On new Galilei- and Poincaré-invariant nonlinear equations for electromagnetic field

W.I. FUSHCHYCH, I.M. TSYFRA

Nonlinear systems of differential equations for $\vec{E}$ and $\vec{H}$ which are compatible with the Galilei relativity principle are proposed. It is proved that the Schrödinger equation together with the nonlinear equation of hydrodynamic type for $\vec{E}$ and $\vec{H}$ are invariant with respect to the Galilei algebra. New Poincaré-invariant equations for electromagnetic field are constructed.

1. It is usually accepted to think that the classical Galilei relativity principle does not take place in electrodynamics. This postulate was accepted more then 100 years ago and it is even difficult to state the following problems:

1. Do systems of differential equations for vector-functions $(\vec{E},\vec{H})$ or $(\vec{D},\vec{B})$ which are invariant under the Galilei algebra exist?
2. Is it possible to construct a successive Galilei-invariant electrodynamics?
3. Do the new relativity principles different from Galilei or Poincaré–Lorentz–Einstein ones exist?

The positive answers to this questions are given in [1–6]. But from the physical and mathematical points of view this fundamental problems still require detailed investigations. In the paper we continue these investigations. Further we give theorems on local symmetries of the following systems of differential equations

$$\begin{aligned}&\frac{\partial\vec{D}}{\partial t}=\operatorname{rot}\vec{H},\quad \frac{\partial\vec{B}}{\partial t}=-\operatorname{rot}\vec{E},\\ &\operatorname{div}\vec{D}=0,\qquad \operatorname{div}\vec{B}=0;\end{aligned}\tag{1}$$

$$\begin{aligned}&a_1\vec{D}+a_2\Box\vec{D}=F_1\big(\vec{E}^2,\vec{B}^2,\vec{B}\vec{E}\big)\vec{E}+F_2\big(\vec{E}^2,\vec{B}^2,\vec{B}\vec{E}\big)\vec{B},\\ &b_1\vec{H}+b_2\Box\vec{H}=R_1\big(\vec{E}^2,\vec{B}^2,\vec{B}\vec{E}\big)\vec{E}+R_2\big(\vec{E}^2,\vec{B}^2,\vec{B}\vec{E}\big)\vec{B};\end{aligned}\tag{2}$$

$$\frac{\partial\vec{E}}{\partial t}=\operatorname{rot}\vec{H}+N_1\vec{\nabla}P_1,\quad \frac{\partial\vec{H}}{\partial t}=-\operatorname{rot}\vec{E}+N_2\vec{\nabla}P_2,\tag{3}$$

$$\operatorname{div}\vec{E}=N_1\frac{\partial P_1}{\partial t},\quad \operatorname{div}\vec{H}=N_2\frac{\partial P_2}{\partial t},\tag{4}$$

where N_1, N_2, P_1, P_2 are functions of $w_1=\vec{E}^2-\vec{H}^2$, $w_2=\vec{E}\vec{H}$;

$$\begin{aligned}&\frac{\partial E_k}{\partial t}+H_l\frac{\partial E_k}{\partial x_l}=\frac{\partial F_1(\Psi^\dagger\Psi)}{\partial x_k},\\ &\frac{\partial H_k}{\partial t}+E_l\frac{\partial H_k}{\partial x_l}=\frac{\partial F_2(\Psi^\dagger\Psi)}{\partial x_k},\quad k=1,2,3;\end{aligned}\tag{5}$$

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$$i\frac{\partial\Psi}{\partial t}=\left\{-\frac{1}{2m}\left[\partial_l-ie\lambda(\vec{E}-\vec{H})\left(\frac{\partial\vec{E}}{\partial x_l}-\frac{\partial\vec{H}}{\partial x_l}\right)\right]^2+\right.$$
$$\left.+e\lambda(\vec{E}-\vec{H})\left(\frac{\partial\vec{E}}{\partial t}-\frac{\partial\vec{H}}{\partial t}\right)\right\}\Psi-\frac{e}{2m}\vec{\sigma}(\vec{E}-\vec{H})\Psi, \tag{6}$$

