Implicit and parabolic ansatzes: some new ansatzes for old equations

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We give a survey of some results on new types of solutions for partial differential equations. First, we describe the method of implicit ansatzes, which gives equations for functions which define implicitly solutions of some partial differential equations. In particular, we find that the family of eikonal equations (in different geometries) has the special property that the equations for implicit ansatzes are also eikonal equations. We also find that the eikonal equation defines implicitly solutions of the Hamilton–Jacobi equation. Parabolic ansatzes are ansatzes which reduce hyperbolic equations to parabolic ones (or to a Schrödinger equation). Their uses in obtaining new types of solutions for equations invariant under $AO(p,q)$ are described. We also give some results on conformally invariant nonlinear wave equations and describe some exact solutions of a conformally invariant nonlinear Schrödinger equation.

1 Introduction

In this talk, I would like to present some results obtained during the past few years in my collaboration with Willy Fushchych and some of his students. The basic themes here are *ansatz* and *symmetry algebras* for partial differential equations.

I wrote this talk after Wilhelm Fushchych' untimely death, but the results I give here were obtained jointly or as a direct result of our collaboration, so it is only right that he appears as an author.

In 1993/1994 during his visits to Linköping and my visits to Kyiv, we managed, amongst other things, to do two things: use light-cone variables to construct new solutions of some hyperbolic equations in terms of solutions of the Schrödinger or heat equations; and to develop the germ of new variation on finding ansatzes. This last piece is an indication of work in progress and it is published here for the first time. I shall begin this talk with this topic first.

2 The method of implicit ansatzes

2.1 The wave and heat equations

Given an equation for one unknown real function (the dependent variable), u, say, and several independent ("geometric") variables, the usual approach, even in terms of symmetries, is to attempt to find ansatzes for u *explicitly*. What we asked was the following: why not try and give u *implicitly*? This means the following: look for some function $\phi(x,u)$ so that $\phi(x,u)=C$ defines u implicitly, where x represents the geometric variables and C is a constant. This is evidently natural, especially if you

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are used to calculating symmetry groups, because one then has to treat u on the same footing as x. If we assume, at least locally, that $\phi_u(x,u) \neq 0$, where $\phi_u = \partial\phi/\partial u$, then the implicit function theorem tells us that $\phi(x,u) = C$ defines u implicitly as a function of x, for some neighbourhood of (x,u) with $\phi_u(x,u) \neq 0$, and that $u_\mu = -\frac{\phi_\mu}{\phi_u}$, where $\phi_\mu = \frac{\partial\phi}{\partial x^\mu}$. Higher derivatives of u are then obtained by applying the correct amount of total derivatives.

The wave equation $\Box u = F(u)$ becomes

$$\phi_u^2 \Box\phi = 2\phi_u\phi_\mu\phi_{\mu u} - \phi_\mu\phi_\mu\phi_{uu} - \phi_u^3 F(u)$$

or

$$\Box\phi = \partial_u\left(\frac{\phi_\mu\phi_\mu}{\phi_u}\right) - \phi_u F(u).$$

This is quite a nonlinear equation. It has exactly the same symmetry algebra as the equation $\Box u = F(u)$, except that the parameters are now arbitrary functions of ϕ. Finding exact solutions of this equation will give u implicitly. Of course, one is entitled to ask what advantages are of this way of thinking. Certainly, it has the disadvantage of making linear equations into very nonlinear ones. The symmetry is not improved in any dramatic way that is exploitable (such as giving a conformally-invariant equation starting from a merely Poincaré invariant one). It can be advantageous when it comes to adding certain conditions. For instance, if one investigates the system

$$\Box u = 0, \quad u_\mu u_\mu = 0,$$

we find that $u_\mu u_\mu = 0$ goes over into $\phi_\mu\phi_\mu = 0$ and the system then becomes

$$\Box\phi = 0, \quad \phi_\mu\phi_\mu = 0.$$

In terms of ordinary Lie ansatzes, this is not an improvement. However, it is not difficult to see that we can make certain non-Lie ansatzes of the anti-reduction type: allow ϕ to be a polynomial in the variable u with coefficients being functions of x. For instance, assume ϕ is a quintic in u: $\phi = Au^5 + Bu + C$. Then we will have the coupled system

$$\Box A = 0, \quad \Box B = 0, \quad \Box C = 0,$$
$$A_\mu A_\mu = B_\mu B_\mu = C_\mu C_\mu = A_\mu B_\mu = A_\mu C_\mu = B_\mu C_\mu = 0.$$

Solutions of *this* system can be obtained using Lie symmetries. The exact solutions of

$$\Box u = 0, \quad u_\mu u_\mu = 0$$

are then obtained in an implicit form which is unobtainable by Lie symmetry analysis alone.

Similarly, we have the system

$$\Box u = 0, \quad u_\mu u_\mu = 1$$

which is transformed into

$$\Box\phi = \phi_{uu}, \quad \phi_\mu\phi_\mu = \phi_u^2$$

or

$$\Box_5\phi = 0, \quad \phi_A\phi_A = 0,$$

where $\Box_5 = \Box - \partial_u^2$ and A is summed from 0 to 4.

