Reduction of self-dual Yang–Mills equations with respect to subgroups of the extended Poincaré group

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For the vector potential of the Yang–Mills field in the Minkowski space $R(l,3)$, we construct the ansatze that are invariant under three-parameter subgroups of the extended Poincaré group $\tilde{P}(1,3)$. We perform the symmetry reduction of self-dual Yang–Mills equations to systems of ordinary differential equations.

1 Introduction

Classical $SU(2)$-invariant Yang–Mills equations (YME) comprise a system of twelve nonlinear partial differential equations (PDE) of the second order in the Minkowski space $R(1,3)$. On the other hand, once the Yang–Mills potentials satisfy the self-duality conditions, the YME are automatically satisfied. This allows one to construct a broad subclass of solutions to the YME using the condition of self-duality, which amounts to a system of nine first-order PDE,

$$F_{\mu\nu} = \frac{i}{2}\varepsilon_{\mu\nu\gamma\delta}F^{\gamma\delta}, \tag{1}$$

where $F_{\mu\nu} = \partial^\mu \vec{A}_\nu - \partial^\nu \vec{A}_\mu + e\vec{A}_\mu \times \vec{A}_\nu$ is the Yang–Mills strength-tensor, $\varepsilon_{\mu\nu\gamma\delta}$ is the rank-four antisymmetric tensor, and e is the gauge coupling constant, with $\mu,\nu,\gamma,\delta, = \overline{0,3}$. Equations (1) are called the self-dual Yang–Mills equations (SDYME).

Self-duality properties have allowed exact solutions to YME to be explicitly constructed, starting with the ansatze for the Yang–Mills fields proposed by Wu and Yang, Rosen, 't Hooft, Corrigan and Fairlie, Wilczek, and Witten. One should also note the Atiyah–Drinfeld–Hitchin–Manin construction that has been applied in the construction of instanton solutions to YME (see reviews [1, 2] and the bibliographies cited therein).

Recently, increasing interest has been given to SDYME and the corresponding Lax pairs in the Euclidean space $R(4)$ in view of the possibility of reducing them to classical integrable equations (Euler–Arnold, Burgers, Kadomtsev–Petviashvili, Liouville, and others). This problem was considered, in particular, in [3–5], where reduction with respect to translations was performed. In [6], SDYME were reduced with respect to all subgroups of the Euclidean group $E(4)$, while in [7, 8], SDYME and the corresponding Lax pairs in four- dimensional Minkowski space with the signature $(++--)$ were reduced with respect to Abelian subgroups of the Poincaré group $P(2,2)$.

In this paper, we continue our investigation of the problem of the symmetry reduction of YME and SDYME in the Minkowski space $R(1,3)$. It is known [9] that the maximal symmetry group (according to Lie) of the YME is the group $C(1,3)\otimes SU(2)$;

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this group also preserves SDYME (1). The presence of high symmetry allows one to apply the method of symmetry reduction [10, 11] to the equations and, further, to obtain exact solutions. Several conformally invariant solutions of YME were found in [12] (see, also, [13]). A systematic investigation of conformally invariant reductions of YME and SDYME was initiated in [14, 15], where YME and SDYME (1) were reduced, with respect to three-parameter subgroups of the Poincaré group $P(1,3)$, to systems of ordinary differential equations (ODE) and new solutions to the YME were constructed. The unified form of the $P(1,3)$-invariant ansatze made it possible [16] to perform a direct reduction of the YME to systems of ODE and to obtain conditionally invariant solutions of the YME. In this paper, we consider the symmetry reduction of SDYME (1) to systems of ODE that correspond to three-parameter subgroups of the extended Poincaré group $\tilde{P}(1,3)$.

The paper is organized as follows. In Section 2, we consider the general procedure for constructing linear ansatze. Section 3 is devoted to the derivation of the unified form of $\tilde{P}(1,3)$-invariant ansatze and to the reduction of SDYME (1) to systems of ODE. In the last section, we consider some of the reduced systems and obtain exact real solutions of (1).

2 Linear form of $\tilde{P}(1,3)$-invariant ansatze

As noted above, SDYME (1) are invariant under the conformal group $C(1,3)$, in which the generators

$$\begin{aligned} &P_\mu = \partial_\mu, \quad J_{\mu\nu} = x^\mu\partial_\nu - x^\nu\partial_\mu + A^{m\mu}\frac{\partial}{\partial A^m_\nu} - A^{m\nu}\frac{\partial}{\partial A^m_\mu}, \\ &D = x_\mu\partial_\mu - A^m_\mu\frac{\partial}{\partial A^m_\mu}, \end{aligned} \tag{2}$$

span a subgroup isomorphic to the extended Poincaré group $\tilde{P}(1,3)$. Here, $\partial_\mu = \frac{\partial}{\partial x_\mu}$, with $\mu,\nu = \overline{0,3}$ and $m,n = \overline{0,3}$. Here and henceforth, we sum over repeated indices (from 0 to 3 for the indices $\mu,\nu,\gamma,\delta,\sigma = \overline{0,3}$, and from 1 to 3 for $m,n = \overline{1,3}$). The indices μ,ν,γ,δ, and σ are raised and lowered by the metric tensor $g_{\mu\nu} = \operatorname{diag}(1,-1,-1,-1)$.

Let $A\tilde{P}(1,3)$ be the extended Poincaré algebra whose basis is given by generators (2) and let $A\tilde{P}(1,3)$ be the extended Poincaré algebra generated by the vector fields

$$P^{(1)}_\mu = \partial_\mu, \quad J^{(1)}_{\mu\nu} = x^\mu\partial_\nu - x^\nu\partial_\mu, \quad D = x_\mu\partial_\mu.$$

In the classical approach, due to Lie [10, 11], symmetry reduction of SDYME (1) to systems of ODE is associated with those subalgebras L of $A\tilde{P}(1,3)$ that satisfy the condition $r = r^{(1)} = 3$, where r is the rank of L and $r^{(1)}$ is the rank of the projection of L onto $A\tilde{P}^{(1)}(1,3)$. As can be easily seen, we have $\dim L = r = 3$, which means that in order to perform the reduction, we need to know the three-dimensional subalgebras of $A\tilde{P}(1,3)$ satisfying the above condition. Taking into account that SDYME (1) are invariant under the conformal group $C(1,3)$, we can restrict ourselves to the three-dimensional subalgebras of $A\tilde{P}(1,3)$ determined up to conformal conjugation. Such subalgebras of the $A\tilde{P}(1,3)$ algebra are known [17, 18]. Since the case of the Poincaré algebra $AP(1,3)$ has been considered in [14, 15], we limit ourselves to those subalgebras of $A\tilde{P}(1,3)$ that are not $C(1,3)$-conjugates to the subalgebras of $AP(1,3)$. We use the results and notation of [18], in particular, the fact that the list of three-dimensional subalgebras of $A\tilde{P}(1,3)$ that are not conjugate to the three-dimensional subalgebras of $AP(1,3)$ is exhausted, up to $C(1,3)$-conjugation, by the following algebras:

