\dfrac{y_1^2}{t^2+1}$, $\omega_2=\dfrac{y_2^2}{t^2+1}$	$4\omega_1\phi_{11}+4\omega_2\phi_{22}+2\phi_1+2\phi_2+$ $+\left(a-\dfrac{\omega_1+\omega_2}{4}\right)\phi=\lambda\lvert\phi\rvert^2\phi$
$\langle S+T+aJ_{12}+2bM\rangle$ $(a>0,\ b\in\mathbb{R})$	$v=\dfrac{1}{\sqrt{t^2+1}}\exp\left[i\left(-b\arctan t+\dfrac{t(y_1^2+y_2^2)}{4(t^2+1)}\right)\right]\phi(\omega_1,\omega_2)$	$\omega_1=\dfrac{y_1^2+y_2^2}{t^2+1}$, $\omega_2=\arctan\dfrac{y_1}{y_2}+a\arctan t$	$4\omega_1\phi_{11}+\dfrac{1}{\omega_1}\phi_{22}+4\phi_1+$ $+ia\phi_2+\left(b-\dfrac{\omega_1}{4}\right)\phi=\lambda\lvert\phi\rvert^2\phi$
$\langle S+T+J_{12}+a(G_1+P_2)\rangle$ $(a\geq 0)$	$v=\dfrac{1}{\sqrt{t^2+1}}\exp\left[\dfrac{i}{4}\left(\dfrac{t^2-1}{t}\omega_1^2+\dfrac{y_1^2}{t}\right)\right]\phi(\omega_1,\omega_2)$	$\omega_1=\dfrac{y_1+ty_2}{t^2+1}$, $\omega_2=\dfrac{ty_1-y_2}{t^2+1}-a\arctan t$	$\phi_{11}+\phi_{22}+i(2\omega_1-a)\phi_2-$ $-\omega_1^2\phi=\lambda\lvert\phi\rvert^2\phi$

It has interesting symmetry properties, with its symmetry algebra containing both the Poincaré and Galilei algebras. We intend to return to this equation in future publications.

Finally, let us note that our ansatz relates the Schrödinger equation with any equation related to the wave equation, such as the Dirac equation. Indeed, the Dirac equation is

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0,$$

so that we may represent Ψ as

$$\Psi = (i\gamma^\mu \partial_\mu + m)\Phi, \tag{27}$$

where Φ is a four-component vector of functions satisfying

$$\Box\Phi + m^2\Phi = 0.$$

Clearly, each of the components can be related (independently) to the Schrödinger equation by using our ansatz (7). In this way, we can use (27) to construct solutions of the Dirac equation from the Schrödinger equation. Similarly, we can use the complex heat equation

$$\frac{\partial v}{\partial t} = \Delta v$$

to construct solutions of the Dirac equation. Instead of ansatz (6), which uses the operator M, we have the ansatz

$$\Psi = e^{k(\delta x)} v(\varepsilon x, \alpha x, \beta x), \quad \bar{\Psi} = e^{k(\delta x)} \bar{v}(\varepsilon x, \alpha x, \beta x),$$

which uses the operator L of Theorem 1. Exact solutions of the complex heat equation in 1+2 space-time dimensions can be obtained from those of the real heat equation given in [19]. Thus we see that solutions of the Dirac equation can be obtained from the Schrödinger and heat equations, or a mixture of both.

6 Appendix

In the following tables we give inequivalent ansatzes for equations (9), (10) and (11) constructed from one- and two-dimensional subalgebras of the corresponding algebras of invariance. This is organized as follows: we consider subalgebras in the ascending chain $AG(1,2) \subset AG_1(1,2) \subset AG_2(1,2)$ (strictly speaking, this is incorrect, since the dilatation operator D has a different representation in $AG_1(1,2)$ and $AG_2(1,2)$, but but here we treat the inclusions as abstract Lie algebra inclusions up to isomorphism). In Tables 1, 2 and 3, we give a list of inequivalent two-dimensional subalgebras, with the corresponding ansatzes and reduced equations (these are ordinary differential equations); in Tables 4, 5 and 6, we do the same for one-dimensional subalgebras of the chain, the reduced equations being partial differential equations. The reductions have been verified using MAPLE.

In order to avoid repetition in the reduced equations, we shall, in the following, regard the function F in equation (9) as being arbitrary; in equation (10), k is an arbitrary real number, so that with this convention equation (10) is a particular case of equation (9), and equation (11) is a particular case of equation (10). Further, in performing the symmetry reductions of (9) for arbitrary F, we use the inequivalent subalgebras (of dimensions 1 and 2) of $AG_1(1,2)$ the symmetry reduction of (10) is done using those subalgebras of $AG_2(1,2)$ which are not equivalent to subalgebras of $AG(1,2)$; the reductions of (11) are done with respect to subalgebras of $AG_2(1,2)$ which are not equivalent to subalgebras of $AG_1(1,2)$.

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It is with great sadness that we announce that Professor W.I. Fushchych died on April 7th 1997, after a short illness. This is a tremendous loss for his family, his many students, and for the scientific community. His deep contributions to the field of symmetry analysis of differential equations have made the Kyiv school of symmetries known throughout the world. We take this oportunity to express our deep sense of loss as well as our gratitude for all the encouragement in research Wilhelm Fushchych gave during the years we knew him.

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