It is evident, however, that the extension of this method to a system of equations is complicated to say the least, and I only say that we have not contemplated going beyond the present case of just one unknown function.

We can treat the heat equation $u_t = \triangle u$ in the same way: the equation for the surface ϕ is

$$\phi_t = \triangle\phi - \frac{\partial}{\partial u}\left(\frac{\nabla\phi\cdot\nabla\phi}{\phi_u}\right).$$

If we now add the condition $\phi_u = \nabla\phi\cdot\nabla\phi$, then we obtain the system

$$\phi_t = \triangle\phi, \quad \phi_u = \nabla\phi\cdot\nabla\phi$$

so that ϕ is a solution to both the heat equation and the Hamilton–Jacobi equation, but with different propagation parameters.

If we, instead, add the condition $\phi_u^2 = \nabla\phi\cdot\nabla\phi$, we obtain the system

$$\phi_t = \triangle\phi - \phi_{uu}, \quad \phi_u^2 = \nabla\phi\cdot\nabla\phi.$$

The first of these is a new type of equation: it is a relativistic heat equation with a very large symmetry algebra which contains the Lorentz group as well as Galilei type boosts; the second equation is just the eikonal equation. The system is evidently invariant under the Lorentz group acting in the space parametrized by $(x^1,\ldots,x^n,u)$, and this is a great improvement in symmetry on the original heat equation.

It follows from this that we can obtain solutions to the heat equation using Lorentz-invariant ansatzes, albeit through a modified equation.

2.2 Eikonal equations

Another use of this approach is seen in the following. First, let us note that there are three types of the eikonal equation

$$u_\mu u_\mu = \lambda,$$

namely the time-like eikonal equation when $\lambda = 1$, the space-like eikonal one when $\lambda = -1$, and the isotropic eikonal one when $\lambda = 0$. Representing these implicitly, we find that the time-like eikonal equation in $1+n$ time-space

$$u_\mu u_\mu = 1$$

goes over into the isotropic eikonal one in a space with the metric $(1, \underbrace{-1,\ldots,-1}_{n+1})$

$$\phi_\mu\phi_\mu = \phi_u^2.$$

The space-like eikonal equation

$$u_\mu u_\mu = -1$$

goes over into the isotropic eikonal one in a space with the metric $(1, 1, \underbrace{-1, \ldots, -1}_{n})$

$$\phi_\mu \phi_\mu = -\phi_u^2$$

whereas

$$u_\mu u_\mu = 0$$

goes over into

$$\phi_\mu \phi_\mu = 0.$$

Thus, we see that, from solutions of the isotropic eikonal equation, we can construct solutions of time- and space-like eikonal ones in a space of one dimension less. We also see the importance of studying equations in higher dimensions, in particular in spaces with the relativity groups $SO(1,4)$ and $SO(2,3)$.

It is also possible to use the isotropic eikonal to construct solutions of the Hamilton–Jacobi equation in $1+n$ dimensions

$$u_t + (\nabla u)^2 = 0$$

which goes over into

$$\phi_u \phi_t = (\nabla \phi)^2$$

and this equation can be written as

$$\left(\frac{\phi_u + \phi_t}{2}\right)^2 - \left(\frac{\phi_u - \phi_t}{2}\right)^2 = (\nabla \phi)^2$$

which, in turn, can be written as

$$g^{AB} \phi_A \phi_B = 0$$

with $A, B = 0, 1, \ldots, n+1$, $g^{AB} = \mathrm{diag}\,(1, -1, \ldots, -1)$ and

$$\phi_0 = \frac{\phi_u + \phi_t}{2}, \quad \phi_{n+1} = \frac{\phi_u - \phi_t}{2}.$$

It is known that the isotropic eikonal and the Hamilton–Jacobi equations have the conformal algebra as a symmetry algebra (see [15]), and here we see the reason why this is so. It is not difficult to see that we can recover the Hamilton–Jacobi equation from the isotropic eikonal equation on reversing this procedure.

This procedure of reversal is extremely useful for hyperbolic equations of second order. As an elementary example, let us take the free wave equation for one real function u in $3+1$ space-time:

$$\partial_0^2 u = \partial_1^2 u + \partial_2^2 u + \partial_3^2 u$$

and write it now as

$$(\partial_0 + \partial_3)(\partial_0 - \partial_3) u = \partial_1^2 u + \partial_2^2 u$$

or

$$\partial_\sigma \partial_\tau u = \partial_1^2 u + \partial_2^2 u,$$

where $\sigma = \frac{x^0 - x^3}{2}$, $\tau = \frac{x^0 + x^3}{2}$. Now assume $u = e^\sigma \Psi(\tau, x^1, x^2)$. With this assumption, we find

$$\partial_\tau \Psi = \left(\partial_1^2 + \partial_2^2\right) \Psi$$

which is the heat equation. Thus, we can obtain a class of solutions of the free wave equation from solutions of the free heat equation. This was shown in [1]. The ansatz taken here seems quite arbitrary, but we were able to construct it using Lie point symmetries of the free wave equation. A similar ansatz gives a reduction of the free complex wave equation to the free Schrödinger equation. We have not found a way of reversing this procedure, to obtain the free wave equation from the free heat or Schrödinger equations. The following section gives a brief description of this work.