$$\begin{aligned}
&L_1 = \langle D, P_0, P_3\rangle, && L_2 = \langle J_{12} + \alpha D, P_0, P_3\rangle,\\
&L_3 = \langle J_{12}, D, P_0\rangle, && L_4 = \langle J_{12}, D, P_3\rangle,\\
&L_5 = \langle J_{03} + \alpha D, P_0, P_3\rangle, && L_6 = 2\langle J_{03} + \alpha D, P_1, P_2\rangle,\\
&L_7 = \langle J_{03} + \alpha D, M, P_1\rangle\ (\alpha \neq 0), && L_8 = \langle J_{03} + D + 2T, P_1, P_2\rangle,\\
&L_9 = \langle J_{02} + D + 2T, M, P_1\rangle, && L_{10} = \langle J_{03}, D, P_1\rangle,\\
&L_{11} = \langle J_{03}, D, M\rangle, && L_{12} = \langle J_{12} + \alpha J_{03} + \beta D, P_0, P_3\rangle,\\
&L_{13} = \langle J_{12} + \alpha J_{03} + \beta D, P_1, P_2\rangle, &&\\
&L_{14} = \langle J_{12} + \alpha(J_{03} + D + 2T), P_1, P_2\rangle, && L_{15} = \langle J_{12} + \alpha J_{03}, D, M\rangle,\\
&L_{16} = \langle J_{03} + \alpha D, J_{12} + \beta D, M\rangle, \quad (0 \le |\alpha| \le 1, \beta \ge 0, |\alpha| + |\beta| \neq 0), &&\\
&L_{17} = \langle J_{03} + D + 2T, J_{12} + \alpha T, M\rangle \quad (\alpha \ge 0), &&\\
&L_{18} = \langle J_{03} + D, J_{12} + 2T, M\rangle, && L_{19} = \langle J_{03}, J_{12}, D\rangle,\\
&L_{20} = \langle G_1, J_{03} + \alpha D, P_2\rangle \quad (0 < |\alpha| \le 1), && L_{21} = \langle J_{03} + D, G_1 + P_2, M\rangle,\\
&L_{22} = \langle J_{03} - D + M, G_1, P_2\rangle, && L_{23} = \langle J_{03} + 2D, G_1 + 2T, M\rangle,\\
&L_{24} = \langle J_{03} + 2D, G_1 + 2T, P_2\rangle. &&
\end{aligned} \tag{3}$$

Here, $M = P_0+P_3$, $G_1 = J_{01}-J_{13}$, and $T = \frac{1}{2}(P_0-P_3)$; also, $\alpha, \beta > 0$ unless explicitly stated otherwise. In what follows, α and β take on the values given in list (3).

Note that all of the subalgebras L_j ($j = \overline{1,24}$) satisfy the condition $r = r^{(1)} = 3$.

Let us demonstrate that, similar to [14, 15, 19], the ansatz for the $\vec{A}_\mu$ fields can be taken, without any loss of generality, in the linear form

$$\vec{A}_\mu(x) = \Lambda(x)\vec{B}_\mu(\omega), \tag{4}$$

where $\Lambda(x)$ is a known square nondegenerate order-12 matrix and $\vec{B}_\mu(\omega)$ are new unknown vector-functions of the independent variable $\omega = \omega(x)$, with $x = (x_0, x_1, x_2, x_3) \in R(1,3)$.

Obviously, the fact that the sought for ansatz is linear requires that the algebra L_j contain an invariant $\omega(x)$ independent of $\vec{A}_\mu$, as well as twenty linear invariants of the form

$$f^m_{\mu 0}(x)A^m_0 + f^m_{\mu 1}(x)A^m_1 + f^m_{\mu 2}(x)A^m_2 + f^m_{\mu 3}(x)A^m_3,$$

which are functionally dependent as functions of A^m_0, A^m_1, A^m_2, and A^m_3. These invariants can be considered as components of a vector $F\vec{A}$, where $F = (f^m_{\mu\nu}(x))$, while

$$\vec{A} = \begin{pmatrix} \vec{A}_0 \\ \vec{A}_1 \\ \vec{A}_2 \\ \vec{A}_3 \end{pmatrix}.$$

Here, the matrix F is nondegenerate in some domain in $R(1,3)$. According to the theorem on the conditional existence of invariant solutions [11], the ansatz $F\vec{A} = \vec{B}(\omega)$ results in a reduction of system (1) to a system of ODE that relates the independent variable ω, the sought for functions B_μ^m, and the first derivatives thereof. Setting $\Lambda = F^{-1}(x)$, we arrive at ansatz (4).

Let $L = \langle X_1, X_2, X_3\rangle$ be one of the subalgebras of $A\tilde{P}(1,3)$ from list (3), with X_k being an operator of form (2), i.e.,

$$X_k = \xi_{km}(x)\partial_\mu + \rho_{m\sigma\lambda}(x)A_\lambda^m \frac{\partial}{\partial A_\sigma^m} \quad (k = 1, 2, 3).$$

The function $f_{\delta\gamma}^n(x)A_\gamma^n$ is an invariant of the operator X_k if and only if

$$\xi_{k\mu}(x)\frac{\partial f_{\delta\gamma}^n(x)}{\partial x_\mu}A_\gamma^n + \rho_{k\sigma\lambda}(x)A_\lambda^n f_{\delta\sigma}^n(x) = 0$$

or

$$\xi_{k\mu}(x)\frac{\partial f_{\delta\gamma}^n(x)}{\partial x_\mu} + f_{\delta\sigma}^n(x)\rho_{k\sigma\gamma}(x) = 0 \tag{5}$$

for all values of γ. Let $F(x) = (f_{\delta\sigma}^n(x))$ and $\Gamma_k(x) = (\rho_{k\sigma\gamma}(x))$ be square matrices of order 12. Then the second term on the left-hand side of (5) is an element of the matrix $F(x)\Gamma_k(x)$.

These observations lead us to the following theorem.

Theorem 1. *The system of functions $f_{\delta\gamma}^n(x)A_\gamma^n$ is a system of functional invariants of a subalgebra L if and only if $F = (f_{\delta\sigma}^n(x))$ is a nondegenerate matrix in some domain of $R(1,3)$ and satisfies the system of equations*

$$\xi_{k\mu}(x)\frac{\partial F(x)}{\partial x_\mu} + F(x)\Gamma_k(x) = 0 \quad (k = 1, 2, 3). \tag{6}$$

Similarly, the function $\omega(x)$ is an invariant of the operator X_k if and only if $X_k\omega = 0$, i.e.,

$$\xi_{k\mu}(x)\frac{\partial \omega}{\partial x_\mu} = 0. \tag{7}$$

Since all of the algebras L_j satisfy the condition

$$\operatorname{rank}\|\xi_{k\mu}(x)\| = 3,$$

systems (6) and (7) are compatible.

Theorem 1 assigns a matrix Γ_k to every generator X_k of the subalgebra L of $A\tilde{P}(1,3)$. Let us indicate the explicit form of these matrices for all generators (2) of the algebra $A\tilde{P}(1,3)$.

Since the operator P_μ is independent of $\frac{\partial}{\partial A_\mu^m}$, the corresponding Γ is a zero matrix. Denote by $-S_{\mu\nu}$ the Γ-matrix that corresponds to the operator $J_{\mu\nu}$. It is easy to verify that

$$S_{01} = \begin{pmatrix} 0 & -I & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad S_{02} = \begin{pmatrix} 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_{03} = \begin{pmatrix} 0 & 0 & 0 & -I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 \end{pmatrix}, \quad S_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix}, \quad S_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \end{pmatrix},$$

where 0 is the zero and I is the unit matrix of order 3.

The D operator corresponds to the matrix $-E$, where E is the unit order-12 matrix.

The above matrices determine a matrix representation of the algebra $A\tilde{Q}(1,3) = AQ(1,3) \oplus \langle D\rangle$, because

$$[S_{\mu\nu}, S_{\delta\gamma}] = g_{\mu\gamma}S_{\nu\delta} + g_{\nu\delta}S_{\mu\gamma} - g_{\mu\delta}S_{\nu\gamma} - g_{\nu\gamma}S_{\mu\delta}, \quad [E, S_{\mu\nu}] = 0.$$

Let $a = (1,0,0,0)$, $b = (0,1,0,0)$, $c = (0,0,1,0)$, $d = (0,0,0,1)$, and $k = a + d$. Denote by a_μ, b_μ, c_μ, and d_μ, the μth component of the vectors a, b, c, and d, respectively. Then,

$$x_0 = ax = a_\mu x^\mu, \quad x_1 = -bx = -b_\mu x^\mu,$$
$$x_2 = -cx = -c_\mu x^\mu, \quad x_3 = -dx = -d_\mu x^\mu.$$

Theorem 2. *For every subalgebra L_j $(j = 1, \ldots, 24)$ from list (3), there exists a linear ansatz (4), in which ω is a solution to system (7) and*

$$\Lambda^{-1} = \exp\{-\log\theta E\}\exp\{\theta_0 S_{03}\}\exp\{-\theta_1 S_{12}\}\exp\{-2\theta_2(S_{01} - S_{13})\}.$$