$\vec{\sigma}$ are the Pauli matrices, Ψ is a wave function;

$$i\frac{\partial\Psi}{\partial t}=\left\{-\frac{1}{2m}\left[\partial_l-ie\left(\lambda_1\frac{E_l}{\sqrt{\vec{E}^2}}+\lambda_2\frac{H_l}{\sqrt{\vec{H}^2}}\right)\right]^2+\right.$$
$$\left.+e\lambda_1\left(\frac{\lambda_1}{\sqrt{\vec{E}^2}}+\frac{\lambda_2}{\sqrt{\vec{H}^2}}\right)\right\}\Psi-\frac{e}{2m}\beta\left[\vec{\sigma}\left(\lambda_3\frac{\vec{E}}{\sqrt{\vec{E}^2}}+\lambda_4\frac{\vec{H}}{\sqrt{\vec{H}^2}}\right)\right]\Psi, \tag{7}$$

where λ, λ_1, λ_2, λ_3, λ_4, β are functions of $\vec{E}^2$, $\vec{H}^2$, $\vec{E}\vec{H}$.

$$\left(\frac{\partial}{\partial t}+v_l\frac{\partial}{\partial x_l}\right)m(\vec{v}^2)\vec{v}=a_1(\vec{E}+\vec{v}\times H)+a_2(\vec{H}-\vec{v}\times E), \tag{8}$$

where $\vec{v}=(v_1,v_2,v_3)$, a_1, a_2 are smooth functions of $\vec{v}^2$, $\vec{E}^2$, $\vec{H}^2$, $\vec{v}\vec{E}$, $\vec{v}\vec{H}$, $\vec{E}(\vec{v}\times\vec{H})$, $\vec{H}(\vec{v}\times\vec{E})$.

Equation (8) can be considered as a hydrodynamics generalization of the classical Newton–Lorentz equation of motion.

2. To study symmetries of the above equations (1)–(4), we use in principle the standard Lie scheme and therefore all statements are given without proofs. But it should be noted that the proofs of theorems require nonstandard steps and long cumbersome calculations which are omitted here.

As proved in [9], system (1) of undetermined equations for $\vec{D}$, $\vec{B}$, $\vec{E}$, $\vec{H}$ is invariant with respect to the infinite-dimensional algebra which contains the Poincaré, Galilei and conformal algebras as subalgebras. This fact allows us to impose some conditions on functional dependence of $\vec{D}$, $\vec{B}$, $\vec{E}$, $\vec{H}$ and to select equations invariant under the Galilei algebra $AG(1,3)$.

Theorem 1. *System (1) is invariant with respect to the Galilei algebra $AG(1,3)$ with basis operators*

$$P_0=\partial_t=\frac{\partial}{\partial t},\quad P_a=\partial_{x_a}=\frac{\partial}{\partial x_a},$$
$$J_{ab}=x_a\partial_{x_b}-x_b\partial_{x_a}+E_a\partial_{E_b}-E_b\partial_{E_a}+H_a\partial_{H_b}-H_b\partial_{H_a}+$$
$$+D_a\partial_{D_b}-D_b\partial_{D_a}+B_a\partial_{B_b}-B_b\partial_{B_a},$$
$$G_a=t\partial_{x_a}+\varepsilon_{abc}\left(B_b\partial_{E_c}-D_b\partial_{H_c}\right)$$

if

$$\vec{D}=N(\vec{B}^2,\vec{B}\vec{E})\vec{B},\quad \vec{H}=-N(\vec{B}^2,\vec{B}\vec{E})\vec{E}+M(\vec{B}^2,\vec{B}\vec{E})\vec{B}, \tag{9}$$

where M, N are arbitrary functions of their variables.