3 Parabolic ansatzes for hyperbolic equations: light-cone coordinates and reduction to the heat and Schrödinger equations

Although it is possible to proceed directly with the ansatz just made to give a reduction of the wave equation to the Schrödinger equation, it is useful to put it into perspective using symmetries: this will show that the ansatz can be constructed by the use of infinitesimal symmetry operators. To this end, we quote two results:

Theorem 1. *The maximal Lie point symmetry algebra of the equation*

$$\Box u = m^2 u,$$

where u is a real function, has the basis

$$P_\mu = \partial_\mu, \quad I = u\partial_u, \quad J_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$$

when $m \neq 0$, and

$$P_\mu = \partial_\mu, \quad I = u\partial_u, \quad J_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu,$$
$$D = x^\mu \partial_\mu, \quad K_\mu = 2x_\mu D - x^2 \partial_\mu - 2x_\mu u \partial_u$$

when $m = 0$, where

$$\partial_u = \frac{\partial}{\partial u}, \quad \partial_\mu = \frac{\partial}{\partial x^\mu}, \quad x_\mu = g_{\mu\nu} x^\nu,$$
$$g_{\mu\nu} = \mathrm{diag}\,(1, -1, \ldots, -1), \quad \mu, \nu = 0, 1, 2, \ldots, n.$$

We notice that in both cases ($m = 0$, $m \neq 0$), the equation is invariant under the operator I, and is consequently invariant under $\alpha^\mu \partial_\mu + kI$ for all real constants k and real, constant four-vectors α. We choose a hybrid tetradic basis of the Minkowski space: α: $\alpha^\mu \alpha_\mu = 0$; ϵ: $\epsilon^\mu \epsilon_\mu = 0$; β: $\beta^\mu \beta_\mu = -1$; δ: $\delta^\mu \delta_\mu = -1$; and $\alpha^\mu \epsilon_\mu = 1$, $\alpha^\mu \beta_\mu = \alpha^\mu \delta_\mu = \epsilon^\mu \beta_\mu = \epsilon^\mu \delta_\mu = 0$. We could take, for instance, $\alpha = \frac{1}{\sqrt{2}}(1, 0, 0, 1)$,

$\epsilon = \frac{1}{\sqrt{2}}(1,0,0,-1)$, $\beta = (0,1,0,0)$, $\delta = (0,0,1,0)$. Then the invariance condition (the so-called invariant-surface condition),

$$(\alpha^\mu \partial_\mu + kI)u = 0,$$

gives the Lagrangian system

$$\frac{dx^\mu}{\alpha^\mu} = \frac{du}{ku}$$

which can be written as

$$\frac{d(\alpha x)}{0} = \frac{d(\beta x)}{0} = \frac{d(\delta x)}{0} = \frac{d(\epsilon x)}{1} = \frac{du}{ku}.$$

Integrating this gives us the general integral of motion of this system

$$u - e^{k(\epsilon x)}\Phi(\alpha x, \beta x, \delta x)$$

and, on setting this equal to zero, this gives us the ansatz

$$u = e^{k(\epsilon x)}\Phi(\alpha x, \beta x, \delta x).$$

Denoting $\tau = \alpha x$, $y_1 = \beta x$, $y_2 = \delta x$, we obtain, on substituting into the equation $\Box u = m^2 u$,

$$2k\partial_\tau \Phi = \triangle\Phi + m^2\Phi,$$

where $\triangle = \frac{\partial^2}{\partial_{y_1^2}} + \frac{\partial^2}{\partial_{y_2^2}}$. This is just the heat equation (we can gauge away the linear term by setting $\Phi = e^{\frac{m^2\tau}{2k}}\Psi$). The solutions of the wave equation we obtain in this way are given in [1].

The second result is the following:

Theorem 2. *The Lie point symmetry algebra of the equation*

$$\Box\Psi + \lambda F(|\Psi|)\Psi = 0$$

has basis vector fields as follows:

(i) when $F(|\Psi|) = \text{const}\,|\Psi|^2$*:*

$$\partial_\mu, \quad J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu, \quad K_\mu = 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu - 2x_\mu\left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right),$$
$$D = x^\nu\partial_\nu - \left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right), \quad M = i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right),$$

where $x^2 = x_\mu x^\mu$.