Moreover, the functions θ, θ_0, θ_1, θ_2 and ω can be represented as follows:

$L_1:\quad \theta = |bx|^{-1}, \quad \theta_0 = \theta_1 = \theta_2 = 0, \quad \omega = cx(bx)^{-1},$

$L_2:\quad \theta = \Psi_1^{-\frac{1}{2}}, \quad \theta_0 = \theta_2 = 0, \quad \theta_1 = \Phi, \quad \omega = \log\Psi_1 + 2\Phi,$

$L_3:\quad \theta = |dx|^{-1}, \quad \theta_0 = \theta_2 = 0, \quad \theta_1 = \Phi, \quad \omega = \Psi_1(dx)^{-2},$

$L_4:\quad \theta = |ax|^{-1}, \quad \theta_0 = \theta_2 = 0, \quad \theta_1 = \Phi, \quad \omega = \Psi_1(ax)^{-2},$

$L_5:\quad \theta = |bx|^{-1}, \quad \theta_0 = \alpha^{-1}\log|bx|, \quad \theta_1 = \theta_2 = 0, \quad \omega = cx(bx)^{-1},$

$L_6:\quad \theta = |\Psi_2|^{-\frac{1}{2}}, \quad \theta_0 = \dfrac{1}{2}\log|(ax - dx)(kx)^{-1}|, \quad \theta_1 = \theta_2 = 0,$
$\qquad \omega = (1-\alpha)\log|ax - dx| + (1+\alpha)\log|kx|,$

$L_7:\quad \theta = |cx|^{-1}, \quad \theta_0 = \alpha^{-1}\log|cx|, \quad \theta_1 = \theta_2 = 0, \quad \omega = |kx|^{\alpha}|cx|^{1-\alpha},$

$L_8:\quad \theta = |ax - dx|^{-\frac{1}{2}}, \quad \theta_0 = \dfrac{1}{2}\log|ax - dx|, \quad \theta_1 = \theta_2 = 0,$
$\qquad \omega = kx - \log|ax - dx|,$

$L_9:\quad \theta = |cx|^{-1}, \quad \theta_0 = \log|cx|, \quad \theta_1 = \theta_2 = 0, \quad \omega = kx - 2\log|cx|,$

$L_{10}:\quad \theta = |cx|^{-1}, \quad \theta_0 = \log|(ax - dx)(cx)^{-1}|, \quad \theta_1 = \theta_2 = 0,$
$\qquad \omega = \Psi_2(cx)^{-2},$

$L_{11}:\quad \theta = |cx|^{-1}, \quad \theta_0 = -\log|(kx(cx)^{-1}|, \quad \theta_1 = \theta_2 = 0, \quad \omega = cx(bx)^{-1},$

$$L_{12}:\quad \theta=\Psi_1^{-\frac12},\quad \theta_0=-\alpha\Phi,\quad \theta_1=\Phi,\quad \theta_2=0,\quad \omega=\log\Psi_1+2\beta\Phi,$$

$$L_{13}:\quad \theta=|\Psi_2|^{-\frac12},\quad \theta_0=\frac12\log|(ax-dx)(kx)^{-1}|,$$
$$\theta_1=-\frac{1}{2\alpha}\log|(ax-dx)(kx)^{-1}|,\quad \theta_2=0,$$
$$\omega=(\alpha-\beta)\log|ax-dx|+(\alpha+\beta)\log|kx|,$$

$$L_{14}:\quad \theta=|ax-dx|^{-\frac12},\quad \theta_0=\frac12\log|ax-dx|,\quad \theta_1=-\frac12\log|ax-dx|,$$
$$\theta_2=0,\quad \omega=kx-\log|ax-dx|,$$

$$L_{15}:\quad \theta=\Psi_1^{-\frac12},\quad \theta_0=-\alpha\Phi,\quad \theta_1=\Phi,\quad \theta_2=0,\quad \omega=\log[\Psi_1(kx)^{-2}]+2\alpha\Phi,$$

$$L_{16}:\quad \theta=\Psi_1^{-\frac12},\quad \theta_0=\frac12\log|\Psi_1(kx)^{-2}],\quad \theta_1=\Phi,\quad \theta_2=0,$$
$$\omega=\log[\Psi_1^{1-\alpha}(kx)^{2\alpha}]+2\beta\Phi,$$

$$L_{17}:\quad \theta=\Psi_1^{-\frac12},\quad \theta_0=\frac12\log\Psi_1,\quad \theta_1=\Phi,\quad \theta_2=0,\quad \omega=kx-\log\Psi_1+2\alpha\Phi,$$

$$L_{18}:\quad \theta=\Psi_1^{-\frac12},\quad \theta_0=\frac12\log\Psi_1,\quad \theta_1=\Phi,\quad \theta_2=0,\quad \omega=kx+2\Phi,$$

$$L_{19}:\quad \theta=\Psi_1^{-\frac12},\quad \theta_0=-\frac12\log|kx(ax-dx)^{-1}|,\quad \theta_1=\Phi,\quad \theta_2=0,$$
$$\omega=\Psi_1|\Psi_2|^{-1},$$

$$L_{20}:\quad \theta=|\Psi_3|^{-\frac12},\quad \theta_0=\frac{1}{2\alpha}\log|\Psi_3|,\quad \theta_1=0,\quad \theta_2=\frac12 bx(kx)^{-1},$$
$$\omega=|kx|^{2\alpha}|\Psi_3|^{1-\alpha},$$

$$L_{21}:\quad \theta=|cxkx-bx|^{-1},\quad \theta_0=\log|cxkx-bx|^{-1},\quad \theta_1=0,\quad \theta_2=\frac12 cx,$$
$$\omega=kx,$$

$$L_{22}:\quad \theta=|kx|^{-\frac12},\quad \theta_0=-\frac12\log|kx|,\quad \theta_1=0,\quad \theta_2=\frac12 bx(kx)^{-1},$$
$$\omega=ax-dx+\log|kx|-(bx)^2(kx)^{-1},$$

$$L_{23}:\quad \theta=|cx|^{-1},\quad \theta_0=\frac12\log|cx|,\quad \theta_1=0,\quad \theta_2=-\frac14 kx,$$
$$\omega=[4bx+(kx)^2](cx)^{-1},$$

$$L_{24}:\quad \theta=|4bx+(kx)^2|^{-1},\quad \theta_0=\frac12\log|4bx+(kx)^2|,\quad \theta_1=0,\quad \theta_2=-\frac14 kx,$$
$$\omega=\left[ax-dx+bxkx+\frac16(kx)^3\right]^2[4bx+(kx)^2]^{-3}.$$

Here, $\Phi=\arctan\frac{cx}{bx}$, $\Psi_1=(bx)^2+(cx)^2$, $\Psi_2=(ax)^2-(dx)^2$, *and* $\Psi_3=(ax)^2-(bx)^2-(dx)^2$.

Proof. All of the cases are analyzed similarly, so we can limit ourselves to the subalgebra $L_2=\langle J_{12}+\alpha D,P_0,P_3\rangle$.