Choosing concrete form of M and N, we obtain families of Galilei-invariant equations (1) with conditions (9). So, when $N = \vec{B}\vec{E}$, $M = 1$, then (9) takes the form

$$\vec{D} = \frac{(\vec{E}\vec{H})^2}{(1-\vec{E}^2)^2}\vec{E} + \frac{\vec{E}\vec{H}}{1-\vec{E}^2}\vec{H}, \quad \vec{B} = \frac{\vec{E}\vec{H}}{1-\vec{E}^2}\vec{E} + \vec{H}.$$

Corollary 1. *The transformation rule for $\vec{E}$ and $\vec{H}$ has the form*

$$\begin{aligned} &\vec{E} \to \vec{E}' = \vec{E} + \vec{u} \times \vec{B}, \quad \vec{H} \to \vec{H}' = \vec{H} - \vec{u} \times \vec{D}, \\ &\vec{D} \to \vec{D}' = \vec{D}, \quad \vec{B} \to \vec{B}' = \vec{B} \end{aligned}$$

under Galilei transformations, where $\vec{u}$ is a velocity of an inertial system with respect to another inertial system.

Theorem 2. *System (1), (2) is invariant with respect to the Poincaré algebra $AP(1,3)$ with basis elements*

$$\begin{aligned} &P_0 = \partial_{x_0}, \quad P_a = \partial_{x_a}, \\ &J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a} + E_a\partial_{E_b} - E_b\partial_{E_a} + H_a\partial_{H_b} - H_b\partial_{H_a} + \\ &\qquad + D_a\partial_{D_b} - D_b\partial_{D_a} + B_a\partial_{B_b} - B_b\partial_{B_a}, \\ &J_{0a} = x_0\partial_{x_a} + x_a\partial_{x_0} + \varepsilon_{abc}\left(D_b\partial_{H_c} + E_b\partial_{B_c} - H_b\partial_{D_c} - B_b\partial_{E_c}\right) \end{aligned}$$

if and only if

$$\begin{aligned} &F_1 = R_2 = M(\vec{B}^2 - \vec{E}^2, \vec{B}\vec{E}), \quad F_2 = -R_1 = N(\vec{B}^2 - \vec{E}^2, \vec{B}\vec{E}), \\ &a_1 = b_1 = a(\vec{B}^2 - \vec{E}^2, \vec{B}\vec{E}), \quad a_2 = b_2 = b(\vec{B}^2 - \vec{E}^2, \vec{B}\vec{E}). \end{aligned}$$

Theorem 3. *System (3) is invariant with respect to the Poincaré algebra $AP(1,3)$ with basis elements*

$$\begin{aligned} &P_0 = \partial_t, \quad P_a = \partial_{x_a}, \\ &J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a} + E_a\partial_{E_b} - E_b\partial_{E_a} + H_a\partial_{H_b} - H_b\partial_{H_a}, \\ &J_{0a} = t\partial_{x_a} + \varepsilon_{abc}\left(E_b\partial_{H_c} - H_b\partial_{E_c}\right) \end{aligned}$$

if and only if $\vec{E}$ and $\vec{H}$ satisfy system (4).

System (5) was proposed in [4] and its symmetry has been studied in [10], when $F_1 = 0$, $F_2 = 0$.

Corollary 2. *System (5), (6) can be considered as a system of equations describing the interaction of electromagnetic field with a Schrödinger field of spin $s = 1/2$.*

Theorem 4. *System (5), (6) is invariant with respect to the Galilei algebra $AG(1,3)$ whose basis elements are given by formulas*

$$\begin{aligned} &P_0 = \partial_t, \quad P_a = \partial_{x_a}, \\ &J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a} + E_a\partial_{E_b} - E_b\partial_{E_a} + \\ &\qquad + H_a\partial_{H_b} - H_b\partial_{H_a} + \frac{1}{4}\left([\sigma_a,\sigma_b]\Psi\right)_n\partial_{\Psi_n}, \\ &G_a = t\partial_{x_a} + \partial_{E_a} + \partial_{H_a} + imx_a\Psi_k\partial_{\Psi_k}. \end{aligned} \tag{10}$$

if λ is a function of $W = \left(\vec{E} - \vec{H}\right)^2$.