(ii) when $F(|\Psi|) = \text{const}\,|\Psi|^k$, $k \neq 0, 2$*:*

$$\partial_\mu, \quad J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu, \quad D_{(k)} = x^\nu\partial_\nu - \frac{2}{k}\left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right),$$
$$M = i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right).$$

(iii) when $F(|\Psi|) \neq \text{const}\,|\Psi|^k$ *for any* k*, but* $\dot{F} \neq 0$*:*

$$\partial_\mu, \quad J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu, \quad M = i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right).$$

(iv) when $F(|\Psi|) = \text{const} \neq 0$*:*

$$\begin{aligned} &\partial_\mu, \quad J_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu, \quad M = i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right), \quad L = \left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right), \\ &L_1 = i\left(\overline{\Psi}\partial_\Psi - \Psi\partial_{\overline{\Psi}}\right), \quad L_2 = \overline{\Psi}\partial_\Psi + \Psi\partial_{\overline{\Psi}}, \quad B\partial_\Psi, \end{aligned}$$

where B *is an arbitrary solution of* $\Box\Psi = F\Psi$*.*

(v) when $F(|\Psi|) = 0$*:*

$$\begin{aligned} &\partial_\mu, \quad J_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu, \quad K_\mu = 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu - 2x_\mu\left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right), \\ &D = x^\mu \partial_\mu, \quad M = i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right), \quad L = \left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right), \\ &L_1 = i\left(\overline{\Psi}\partial_\Psi - \Psi\partial_{\overline{\Psi}}\right), \quad L_2 = \overline{\Psi}\partial_\Psi + \Psi\partial_{\overline{\Psi}}, \quad B\partial_\Psi, \end{aligned}$$

where B *is an arbitrary solution of* $\Box\Psi = 0$*.*

In this result, we see that in all cases we have $M = i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right)$ as a symmetry operator. We can obtain the ansatz

$$\Psi = e^{ik(\epsilon x)}\Phi(\alpha x, \beta x, \delta x)$$

in the same way as for the real wave equation, using M in place of I. However, now we have an improvement in that our complex wave equation may have a nonlinear term which is invariant under M (this is not the case for I). Putting the ansatz into the equation gives us a nonlinear Schrödinger equation:

$$i\partial_\tau \Phi = -\triangle\Phi + \lambda F(|\Phi|)\Phi$$

when $k = -1/2$. Solutions of the hyperbolic equation which this nonlinear Schrödinger equation gives is described in [2] (but it does not give solutions of the free Schrödinger equation).

The above two results show that one can obtain ansatzes (using symmetries) to reduce some hyperbolic equations to the heat or Schrödinger equations. The more interesting case is that of complex wave functions, as this allows some nonlinearities. There is a useful way of characterizing those complex wave equations which admit the symmetry M: if we use the amplitude-phase representation $\Psi = Re^{i\theta}$ for the wave function, then our operator M becomes ∂_θ, and we can then see that it is those equations which, written in terms of R and θ, do not contain any pure θ terms (they are present as derivatives of θ). To see this, we only need consider the nonlinear wave equation again, in this representation:

$$\begin{aligned} &\Box R - R\theta_\mu\theta_\mu + \lambda F(R)R = 0, \\ &R\Box\theta + 2R_\mu\theta_\mu = 0 \end{aligned}$$

when λ and F are real functions. The second equation is easily recognized as the continuity equation:

$$\partial_\mu(R^2\theta_\mu) = 0$$

(it is also a type of conservation of angular momentum). Clearly, the above system does not contain θ other than in terms of its derivatives, and therefore it must admit ∂_θ as a symmetry operator.

Writing an equation in this form has another advantage: one sees that the important part of the system is the continuity equation, and this allows us to consider other systems of equations which include the continuity equation, but have a different first equation. It is a form which can make calculating easier.

Having found the above reduction procedure and an operator which gives us the reducing ansatz, it is then natural to ask if there are other hyperbolic equations which are reduced down to the Schrödinger or diffusion equation. Thus, one may look at hyperbolic equations of the form

$$\Box\Psi = H(\Psi, \Psi^*)$$

which admit the operator M. An elementary calculation gives us that $H = F(|\Psi|)\Psi$. The next step is to allow H to depend upon derivatives:

$$\Box\Psi = F(\Psi, \Psi^*, \Psi_\mu, \Psi^*_\mu)\Psi$$

and we make the assumption that F is real. Now, it is convenient to do the calculations in the amplitude-phase representation, so our functions will depend on R, θ, R_μ, θ_μ. However, if we want the operator M to be a symmetry operator, the functions may not depend on θ although they may depend on its derivatives, so that F must be a function of $|\Psi|$, the amplitude. This leaves us with a large class of equations, which in the amplitude-phase form are

$$\Box R = F(R, R_\mu, \theta_\mu)R, \tag{1}$$

$$R\Box\theta + 2R_\mu\theta_\mu = 0 \tag{2}$$

and we easily find the solution

$$F = F(R, R_\mu R_\mu, \theta_\mu\theta_\mu, R_\mu\theta_\mu)$$

when we also require the invariance under the Poincaré algebra (we need translations for the ansatz and Lorentz transformations for the invariance of the wave operator).

We can ask for the types of systems (1), (2) invariant under the algebras of Theorem 2, and we find:

Theorem 3. *(i) System (1), (2) is invariant under the algebra* $\langle P_\mu, J_{\mu\nu}\rangle$.