According to Theorem 1, the entries of column $\Lambda^{-1}\vec{A}$ are invariants of the subalgebra L_2 if and only if

$$-x_1\frac{\partial\Lambda}{\partial x_2}+x_2\frac{\partial\Lambda}{\partial x_1}+\alpha\left(x_\mu\frac{\partial\Lambda}{\partial x_\mu}\right)-\Lambda(S_{12}+\alpha E)=0,\quad \frac{\partial\Lambda}{\partial x_0}=0,\quad \frac{\partial\Lambda}{\partial x_3}=0.(8)$$

The last two equations in (8) demonstrate that $\Lambda = \Lambda(x_1, x_2)$, while the first equation implies that one can set $\theta_0 = \theta_2 = 0$ in the expression for Λ. By the Campbell–Hausdorff formula, we have, in this case,

$$\xi_\mu \frac{\partial \Lambda}{\partial x_\mu} = -\Lambda \xi_\mu \left(\frac{\partial \theta}{\partial x_\mu} + \frac{\partial \theta_1}{\partial x_\mu} \right).$$

Hence, the common factor of Λ can be canceled from the left on the left-hand side of the first equation in (8), which gives an equation whose left-hand side can be represented as a combination of the matrices E and S_{12}. Equating the coefficients in these combinations to zero, we arrive at the system of equations below:

$$\begin{gathered} \frac{1}{\theta}\left\{ x_1 \frac{\partial \theta}{\partial x_2} - x_2 \frac{\partial \theta}{\partial x_1} - \alpha \left(x_1 \frac{\partial \theta}{\partial x_1} + x_2 \frac{\partial \theta}{\partial x_2} \right) \right\} - \alpha = 0, \\ x_1 \frac{\partial \theta_1}{\partial x_2} - x_2 \frac{\partial \theta_1}{\partial x_1} - \alpha \left(x_1 \frac{\partial \theta_1}{\partial x_1} + x_2 \frac{\partial \theta_1}{\partial x_2} \right) - 1 = 0, \end{gathered} \tag{9}$$

which is equivalent to (8). It is not difficult to verify that system (9) is satisfied by the functions

$$\theta = (x_1^2 + x_2^2)^{-\frac{1}{2}} = [(bx)^2 + (cx)^2]^{-\frac{1}{2}}, \quad \theta_1 = \arctan \frac{x_2}{x_1} = \arctan \frac{cx}{bx}.$$

Equations (7) for $\omega(x)$ are of the form

$$-x_1 \frac{\partial \omega}{\partial x_1} + x_2 \frac{\partial \omega}{\partial x_2} + \alpha \left(x_\mu \frac{\partial \omega}{\partial x_\mu} \right) = 0, \quad \frac{\partial \omega}{\partial x_0} = 0, \quad \frac{\partial \omega}{\partial x_1} = 0.$$

This implies that

$$\omega = \log(x_1^2 + x_2^2) + 2 \arctan \frac{x_2}{x_1} = \log[(bx)^2 + (cx)^2] + 2 \arctan \frac{cx}{bx},$$

which proves the theorem.

3 Covariant form of the linear ansatz and symmetry reduction of SDYME

By Theorem 2, the ansatze that correspond to the subalgebras L_j $(j = 1, \ldots, 24)$, are of the linear form (4), where

$$\Lambda(x) = \exp\{2\theta_2(S_{01} - S_{13})\} \exp\{\theta_1 S_{12}\} \exp\{-\theta_0 S_{03}\} \exp\{\log \theta E\}.$$

Thus, it follows that

$$\Lambda = \theta \begin{pmatrix} [\cosh \theta_0 + 2\theta_2^2 e^{-\theta_0}] & 2[-\theta_2 \cos \theta_1] & 2[\theta_2 \sin \theta_1] & [\sinh \theta_0 + 2\theta_2^2 e^{-\theta_0}] \\ 2[-\theta_2 e^{-\theta_0}] & [\cos \theta_1] & [-\sin \theta_1] & 2[\theta_2 e^{-\theta_0}] \\ [0] & [\sin \theta_1] & [\cos \theta_1] & [0] \\ [\sinh \theta_0 + 2\theta_2^2 e^{-\theta_0}] & 2[-\theta_2 \cos \theta_1] & 2[\theta_2 \sin \theta_1] & [\cosh \theta_0 - 2\theta_2^2 e^{-\theta_0}] \end{pmatrix},$$

where $[f]$ denotes $[f] = f \cdot I$ and I is a unit matrix of order 3.

In view of the above, ansatz (4) can be represented in the following form:

$$\begin{aligned}
\vec{A}_0 &= \theta[\cosh\theta_0\vec{B}_0 + \sinh\theta_0\vec{B}_3 + 2\theta_2^2e^{-\theta_0}(\vec{B}_0 - \vec{B}_3) + 2\theta_2(\sin\theta_1\vec{B}_2 - \cos\theta_1\vec{B}_1)],\\
\vec{A}_1 &= \theta[\cos\theta_1\vec{B}_1 - \sin\theta_1\vec{B}_2 - 2\theta_2e^{-\theta_0}(\vec{B}_0 - \vec{B}_3)],\\
\vec{A}_2 &= \theta[\sin\theta_1\vec{B}_1 + \cos\theta_1\vec{B}_2],\\
\vec{A}_3 &= \theta[\sinh\theta_0\vec{B}_0 + \cosh\theta_0\vec{B}_3 + 2\theta_2e^{-\theta_0}(\vec{B}_0 - \vec{B}_3) + \theta_2(\sin\theta_1\vec{B}_2 - \cos\theta_1\vec{B}_1)],
\end{aligned} \tag{10}$$

and, as is not difficult to verify,

$$\begin{aligned}
&\vec{A}_\mu = a_\mu\vec{A}_0 + b_\mu\vec{A}_1 + c_\mu\vec{A}_2 + d_\mu\vec{A}_3,\\
&\vec{B}_0 = a_\nu\vec{B}^\nu, \quad \vec{B}_1 = -b_\nu\vec{B}^\nu, \quad \vec{B}_2 = -c_\nu\vec{B}^\nu, \quad \vec{B}_3 = -d_\nu\vec{B}^\nu,
\end{aligned}$$

where a_μ, b_μ, c_μ, and d_μ are the μth components of the vectors a, b, c, and d, respectively, given in Section 2.

In these notations, the linear ansatz (10), as well as the linear ansatz (4) can be represented as

$$\begin{aligned}
\vec{A}_\mu(x) = \theta a_{\mu\nu}(x)\vec{B}^\nu(\omega) = \theta\{&(a_\mu a_\nu - d_\mu d_\nu)\cosh\theta_0 + (d_\mu a_\nu - d_\nu a_\mu)\sinh\theta_0 +\\
&+ 2(a_\mu + d_\mu)[\theta_2\cos\theta_1 b_\nu - \theta_2\sin\theta_1 c_\nu + \theta_2^2e^{-\theta_0}(a_\nu + d_\nu)] +\\
&+ (b_\mu c_\nu - b_\nu c_\mu)\sin\theta_1 - (c_\mu c_\nu + b_\mu b_\nu)\cos\theta_1 -\\
&- 2e^{-\theta_0}\theta_2 b_\mu(a_\nu + d_\nu)\}\vec{B}^\nu)\}\vec{B}^\nu(\omega).
\end{aligned} \tag{11}$$

The values taken by the functions θ, θ_0, θ_1, θ_2, and ω in (11) are given in Theorem 2 for each of the subalgebras L_j $(j = 1, \ldots, 24)$.

Thus, we have written the $\tilde{P}(1,3)$-invariant ansatz for the $\vec{A}_\mu(x)$ fields in a manifestly covariant form.

Let us note that ansatz (11) can be obtained from (10) by applying the proliferation formulas that correspond to the Lorentz group $AO(1,3)$ to the functions $\vec{A}_\mu$ from (10) with the generators (2) (see, for instance, [14, 15]). Therefore, the vectors a, b, c, and d can be viewed as a general system of orthonormalized vectors in the Minkowski space $R(1,3)$, which can be expressed as

$$\begin{aligned}
&a_\mu a^\mu = -b_\mu b^\mu = -c_\mu c^\mu = -d_\mu d^\mu = 1,\\
&a_\mu b^\mu = a_\mu c^\mu = a_\mu d^\mu = b_\mu c^\mu = b_\mu d^\mu = c_\mu d^\mu = 0.
\end{aligned}$$

The unified form of the $\tilde{P}(1,3)$-invariant ansatze derived in (11) allows us to perform the reduction of SDYME (1) in the general form.