Theorem 5. *Equation (7) is invariant with respect to the Galilei algebra $AG(1,3)$ with the basis elements P_μ, J_{ab} (10) and*

$$G_a = t\partial_{x_a} - E_a E_k \partial_{E_k} - H_a H_k \partial_{H_k} + imx_a \Psi_k \partial_{\Psi_k}. \tag{11}$$

if λ_1, λ_2, λ_3, λ_4, β are functions of $W = \frac{\vec{E}^2 \vec{H}^2}{(\vec{E}\vec{H})^2}$.

Corollary 3. *Operators G_a (11) give the nonlinear representation of the Galilei algebra. Thus, one can consider system (5), (7) as a basis of the classical Galilei-invariant electrodynamics. The fields $\vec{E}$, $\vec{H}$, Ψ are transformed in the following way*

$$\vec{E} \to \vec{E}' = \frac{\vec{E}}{1+\theta_a E_a},$$
$$\vec{H} \to \vec{H}' = \frac{\vec{H}}{1+\theta_a H_a} \quad \text{no sum over } a,$$
$$\Psi \to \Psi' = \exp\left\{imx_a\theta_a + im\frac{\theta_a^2}{2}t\right\}$$

under transition from one inertial system to another, θ_a is group parameter.

Theorem 6. *System (8) is invariant with respect to the Poincaré algebra $AP(1,3)$ with basis elements*

$$P_0 = \partial_t, \quad P_a = \partial_{x_a},$$
$$J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a} + E_a\partial_{E_b} - E_b\partial_{E_a} + H_a\partial_{H_b} - H_b\partial_{H_a} + v_a\partial_{v_b} - v_b\partial_{v_a}, \tag{12}$$
$$J_{0a} = t\partial_{x_a} + \varepsilon_{abc}\left(E_b\partial_{H_c} - H_b\partial_{E_c}\right) + \partial_{v_a} - v_a\left(v_k\partial_{v_k}\right)$$

if

$$m(\vec{v}^2) = \frac{m_0}{\sqrt{1-\vec{v}^2}}.$$

and a_1, a_2 are functions of W_1, W_2, W_3, where $W_1 = \vec{E}\vec{H}$, $W_2 = \vec{E}^2 - \vec{H}^2$, $W_3 = \frac{1}{1-v^2}[(\vec{v}\vec{E})^2 + (\vec{v}\vec{H})^2 - \vec{v}^2\vec{H}^2 - \vec{E}^2 - 2\vec{E}(\vec{v}\times\vec{H})]$.

Corollary 4. *From this theorem we obtain the dependence of a particle mass from $\vec{v}^2$, as a consequence of Poincaré-invariance of system (8).*

Theorem 7. *System (8) is invariant with respect to the Galilei algebra $AG(1,3)$ with P_μ, J_{ab} from (12) and*

$$G_a = t\partial_{x_a} + \partial_{v_a}$$

only if $m = m_0 = \text{const}$, $a_1 = a_2 = 0$.

Corollary 5. *Operators (12) give a linear representation for $\vec{E}$ and $\vec{H}$ [8] and a nonlinear representation for velocity $\vec{v}$. The explicit form of transformations for $\vec{v}$ generated by G_1 is*

$$v_1 \to v_1' = \frac{v_1+\theta_1}{1+\theta_1 v_1}, \quad v_2 \to v_2' = \frac{v_2}{1+\theta_1 v_1}, \quad v_3 \to v_3' = \frac{v_3}{1+\theta_1 v_1}.$$

Remark 1. In conclusion we note that there exists the nonlinear representation of the Galilei algebra AG(1,3), generated by the operators P_μ, J_{ab} from (12) and

$$G_a^{(1)} = t\partial_{x_a} - E_aE_k\partial_{E_k} - H_aH_k\partial_{H_k} - v_av_k\partial_{v_k}.$$

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