(ii) System (1), (2) is invariant under $\langle P_\mu, J_{\mu\nu}, D\rangle$ *with* $D = x^\sigma\partial_\sigma - \frac{2}{k}R\partial_R$, $k \neq 0$ *if and only if*

$$F = R^k G\left(\frac{R_\mu R_\mu}{R^{2+k}}, \frac{\theta_\mu\theta_\mu}{R^k}, \frac{\theta_\mu R_\mu}{R^{1+k}}\right),$$

where G is an arbitrary continuously differentiable function.

(iii) System (1), (2) is invariant under $\langle P_\mu, J_{\mu\nu}, D_0\rangle$ *with* $D_0 = x^\sigma\partial_\sigma$ *if and only if*

$$F = R_\mu R_\mu G\left(R, \frac{\theta_\mu\theta_\mu}{R_\mu R_\mu}, \frac{\theta_\mu R_\mu}{R_\mu R_\mu}\right),$$

where G is an arbitrary continuously differentiable function.

(iv) System (1), (2) is invariant under $\langle P_\mu, J_{\mu\nu}, D, K_\mu\rangle$ *with* $D = x^\sigma\partial_\sigma - R\partial_R$ *and* $K_\mu = 2x_\mu D - x^2\partial_\mu$ *if and only*

$$F = R^2 G\left(\frac{\theta_\mu\theta_\mu}{R^2}\right),$$

where G is an arbitrary continuously differentiable function of one variable.

The last case contains, as expected, case (i) of Theorem 2 when we choose $G(\xi) = \xi - \lambda R^2$. Each of the resulting equations in the above result is invariant under the operator M and so one can use the ansatz defined by M to reduce the equation but we do not always obtain a nice Schrödinger equation. If we ask now for invariance under the operator $L = R\partial_R$ (it is the operator L of case (v), Theorem 2, expressed in the amplitude-phase form), then we obtain some other types of restrictions:

Theorem 4. *(i) System (1), (2) is invariant under* $\langle P_\mu, J_{\mu\nu}, L\rangle$ *if and only if*

$$F = G\left(\frac{R_\mu R_\mu}{R^2}, \theta_\mu\theta_\mu, \frac{R_\mu\theta_\mu}{R}\right).$$

(ii) System (1), (2) is invariant under $\langle P_\mu, J_{\mu\nu}, D_0, L\rangle$ *with* $D_0 = x^\sigma\partial_\sigma$ *if and only if*

$$F = \frac{R_\mu R_\mu}{R^2} G\left(\frac{R^2\theta_\mu\theta_\mu}{R_\mu R_\mu}, \frac{R\theta_\mu R_\mu}{R_\mu R_\mu}\right).$$

(iii) System (1), (2) is invariant under $\langle P_\mu, J_{\mu\nu}, K_\mu, L\rangle$*, where* $K_\mu = 2x_\mu x^\sigma\partial_\sigma - x^2\partial_\mu - 2x_\mu R\partial_R$*, if and only if*

$$F = \kappa\theta_\mu\theta_\mu,$$

where κ *is a constant.*

The last case (iii) gives us the wave equation

$$\Box\Psi = (\kappa - 1)\frac{j_\mu j_\mu}{|\Psi|^4}\Psi,$$

where $j_\mu = \frac{1}{2i}\left[\bar{\Psi}\Psi_\mu - \Psi\bar{\Psi}_\mu\right]$, which is the current of the wave-function Ψ. For $\kappa = 1$, we recover the free complex wave equation. This equation, being invariant under both M and N, can be reduced by the ansatzes they give rise to. In fact, with the ansatz (obtained with L)

$$\Psi = e^{(\epsilon x)/2}\Phi(\alpha x, \beta x, \delta x)$$

with ϵ, α isotropic 4-vectors with $\epsilon\alpha = 1$, and β, δ two space-like orthogonal 4-vectors, the above equation reduces to the equation

$$\Phi_\tau = \triangle\Phi - (\kappa - 1)\frac{\vec{j}\cdot\vec{j}}{|\Phi|^4}\Phi,$$

where $\tau = \alpha x$ and $\triangle = \partial^2/\partial y_1^2 + \partial^2/\partial y_2^2$ with $y_1 = \beta x$, $y_2 = \delta x$, and we have

$$\vec{j} = \frac{1}{2i}[\bar{\Phi}\nabla\Phi - \Phi\nabla\bar{\Phi}].$$

These results show what nonlinearities are possible when we require the invariance under subalgebras of the conformal algebra in the given representation. The above equations are all related to the Schrödinger or heat equation. There are good reasons for looking at conformally invariant equations, not least physically. As mathematical reasons, we would like to give the following examples. First, note that the equation

$$\Box_{p,q}\Psi = 0, \tag{3}$$

where

$$\square_{p,q} = g^{AB}\partial_A\partial_B, \quad A, B = 1, \dots, p, p+1, \dots, p+q$$

with $g^{AB} = \mathrm{diag}(\underbrace{1, \dots, 1}_{p}, \underbrace{-1, \dots, -1}_{q})$, is invariant under the algebra generated by the operators