Lemma. *The ansatz (11) allows one to reduce SDYME (1) to the system*

$$T_{\mu\nu} = \frac{i}{2}\varepsilon_{\mu\nu\sigma\delta}T^{\sigma\delta}, \tag{12}$$

where

$$\begin{aligned}
T_{\mu\nu} = G_\mu(\omega)\frac{d\vec{B}_\nu(\omega)}{d\omega} - G_\nu(\omega)\frac{d\vec{B}_\mu(\omega)}{d\omega} + H_\mu(\omega)\vec{B}_\nu(\omega) -\\
- H_\nu(\omega)\vec{B}_\mu(\omega) + S_{\mu\nu\gamma}(\omega)\vec{B}^\gamma(\omega) + e\vec{B}_\mu(\omega)\times\vec{B}_\nu(\omega).
\end{aligned} \tag{13}$$

In (13), the functions $G_\mu(\omega)$, $H_\mu(\omega)$, and $S_{\mu\nu\gamma}(\omega)$ are determined from

$$\theta G_\gamma = a_{\mu\gamma}\frac{\partial\omega}{\partial x_\mu}, \quad H_\gamma\theta^2 = a_{\mu\gamma}\frac{\partial\theta}{\partial x_\mu}, \quad \theta S_{\delta\sigma\gamma} = a^\mu_\delta\frac{\partial a_{\mu\gamma}}{\partial x_\nu}a_{\nu\sigma} - a^\mu_\sigma\frac{\partial a_{\mu\gamma}}{\partial x_\nu}a_{\nu\delta}.$$

To prove the lemma, it suffices to substitute ansatz (11) into SDYME (1) and to contract the resulting expression with the tensor $a^\mu_\sigma a^\nu_\delta$, using the fact that $a_{\mu\nu}$ satisfies $a^\mu_\nu a_{\mu\gamma} = g_{\nu\gamma}$.

According to the lemma, the construction of the reduced systems associated with subalgebras L_j is tantamount to finding the functions $G_\gamma(\omega)$, $H_\gamma(\omega)$, and $S_{\delta\sigma\gamma}(\omega)$ for every such subalgebra. We skip the cumbersome calculations and give only the explicit form of these functions for each of the subalgebras L_j in the following list:

$$L_1: \quad G_\gamma = \epsilon_1(c_\gamma - b_\gamma\omega), \quad H_\gamma = -\epsilon_1 b_\gamma, \quad S_{\delta\sigma\gamma} = 0,$$

$$L_2: \quad G_\gamma = 2(b_\gamma + c_\gamma), \quad H_\gamma = -b_\gamma, \quad S_{\delta\sigma\gamma} = (b_\delta c_\sigma - b_\sigma c_\delta)c_\gamma,$$

$$L_3: \quad G_\gamma = 2\sqrt{\omega}(b_\gamma - \epsilon_2\sqrt{\omega}d_\gamma), \quad H_\gamma = -\epsilon_2 d_\gamma, \quad S_{\delta\sigma\gamma} = \frac{1}{\sqrt{\omega}}(c_\sigma b_\delta - b_\sigma c_\delta)c_\gamma,$$

$$L_4: \quad G_\gamma = 2\sqrt{\omega}(b_\gamma - \epsilon_3\sqrt{\omega}a_\gamma), \quad H_\gamma = -\epsilon_3 a_\gamma, \quad S_{\delta\sigma\gamma} = \frac{1}{\sqrt{\omega}}(c_\sigma b_\delta - b_\sigma c_\delta)c_\gamma,$$

$$L_5: \quad G_\gamma = \epsilon_1(c_\gamma - b_\gamma\omega), \quad H_\gamma = -\epsilon_1 b_\gamma,$$
$$S_{\delta\sigma\gamma} = \epsilon_1\alpha^{-1}[b_\sigma(d_\delta a_\gamma - d_\gamma a_\delta) - b_\delta(d_\sigma a_\gamma - d_\gamma a_\sigma)],$$

$$L_6: \quad G_\gamma = \epsilon_4(1-\alpha)(a_\gamma - d_\gamma) + \epsilon_5(1+\alpha)k_\gamma,$$
$$H_\gamma = -\frac{1}{2}\epsilon_6[\epsilon_5(a_\gamma - d_\gamma) + \epsilon_4 k_\gamma],$$
$$S_{\delta\sigma\gamma} = \frac{1}{2}[\epsilon_4(a_\gamma - d_\gamma) - \epsilon_5 k_\gamma](a_\sigma d_\delta - a_\delta d_\sigma),$$

$$L_7: \quad G_\gamma = \omega[\epsilon_5\alpha k_\gamma\omega^{-\frac{1}{\alpha}} + \epsilon_7(1-\alpha)c_\gamma], \quad H_\gamma = -\epsilon_7 c_\gamma,$$
$$S_{\delta\sigma\gamma} = \epsilon_7\alpha^{-1}[c_\sigma(a_\gamma d_\delta - d_\gamma a_\delta) - c_\delta(a_\gamma d_\sigma - d_\gamma a_\sigma)],$$

$$L_8: \quad G_\gamma = k_\gamma - \epsilon_4(a_\gamma - d_\gamma), \quad H_\gamma = -\frac{1}{2}\epsilon_4(a_\gamma - d_\gamma),$$
$$S_{\delta\sigma\gamma} = \frac{1}{2}\epsilon_4[(a_\gamma - d_\gamma)(a_\sigma d_\delta - a_\delta d_\sigma)],$$

$$L_9: \quad G_\gamma = k_\gamma - 2\epsilon_7 c_\gamma, \quad H_\gamma = -\epsilon_7 c_\gamma,$$
$$S_{\delta\sigma\gamma} = \epsilon_7[c_\sigma(a_\gamma d_\delta - d_\gamma a_\delta) - c_\delta(a_\gamma d_\sigma - d_\gamma a_\sigma)],$$

$$L_{10}: \quad G_\gamma = \epsilon_4[(a_\gamma - d_\gamma)\omega + k_\gamma] - 2\epsilon_7 c_\gamma\omega, \quad H_\gamma = -\epsilon_7 c_\gamma,$$
$$S_{\delta\sigma\gamma} = \epsilon_4(a_\gamma - d_\gamma)(a_\sigma d_\delta) - a_\delta d_\sigma) - \epsilon_7 c_\sigma(a_\gamma d_\delta - d_\gamma a_\delta) +$$
$$+ \epsilon_7 c_\delta(a_\gamma d_\sigma - d_\gamma a_\sigma),$$

$$L_{11}: \quad G_\gamma = \epsilon_7\omega(c_\gamma - b_\gamma\omega), \quad H_\gamma = -\epsilon_7 c_\gamma,$$
$$S_{\delta\sigma\gamma} = \epsilon_7[c_\sigma(a_\gamma d_\delta - d_\gamma a_\delta) - c_\delta(a_\gamma d_\sigma - d_\gamma a_\sigma)] - \epsilon_5 k_\gamma(a_\sigma d_\delta - d_\sigma a_\delta),$$

$$L_{12}: \quad G_\gamma = 2(b_\gamma + \beta c_\gamma), \quad H_\gamma = -b_\gamma,$$
$$S_{\delta\sigma\gamma} = c_\gamma(c_\sigma b_\delta - c_\delta b_\sigma) - \alpha[c_\sigma(d_\delta a_\gamma - a_\delta d_\gamma) - c_\delta(d_\sigma a_\gamma - a_\sigma d_\gamma)],$$

$$L_{13}: \quad G_\gamma = \epsilon_4(\alpha-\beta)(a_\gamma - d_\gamma) + \epsilon_5(\alpha+\beta)k_\gamma,$$
$$H_\gamma = -\frac{1}{2}\epsilon_6[\epsilon_4 k_\gamma + \epsilon_5(a_\gamma - d_\gamma)],$$

$$S_{\delta\sigma\gamma} = \frac{1}{2}[\epsilon_4(a_\gamma - d_\gamma) - \epsilon_5 k_\gamma](a_\sigma d_\delta - a_\delta d_\sigma) - \frac{1}{2\alpha}[(\epsilon_4(a_\sigma - d_\sigma) - \epsilon_5 k_\sigma)(b_\delta c_\gamma - c_\delta b_\gamma) - (\epsilon_4(a_\delta - d_\delta) - \epsilon_5 k_\delta)(b_\sigma c_\gamma - c_\sigma b_\gamma)],$$

$L_{14}:$
$$G_\gamma = k_\gamma - \epsilon_4(a_\gamma - d_\gamma), \quad H_\gamma = -\frac{1}{2}\epsilon_4(a_\gamma - d_\gamma),$$
$$S_{\delta\sigma\gamma} = \frac{1}{2}\epsilon_4[(a_\gamma - d_\gamma)(a_\sigma d_\delta - a_\delta d_\sigma) - (a_\sigma - d_\sigma)(b_\delta c_\gamma - c_\delta b_\gamma) + (a_\delta - d_\delta)(b_\sigma c_\gamma - c_\sigma b_\gamma)],$$

$L_{15}:$
$$G_\gamma = 2(b_\gamma + \alpha c_\gamma - k_\gamma e^{\frac{1}{2}\omega}), \quad H_\gamma = -b_\gamma,$$
$$S_{\delta\sigma\gamma} = c_\gamma(c_\sigma b_\delta - c_\delta b_\sigma) - \alpha[c_\sigma(d_\delta a_\gamma - a_\delta d_\gamma) - c_\delta(d_\sigma a_\gamma - a_\sigma d_\gamma)],$$

$L_{16}:$
$$G_\gamma = 2[(1-\alpha)b_\gamma + \alpha k_\gamma + \beta c_\gamma], \quad H_\gamma = -b_\gamma,$$
$$S_{\delta\sigma\gamma} = c_\gamma(c_\sigma b_\delta - c_\delta b_\sigma) - k_\gamma(a_\sigma d_\delta - a_\delta d_\sigma) + b_\sigma(d_\delta a_\gamma - a_\delta d_\gamma) - b_\delta(d_\sigma a_\gamma - a_\sigma d_\gamma),$$

$L_{17}:$
$$G_\gamma = k_\gamma - 2b_\gamma + 2\alpha c_\gamma, \quad H_\gamma = -b_\gamma,$$
$$S_{\delta\sigma\gamma} = b_\sigma(d_\delta a_\gamma - a_\delta d_\gamma) - b_\sigma(d_\sigma a_\gamma - a_\sigma d_\gamma) + c_\gamma(c_\sigma b_\delta - c_\delta b_\sigma),$$

$L_{18}:$
$$G_\gamma = k_\gamma + 2c_\gamma, \quad H_\gamma = -b_\gamma,$$
$$S_{\delta\sigma\gamma} = b_\sigma(d_\delta a_\gamma - a_\delta d_\gamma) - b_\delta(d_\sigma a_\gamma - a_\sigma d_\gamma) + c_\gamma(c_\sigma b_\delta - c_\delta b_\sigma),$$

$L_{19}:$
$$G_\gamma = 2b_\gamma\omega - \epsilon_6\omega\sqrt{\omega}(\epsilon_4 k_\gamma + \epsilon_5(a_\gamma - d_\gamma)), \quad H_\gamma = -b_\gamma,$$
$$S_{\delta\sigma\gamma} = \frac{1}{2}\sqrt{\omega}[\epsilon_4(a_\gamma - d_\gamma) - \epsilon_5 k_\gamma](d_\delta a_\sigma - a_\delta d_\sigma) + c_\gamma(b_\delta c_\sigma - c_\delta b_\sigma)$$

$L_{20}:$
$$G_\gamma = \epsilon_5\omega[(1+\alpha)k_\gamma\omega^{-\frac{1}{2\alpha}} + \epsilon_8(1-\alpha)(a_\gamma - d_\gamma)\omega^{\frac{1}{2\alpha}}],$$
$$H_\gamma = -\frac{1}{2}\epsilon_5[k_\gamma\omega^{-\frac{1}{2\alpha}} + \epsilon_8(a_\gamma - d_\gamma)\omega^{\frac{1}{2\alpha}}],$$
$$S_{\delta\sigma\gamma} = \epsilon_5[\frac{1}{2\alpha}(k_\gamma\omega^{-\frac{1}{2\alpha}} + \epsilon_8(a_\gamma - d_\gamma)\omega^{\frac{1}{2\alpha}})(a_\sigma d_\delta - d_\sigma a_\delta) + b_\gamma(k_\delta b_\sigma - k_\sigma b_\delta)\omega^{-\frac{1}{2\alpha}}],$$

$L_{21}:$
$$G_\gamma = k_\gamma, \quad H_\gamma = -\epsilon_9[c_\gamma\omega - b_\gamma],$$
$$S_{\delta\sigma\gamma} = \epsilon_9[(c_\sigma\omega - b_\sigma)(a_\gamma d_\delta - d_\gamma a_\delta) - (c_\delta\omega - b_\delta)(a_\gamma d_\sigma - d_\gamma a_\sigma)] + c_\sigma(k_\delta b_\gamma - k_\gamma b_\delta) - c_\delta(k_\sigma b_\gamma - k_\gamma b_\sigma),$$

$L_{22}:$
$$G_\gamma = a_\gamma - d_\gamma + \epsilon_5 k_\gamma, \quad H_\gamma = -\frac{1}{2}\epsilon_5 k_\gamma,$$
$$S_{\delta\sigma\gamma} = \epsilon_5[b_\gamma(k_\delta b_\sigma - k_\sigma b_\delta) - \frac{1}{2}k_\gamma(a_\delta d_\sigma - d_\delta a_\sigma)],$$

$L_{23}:$
$$G_\gamma = \epsilon_7(4b_\gamma - \omega c_\gamma), \quad H_\gamma = -\epsilon_7 c_\gamma,$$
$$S_{\delta\sigma\gamma} = \frac{1}{2}\epsilon_7[c_\sigma(a_\gamma d_\delta - d_\gamma a_\delta) - c_\delta(a_\gamma d_\sigma - d_\gamma a_\sigma)] - \frac{1}{2}k_\gamma(k_\delta b_\sigma - k_\sigma b_\delta),$$

$L_{24}:$
$$G_\gamma = \sqrt{|\omega|}\left[\frac{1}{2}k_\gamma + 2\epsilon_{10}(a_\gamma - d_\gamma)\right] - 12\epsilon_{10}\omega b_\gamma, \quad H_\gamma = -4\epsilon_{10}b_\gamma,$$
$$S_{\delta\sigma\gamma} = 2\epsilon_{10}[b_\sigma(a_\gamma d_\delta - d_\delta a_\delta) - b_\delta(a_\gamma d_\sigma - d_\gamma a_\sigma)] - \frac{1}{2}k_\gamma(k_\delta b_\sigma - k_\sigma b_\delta).$$

Here, $\epsilon_k = 1$ for $\varphi > 0$ and $\epsilon_k = -1$ for $\varphi < 0$. The values of the functions φ for every k are given in Table 1.

Table 1

k	φ	k	φ
1	bx	6	$(ax)^2-(dx)^2$
2	dx	7	cx
3	ax	8	$(ax)^2-(bx)^2-(dx)^2$
4	$ax-dx$	9	$cxkx-bx$
5	kx	10	$4bx+(kx)^2$

4 On the exact real solutions of SDYME

Before we proceed to analyzing the reduced systems and constructing their exact solutions, let us make the following remark. Whereas the YME and SDYME are real in four-dimensional Euclidean space, in Minkowski space, the YME are a system of real second-order PDE, while SDYME (1) are a system of complex first-order PDE. Therefore, self-dual solutions to YME in Minkowski space are, in general, complex, which is an undesirable property.

On the other hand, the systems of PDE that represent SDYME (1) (and, hence, the reduced systems (12) and (13), as well) are not completely defined. Moreover, the symmetry reduction of SDYME preserves their symmetric form, which allows one to address the problem of finding real solutions of these equations. Clearly, the necessary condition for building real solutions of the systems of equations (12) and (13) is given by the equations

$$T_{\mu\nu}=0, \tag{14}$$

which lead us to another system of first-order ODE, this time an overdetermined one. By imposing additional conditions on the functions $\vec{B}_\mu$, we have succeeded, in some cases, in reducing system (14) to an integrable form and in obtaining nontrivial real non-Abelian solutions of SDYME (1). In what follows, we describe these cases in some detail.