$$\partial_A, \quad J_{AB} = x_A\partial_B - x_B\partial_A, \quad K_A = 2x_A x^B\partial_B - x^2\partial_A - 2x_A\left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right),$$
$$D = x^B\partial_B, \quad M = i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right), \quad L = \left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right),$$
$$L_1 = i\left(\overline{\Psi}\partial_\Psi - \Psi\partial_{\overline{\Psi}}\right), \quad L_2 = \overline{\Psi}\partial_\Psi + \Psi\partial_{\overline{\Psi}},$$

namely the generalized conformal algebra $AC(p,q) \oplus \langle M, L, L_1, L_2\rangle$ which contains the algebra $ASO(p,q)$. Here, $\oplus$ denotes the direct sum. Using the ansatz which the operator M gives us, we can reduce equation (3) to the equation

$$i\partial_\tau\Phi = \square_{p-1,q-1}\Phi. \tag{4}$$

This equation (4) is known in the literature: it was proposed by Feynman [7] in Minkowski space in the form

$$i\partial_\tau\Phi = (\partial_\mu - A_\mu)(\partial^\mu - A^\mu)\Phi.$$

It was also proposed by Aghassi, Roman and Santilli [8] who studied the representation theory behind the equation. Fushchych and Seheda [9] studied its symmetry properties in the Minkowski space. The solutions of equation (4) give solutions of (3) [14]. We have that equation (4) has a symmetry algebra generated by the following operators

$$T = \partial_\tau, \quad P_A = \partial_A, \quad J_{AB}, \quad G_A = \tau\partial_A - x_A M,$$
$$D = 2\tau\partial_\tau + x^A\partial_A - \frac{p+q-2}{2}L, \quad M = \frac{i}{2}(\Phi\partial_\Phi - \bar{\Phi}\partial_\Phi), \quad L = (\Phi\partial_\Phi + \bar{\Phi}\partial_\Phi),$$
$$S = \tau^2\partial_\tau + \tau x^A\partial_A - \frac{x^2}{2}M - \frac{\tau(p+q-2)}{2}L$$

and this algebra has the structure $[ASL(2,\mathbb{R}) \oplus AO(p-1,q-1)] \uplus \langle L, M, P_A, G_A\rangle$, where $\uplus$ denotes the semidirect sum of algebras. This algebra contains the subalgebra $AO(p-1,q-1) \uplus \langle T, M, P_A, G_A\rangle$ with

$$[J_{AB}, J_{CD}] = g_{BC}J_{AD} - g_{AC}J_{BD} + g_{AD}J_{BC} - g_{BD}J_{AC},$$
$$[P_A, P_B] = 0, \quad [G_A, G_B] = 0, \quad [P_A, G_B] = -g_{AB}M,$$
$$[P_A, J_{BC}] = g_{AB}P_C - g_{AC}P_B, \quad [G_A, J_{BC}] = g_{AB}G_C - g_{AC}G_B,$$
$$[P_A, D] = P_A, \quad [G_A; D] = G_A, \quad [J_{AB}, D] = 0, \quad [P_A, T] = 0, \quad [G_A, T] = 0,$$
$$[J_{AB}, T] = 0, \quad [M, T] = [M, P_A] = [M, G_A] = [M, J_{AB}] = 0,$$

It is possible to show that the algebra with these commutation relations is contained in $AO(p,q)$: define the basis by

$$T = \frac{1}{2}(P_1 - P_q), \quad M = P_1 + P_q, \quad G_A = J_{1A} + J_{qA},$$
$$J_{AB} \quad (A, B = 2, \dots, q-1),$$

and one obtains the above commutation relations. We see now that the algebra $AO(2,4)$ (the conformal algebra $AC(1,3)$) contains the algebra $AO(1,3) \uplus \langle M, P_A, G_A \rangle$ which contains the Poincaré algebra $AP(1,3) = AO(1,3) \uplus \langle P_\mu \rangle$ as well as the Galilei algebra $AG(1,3) = AO(3) \uplus \langle M, P_a, G_a \rangle$ (μ runs from 0 to 3 and a from 1 to 3). This is reflected in the possibility of reducing

$$\Box_{2,4}\Psi = 0$$

to

$$i\partial_\tau \Phi = \Box_{1,3}\Phi$$

which in turn can be reduced to

$$\Box_{1,3}\Phi = 0.$$

4 Two nonlinear equations

In this final section, I shall mention two equations in nonlinear quantum mechanics which are related to each other by our ansatz. They are

$$|\Psi|\Box\Psi - \Psi\Box|\Psi| = -\kappa|\Psi|\Psi \tag{5}$$

and

$$iu_t + \triangle u = \frac{\triangle|u|}{|u|}u. \tag{6}$$

We can obtain equation (6) from equation (5) with the ansatz

$$\Psi = e^{i(\kappa\tau - (\epsilon x)/2)} u(\tau, \beta x, \delta x),$$

where $\tau = \alpha x = \alpha_\mu x^\mu$ and ϵ, α, β, δ are constant 4-vectors with $\alpha^2 = \epsilon^2 = 0$, $\beta^2 = \delta^2 = -1$, $\alpha\beta = \alpha\delta = \epsilon\beta = \epsilon\delta = 0$, $\alpha\epsilon = 1$.