We use the notation $\vec{e}_1=(1,0,0)$, $\vec{e}_2=(0,1,0)$, and $\vec{e}_3=(0,0,1)$. In order to restore the explicit form of systems (13) and (14), we choose $a=(1,0,0,0)$, $b=(0,1,0,0)$, $c=(0,0,1,0)$, and $d=(0,0,0,1)$.

The case of the L_1 algebra. Let us set $\vec{B}_0=\lambda_0\vec{B}$ and $\vec{B}_3=\lambda_3\vec{B}$, where λ_0 and λ_3 are arbitrary real constants such that $\lambda_0^2+\lambda_3^2\neq 0$. Equations (13) and (14) take the following form:

$$\begin{aligned}
&\epsilon_1\frac{d\vec{B}}{d\omega}+e\vec{B}_2\times\vec{B}=0,\\
&\epsilon_1\omega\frac{d\vec{B}_2}{d\omega}+\epsilon_1\frac{d\vec{B}_1}{d\omega}+\epsilon_1\vec{B}_2-e\vec{B}_1\times\vec{B}_2=0,\\
&\epsilon_1\omega\frac{d\vec{B}}{d\omega}+\epsilon_1\vec{B}+e\vec{B}\times\vec{B}_1=0.
\end{aligned}\tag{15}$$

Further, let us assume that, in (15), $\vec{B}=g_m(\omega)\vec{e}_m$, $\vec{B}_1=h_m(\omega)\vec{e}_m$, and $B^2=f(\omega)\vec{e}_2$, $m=1,2,3$. Then the first two equations of (15) yield the following system for the functions g_m, h_m, and f:

$$\begin{aligned}&\epsilon_1\frac{dg_1}{d\omega}+efg_3=0,\quad \epsilon_1\frac{dh_1}{d\omega}+efh_3=0,\quad \epsilon_1\frac{dg_2}{d\omega}=0,\\ &\epsilon_1\omega\frac{df}{d\omega}+\epsilon_1\frac{dh_2}{d\omega}+\epsilon_1 f=0,\quad \epsilon_1\frac{dg_3}{d\omega}-eg_1f=0,\quad \epsilon_1\frac{dh_3}{d\omega}-eh_1f=0.\end{aligned}\tag{16}$$

We set $f = C\omega^{-1}$ in (16), with C being an arbitrary constant. Then, $g_2 = C_1$ and $h_2 = C_2$, where C_1 and C_2 are arbitrary constants, while the functions g_1, g_3, h_1, and h_3 are to be determined from two similar systems of equations, which amounts to solving the Euler equations. In particular, the system of equations for g_1, g_3 reads

$$\epsilon_1\frac{dg_1}{d\omega}+eC\omega^{-1}g_3=0,\quad \epsilon_1\frac{dg_3}{d\omega}-eC\omega^{-1}g_1=0,$$

from which we have the equation

$$\omega^2\frac{d^2g_3}{d\omega^2}+\omega\frac{dg_3}{d\omega}+e^2C^2g_3=0,$$

whose general solution is given by

$$g_3=C_3\sin(eC\log|\omega|+C_4),$$

and, thus,

$$g_1=\epsilon_1C_3\cos(eC\log|\omega|+C_4).$$

Similarly, we obtain

$$h_1=\epsilon_1C_5\cos(eC\log|\omega|+C_6).\quad h_3=C_5\sin(eC\log|\omega|+C_6).$$

where C_3, C_4, C_5, and C_6 are arbitrary integration constants.

Finally, having checked the last of the equations in (15), we obtain the following solution:

$$\vec{B}_0=\lambda_0\vec{B},\quad \vec{B}_3=\lambda_3\vec{B},\quad \vec{B}=g_m(\omega)\vec{e}_m,\quad \vec{B}_1=h_m(\omega)\vec{e}_m,\quad \vec{B}_2=f(\omega)\vec{e}_2,$$

where

$$\begin{aligned}&g_1=\mp\epsilon_1C_3\cos(eC_1\log|\omega|+C_2),\quad g_2=C_3,\\ &g_3=\mp C_3\sin(eC_1\log|\omega|+C_2),\quad h_1=\pm\epsilon_1e^{-1}\sin(eC_1\log|\omega|+C_2),\\ &h_2=-C_1,\quad h_3=\mp e^{-1}\cos(eC_1\log|\omega|+C_2),\quad f=C_1\omega^{-1},\end{aligned}\tag{17}$$

and C_1, C_2, and C_3 are arbitrary constants.

The case of the L_9 algebra. Let $\vec{B}_0=\vec{B}_3=\vec{B}$ and $\vec{B}_1=\vec{0}$. Then the systems of equations (13) and (14) reduce to the equation

$$2\epsilon_7\frac{d\vec{B}}{d\omega}+\frac{d\vec{B}_2}{d\omega}+2\epsilon_7\vec{B}+e\vec{B}\times\vec{B}_2=0.\tag{18}$$

Let us set $\vec{B}_2=f(\omega)\vec{e}_2$ and $\vec{B}=g(\omega)\vec{e}_1+h(\omega)\vec{e}_3$. Then it follows from (18) that

$$2\epsilon_7\frac{dg}{d\omega}+2\epsilon_7g-efh=0,\quad \frac{df}{d\omega}=0,\quad 2\epsilon_7\frac{dh}{d\omega}+2\epsilon_7h+efg=0,$$

which is solved by the functions

$$f = C_1, \quad g = e^{-\omega} C_2 \sin\left(\frac{eC_1}{2}\omega + C_3\right), \quad h = \epsilon_7 e^{-\omega} C_2 \cos\left(\frac{eC_1}{2}\omega + C_3\right), \tag{19}$$

where C_1, C_2, and C_3 are arbitrary integration constants.

The case of the L_{17} algebra. Setting $\vec{B}_0 = \vec{B}_3 = \vec{B}$, we obtain the following reduction of the system of equations (14):

$$\begin{aligned}
&\frac{d\vec{B}_1}{d\omega} + 2\frac{d\vec{B}}{d\omega} + 2\vec{B} + e\vec{B} \times \vec{B}_1 = 0, \\
&2\alpha\frac{d\vec{B}}{d\omega} - \frac{d\vec{B}_2}{d\omega} + e\vec{B}_2 \times \vec{B} = 0, \\
&2\frac{d\vec{B}_2}{d\omega} + 2\alpha\frac{d\vec{B}_1}{d\omega} + 2\vec{B}_2 - e\vec{B}_1 \times \vec{B}_2 = 0.
\end{aligned} \tag{20}$$

In (20), we set $\vec{B}_1 = \lambda_1 \vec{e}_1$, $\vec{B} = f(\omega)\vec{e}^2 + g(\omega)\vec{e}_3$, and $\vec{B}_2 = h(\omega)\vec{e}_2 + u(\omega)\vec{e}_3$, where $\lambda_1 \neq 0$ is an arbitrary constant. Then the functions f, g, h, and u can be determined from the system of equations

$$\begin{aligned}
&2\frac{df}{d\omega} + 2f + e\lambda_1 g = 0, \quad 2\frac{dh}{d\omega} + 2h + e\lambda_1 u = 0, \quad 2\frac{dg}{d\omega} + 2g - e\lambda_1 f = 0, \\
&2\frac{du}{d\omega} + 2u - e\lambda_1 h = 0, \quad hg - uf = 0, \quad 2\alpha\frac{df}{d\omega} - \frac{dh}{d\omega} = 0, \quad 2\alpha\frac{dg}{d\omega} - \frac{du}{d\omega} = 0.
\end{aligned}$$