Equation (5), with $\kappa = m^2c^2/\hbar^2$ was proposed by Vigier and Guéret [11] and by Guerra and Pusterla [12] as an equation for de Broglie's double solution. Equation (6) was considered as a wave equation for a classical particle by Schiller [10] (see also [13]).

For equation (5), we have the following result:

Theorem 5 (Basarab-Horwath, Fushchych, Roman [3, 4]). *Equation (5) with $\kappa > 0$ has the maximal point-symmetry algebra $AC(1, n+1) \oplus Q$ generated by operators*

$$P_\mu,\ J_{\mu\nu},\ P_{n+1},\ J_{\mu n+1},\ D^{(1)},\ K_\mu^{(1)},\ K_{n+1}^{(1)},\ Q,$$

where

$$P_\mu = \frac{\partial}{\partial x^\mu}, \quad J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu, \quad P_{n+1} = \frac{\partial}{\partial x^{n+1}} = i(u\partial_u - u^*\partial_{u^*}),$$
$$J_{\mu n+1} = x_\mu P_{n+1} - x_{n+1}P_\mu, \quad D^{(1)} = x^\mu P_\mu + x^{n+1}P_{n+1} - \frac{n}{2}(\Psi\partial_\Psi + \Psi^*\partial_{\Psi^*}),$$
$$K_\mu^{(1)} = 2x_\mu D^{(1)} - (x_\mu x^\mu + x_{n+1}x^{n+1})P_\mu,$$
$$K_{n+1}^{(1)} = 2x_{n+1}D^{(1)} - (x_\mu x^\mu + x_{n+1}x^{n+1})P_{n+1}, \quad Q = \Psi\partial_\Psi + \Psi^*\partial_{\Psi^*},$$

where the additional variable x^{n+1} is defined as

$$x^{n+1} = -x_{n+1} = \frac{i}{2\sqrt{\kappa}} \ln \frac{\Psi^*}{\Psi}, \quad \kappa > 0.$$

For $\kappa < 0$ the maximal symmetry algebra of (9) is $AC(2,n) \oplus Q$ generated by the same operators above, but with the additional variable

$$x^{n+1} = x_{n+1} = \frac{i}{2\sqrt{-\kappa}} \ln \frac{\Psi^*}{\Psi}, \quad \kappa < 0.$$

In this result, we obtain new nonlinear representations of the conformal algebras $AC(1, n+1)$ and $AC(2,n)$. It is easily shown (after some calculation) that equation (5) is the only equation of the form

$$\Box u = F(\Psi, \Psi^*, \nabla\Psi, \nabla\Psi^*, \nabla|\Psi|\nabla|\Psi|, \Box|\Psi|)\Psi$$

invariant under the conformal algebra in the representation given in Theorem 5. This raises the question whether there are equations of the same form conformally invariant in the standard representation

$$P_\mu = \frac{\partial}{\partial x^\mu}, \quad J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu,$$
$$D = x^\mu P_\mu - \frac{n-1}{2}(\Psi\partial_\Psi + \Psi^*\partial_{\Psi^*}), \quad K_\mu = 2x_\mu D - x^2 P_\mu.$$

There are such equations [3] and [4], for instance:

$$\Box\Psi = |\Psi|^{4/(n-1)} F\left(|\Psi|^{(3+n)/(1-n)}\Box|\Psi|\right)\Psi, \qquad n \neq 1,$$
$$\Box u = \Box|u| F\left(\frac{\Box|u|}{(\nabla|u|)^2}, |u|\right) u, \qquad n = 1,$$
$$4\Box\Psi = \left\{\frac{\Box|\Psi|}{|\Psi|} + \lambda\frac{(\Box|\Psi|)^n}{|\Psi|^{n+4}}\right\}\Psi, \quad n \text{ arbitrary},$$
$$\Box\Psi = (1+\lambda)\frac{\Box|\Psi|}{|\Psi|}\Psi,$$
$$\Box\Psi = \frac{\Box|\Psi|}{|\Psi|}\left(1 + \frac{\lambda}{|\Psi|^4}\right)\Psi,$$
$$\Box\Psi = \frac{\Box|\Psi|}{|\Psi|}\left(1 + \frac{\lambda}{1+\sigma|\Psi|^4}\right)\Psi.$$

Again we see how the representation dictates the equation.

We now turn to equation (6). It is more convenient to represent it in the amplitude-phase form $u = Re^{i\theta}$:

$$\theta_t + \nabla\theta \cdot \nabla\theta = 0, \tag{7}$$

$$R_t + \triangle\theta + 2\nabla\theta \cdot \nabla R = 0. \tag{8}$$

Its symmetry properties are given in the following result:

Theorem 6 (Basarab-Horwath, Fushchych, Lyudmyla Barannyk [5, 6]). *The maximal point-symmetry algebra of the system of equations (7), (8) is the algebra with basis vector fields*

$$
\begin{aligned}
&P_t = \partial_t, \quad P_a = \partial_a, \quad P_{n+1} = \frac{1}{2\sqrt{2}}(2\partial_t - \partial_\theta), \quad N = \partial_R,\\
&J_{ab} = x_a\partial_b - x_b\partial_a, \quad J_{0\,n+1} = t\partial_t - \theta\partial_\theta,\\
&J_{0a} = \frac{1}{\sqrt{2}}\left(x_a\partial_t + (t+2\theta)\partial_{x_a} + \frac{1}{2}x_a\partial_\theta\right),\\
&J_{a\,n+1} = \frac{1}{\sqrt{2}}\left(-x_a\partial_t + (t-2\theta)\partial_{x_a} + \frac{1}{2}x_a\partial_\theta\right),\\
&D = -\left(t\partial_t + x_a\partial_a + \theta\partial_\theta - \frac{n}{2}\partial_R\right),\\
&K_0 = \sqrt{2}\left(\left(t + \frac{\vec{x}^2}{2}\right)\partial_t + (t+2\theta)x_a\partial_{x_a} + \left(\frac{\vec{x}^2}{4} + 2\theta^2\right)\partial_\theta - \frac{n}{2}(t+2\theta)\partial_R\right),\\
&K_{n+1} = -\sqrt{2}\left(\left(t - \frac{\vec{x}^2}{2}\right)\partial_t + (t-2\theta)x_a\partial_{x_a} + \left(\frac{\vec{x}^2}{4} - 2\theta^2\right)\partial_\theta - \frac{n}{2}(t-2\theta)\partial_R\right),\\
&K_a = 2x_aD - (4t\theta - \vec{x}^2)\partial_{x_a}.
\end{aligned}
$$

The above algebra is equivalent to the extended conformal algebra $AC(1,n+1)\oplus\langle N\rangle$. In fact,with new variables

$$
x_0 = \frac{1}{\sqrt{2}}(t+2\theta), \quad x_{n+1} = \frac{1}{\sqrt{2}}(t-2\theta) \tag{9}
$$

the operators in Theorem 1 can be written as

$$
\begin{aligned}
&P_\alpha = \partial_\alpha, \quad J_{\alpha\beta} = x_\alpha\partial_\beta - x_\beta\partial_\alpha, \quad N = \partial_R,\\
&D = -x_\alpha\partial_\alpha + \frac{n}{2}N, \quad K_\alpha = -x_\alpha D - (x_\mu x^\mu)\partial_\alpha.
\end{aligned} \tag{10}
$$

Exact solutions of system (7), (8) using symmetries have been given in [5] and in [6]. Some examples of solutions are the following (we give the subalgebra, ansatz, and the solutions):
$\boldsymbol{A_1} = \langle J_{12} + dN, P_3 + N, P_4\rangle$ $(d \ge 0)$
Ansatz:

$$
\theta = -\frac{1}{2}t + f(\omega), \quad R = x_3 - d\arctan\left(\frac{x_1}{x_2}\right) + g(\omega), \quad \omega = x_1^2 + x_2^2.
$$

Solution:

$$
\begin{aligned}
&\theta = -\frac{1}{2}t + \varepsilon\sqrt{\frac{x_1^2+x_2^2}{2}} + C_1, \quad \varepsilon = \pm 1,\\
&R = x_3 + d\arctan\left(\frac{x_1}{x_2}\right) - \frac{1}{4}\ln(x_1^2+x_2^2) + C_2,
\end{aligned}
$$

where C_1, C_2 are constants.
$\boldsymbol{A_4} = \langle J_{04} + dN, J_{23} + d_2N, P_2 + P_3\rangle$
Ansatz:

$$
\theta = \frac{1}{t}f(\omega), \quad R = d\ln|t| + g(\omega), \quad \omega = x_1.
$$

Solution:

$$\theta=\frac{(x_1+C_1)^2}{4t},\quad R=d\ln|t|-\left(d+\frac{1}{2}\right)\ln|x_1+C_1|+C_2.$$

$\boldsymbol{A_9}=\langle J_{01},J_{02},J_{03},J_{12},J_{13},J_{23}\rangle$
Ansatz:

$$\theta=\frac{1}{4t}f(\omega)+\frac{x_1^2+x_2^2+x_3^2}{4t},\quad R=g(\omega),\quad \omega=\theta-\frac{1}{2}t.$$

Solution:

$$\theta=\frac{\vec{x}^2-4C_1t+8C_1^2}{4t-8C_1},\quad R=-\frac{3}{2}\ln\left|\frac{\vec{x}^2-2(t-2C_1)^2}{t-2C_1}\right|+C_2.$$

$\boldsymbol{A_{14}}=\langle J_{04}+a_1N,D+a_2N,P_3\rangle$, ($a_1,a_2$ arbitrary)
Ansatz:

$$\theta=\frac{x_1^2}{t}f(\omega),\quad R=g(\omega)+a_1\ln|t|-\left(a_1+a_2+\frac{3}{2}\right)\ln|x_1|,\quad \omega=\frac{x_1}{x_2}.$$

Solution:

$$\theta=\frac{x_1^2}{t},\quad R=a_1\ln|t|+\left(a_2-a_1+\frac{1}{2}\right)\ln|x_1|-2(a_2+1)\ln|x_2|+C.$$

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