The general solution of the first four equations is given by the functions

$$\begin{aligned}
&f = C_1 e^{-\omega} \cos\left(\frac{\lambda_1 e}{2}\omega + C_2\right), \quad g = C_1 e^{-\omega} \sin\left(\frac{\lambda_1 e}{2}\omega + C_2\right), \\
&h = C_3 e^{-\omega} \cos\left(\frac{\lambda_1 e}{2}\omega + C_4\right), \quad u = C_3 e^{-\omega} \sin\left(\frac{\lambda_1 e}{2}\omega + C_4\right),
\end{aligned}$$

where C_1, C_2, C_3, and C_4 are arbitrary constants. Having checked the last three equations of the system, we arrive at the following solution of (20):

$$\vec{B}_0 = \vec{B}_3 = \vec{B} = f\vec{e}_2 + g\vec{e}_3, \quad \vec{B}_2 = h\vec{e}_2 + u\vec{e}_3, \quad \vec{B}_1 = C_3\vec{e}_1,$$

where

$$\begin{aligned}
&f = C_1 e^{-\omega} \cos\left(\frac{eC_3}{2}\omega + C_2\right), \quad g = C_1 e^{-\omega} \sin\left(\frac{eC_3}{2}\omega + C_2\right), \\
&h = 2\alpha C_1 e^{-\omega} \cos\left(\frac{eC_3}{2}\omega + C_2\right), \quad u = 2\alpha C_1 e^{-\omega} \sin\left(\frac{eC_3}{2}\omega + C_2\right),
\end{aligned} \tag{21}$$

and C_1, C_2, and C_3 are arbitrary constants, with $C_3 \neq 0$.

The case of the L_{18} algebra. In this case, we set $\vec{B}_0 = \frac{1}{2}\vec{B}_2 = \vec{B}_3 = \vec{B}$. Then Eqs. (14) reduce to the equation

$$\frac{d\vec{B}}{d\omega} + 2\vec{B} + e\vec{B} \times \vec{B}_1 = 0. \tag{22}$$

In (22), let $\vec{B} = \lambda \vec{e}_3$, $\vec{B}_1 = g_m(\omega)\vec{e}_m$, $m = 1, 2, 3$, and $\lambda \neq 0$ be an arbitrary constant. Then we have the equations

$$\frac{dg_1}{d\omega} - e\lambda g_2 = 0, \quad \frac{dg_2}{d\omega} + e\lambda g_1 = 0, \quad \frac{dg_3}{d\omega} + 2\lambda = 0,$$

whose general solution is given by the functions

$$g_1 = C_1 \sin(e\lambda\omega + C_2), \quad g_2 = C_1 \cos(e\lambda\omega + C_2), \quad g_3 = -2\lambda\omega + C_3,$$

with C_1, C_2, and C_3 being arbitrary integration constants. Thus, we have constructed the following solution to (22):

$$\begin{aligned} &\vec{B}_0 = \frac{1}{2}\vec{B}_2 = \vec{B}_3 = C_4\vec{e}_3, \\ &\vec{B}_1 = C_1 \sin(eC_4\omega + C_2)\vec{e}_1 + C_1 \cos(eC_4\omega + C_2)\vec{e}_2 + (C_3 - 2C_4\omega)\vec{e}_3, \end{aligned} \tag{23}$$

where C_1, C_2, C_3, and C_4 are arbitrary integration constants, with $C_4 \neq 0$.

Inserting the solutions of the reduced equations found in (17), (19), (21), and (23) into ansatz (10), we obtain, respectively, the following exact real solutions of SDYME (1):

(1) $$\begin{aligned} \vec{A}_0 &= \lambda_0 |bx|^{-1}[\mp\epsilon_1 C_3 \cos(eC_1 \log|cx(bx)^{-1}| + C_2)\vec{e}_1 + C_3\vec{e}_2 \mp \\ &\mp C_3 \sin(eC_1 \log|cx(bx)^{-1}| + C_2)\vec{e}_3], \\ \vec{A}_1 &= |bx|^{-1}[\pm\epsilon_1 e^{-1} \sin(eC_1 \log|cx(bx)^{-1}| + C_2)\vec{e}_1 - C_1\vec{e}_2 \mp \\ &\mp e^{-1} \cos(eC_1 \log|cx(bx)^{-1}| + C_2)\vec{e}_3], \\ \vec{A}_2 &= \epsilon_1 C_1 (cx)^{-1}\vec{e}_2, \\ \vec{A}_3 &= \lambda_3 |bx|^{-1}[\mp\epsilon_1 C_3 \cos(eC_1 \log|cx(bx)^{-1}| + C_2)\vec{e}_1 + C_3\vec{e}_2 \mp \\ &\mp C_3 \sin(eC_1 \log|cx(bx)^{-1}| + C_2)\vec{e}_3], \end{aligned}$$

(2) $$\begin{aligned} \vec{A}_0 &= \vec{A}_3 = (cx)^2 e^{-kx} C_2 \left[\sin\left(\frac{1}{2}eC_1(kx - 2\log|cx|) + C_3\right)\vec{e}_1 + \right. \\ &\left. + \epsilon_7 \cos\left(\frac{1}{2}eC_1(kx - 2\log|cx|)|C_3\right)\vec{e}_3\right], \\ \vec{A}_1 &= \vec{0}, \quad \vec{A}_2 = C_1|cx|^{-1}\vec{e}_2, \end{aligned}$$

(3) $$\begin{aligned} \vec{A}_0 &= \vec{A}_3 = e^{-\omega} C_1 \left[\cos\left(\frac{1}{2}eC_3\omega + C_2\right)\vec{e}_2 + \sin\left(\frac{1}{2}eC_3\omega + C_2\right)\vec{e}_3\right], \\ \vec{A}_1 &= [(bx)^2 + (cx)^2]^{-1}\left[(bx)C_3\vec{e}_1 - 2\alpha C_1 (cx) e^{-\omega} \times \right. \\ &\left. \times \left(\cos\left(\frac{1}{2}eC_3\omega + C_2\right)\vec{e}_2 + \sin\left(\frac{1}{2}eC_3\omega + C_2\right)\vec{e}_3\right)\right], \\ \vec{A}_2 &= [(bx)^2 + (cx)^2]^{-1}\left[(cx)C_3\vec{e}_1 + 2\alpha C_1 (bx) e^{-\omega} \times \right. \\ &\left. \times \left(\cos\left(\frac{1}{2}eC_3\omega + C_2\right)\vec{e}_2 + \sin\left(\frac{1}{2}eC_3\omega + C_2\right)\vec{e}_3\right)\right], \\ \omega &= kx - \log[(bx)^2 + (cx)^2] + 2\alpha \arctan cx(bx)^{-1}, \end{aligned}$$

$$
\begin{aligned}
(4)\quad & \vec{A}_0 = \vec{A}_3 = C_4\vec{e}_3,\\
& \vec{A}_1 = [(bx)^2 + (cx)^2]^{-1}\big[C_1(bx)(\sin(eC_4\omega + C_2)\vec{e}_1 + \cos(eC_4\omega + C_2)\vec{e}_2 +\\
&\qquad + (C_3 - 2C_4\omega)\vec{e}_3) - 2C_4(cx)\vec{e}_3\big],\\
& \vec{A}_2 = [(bx)^2 + (cx)^2]^{-1}\big[C_1(cx)(\sin(eC_4\omega + C_2)\vec{e}_1 + \cos(eC_4\omega + C_2)\vec{e}_2 +\\
&\qquad + (C_3 - 2C_4\omega)\vec{e}_3) + 2C_4(bx)\vec{e}_3\big], \quad \omega = kx + 2\arctan(cx(bx)^{-1}).
\end{aligned}
$$

The values of ϵ_1 and ϵ_7 are given in Table 1, α is given in the list of subalgebras, and λ_0, λ_3, C_1, C_2, C_3, and C_4 are arbitrary real constants.

Conclusions

In this paper, we have investigated the structure of $\tilde{P}(1,3)$-invariant ansatze for the vector potential of the Yang–Mills field. The linear form we obtained for the ansatze is reduced to a covariant form, which allows us to simplify considerably the procedure for the symmetry reduction of SDYME (1) to systems of ODE. We have demonstrated the possibility of constructing real solutions of SDYME (1).

Let us note that ansatz (11) can also be used for symmetry reduction in the Minkowski space $R(1,3)$.

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