High-order equations of motion in quantum mechanics and Galilean relativity

W.I. FUSHCHYCH, Z.I. SYMENOH

Linear partial differential equations of arbitrary order invariant under the Galilei transformations are described. Symmetry classification of potentials for these equations in two-dimensional space is carried out. High-order nonlinear partial differential equations invariant under the Galilei, extended Galilei and full Galilei algebras are studied.

Non-relativistic quantum mechanics is based on the equation

$$L\Psi \equiv (S+V)\Psi = 0, \tag{1}$$

where $S = p_0 - p_a^2/2m$, $p_0 = i\partial/\partial x_0 = i\partial/\partial t$, $p_a = -i\partial/\partial x_a$, $V = V(\boldsymbol{x}, \Psi^*\Psi)$. In the case where V is a function only of $\boldsymbol{x}$, equation (1) coincides with the standard linear Schrödinger equation.

The fundamental property of (1) (in the case $V = 0$) is the fact that this equation is compatible with the Galilean relativity principle. In other words, equation (1) ($V = 0$) is invariant under the Galilei group $G(1,3)$. The Lie algebra $AG(1,3) = \langle P_0, P_a, J_{ab}, G_a\rangle$ of the Galilei group is generated (see, e.g., [1, 2]) by the operators

$$\begin{aligned} &P_0 = p_0, \quad P_a = p_a, \quad J_{ab} = x_a p_b - x_b p_a, \quad a \neq b, \ a, b = 1, 2, 3, \\ &G_a = t p_a - m x_a. \end{aligned} \tag{2}$$

The operators $\langle G_a\rangle$ generate the standard Galilei transformations

$$t \to t' = t, \quad x_a \to x'_a = x_a + v_a t.$$

Definition 1. *We say that the equation of type (1) is compatible with the Galilei principle of relativity if it is invariant under the operators* $\langle P_0, P_a, J_{ab}, G_a\rangle$.

Let X be one of the operators $\langle P_0, P_a, J_{ab}, G_a\rangle$.

Definition 2. *Equation (1) is invariant under the operator X if the following condition is true:*

$$\underset{(2)}{X} L\Psi\Big|_{L\Psi=0} = 0, \tag{3}$$

where $\underset{(2)}{X}$ is the second Lie prolongation of the operator X [1–4].

The equation of type (3) is a Lie condition of invariance of the equation under the Lie algebra. In our case, it is the condition of invariance under the algebra $AG(1,3)$.

Theorem 1 [1, 2, 5]. *Among linear equations of the first order in t and of the second order in the space variables $\boldsymbol{x}$ there exists the unique equation (1) ($V = \lambda = \mathrm{const}$) invariant under the algebra $AG(1,3)$ with the basic elements (2).*

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Conclusion. *We can regard the theorem formulated above as a method of deriving the Schrödinger equation from the Galilei principle of relativity [5, 6].*

In the present paper, we give the answer on the following question: Do there exist equations not equivalent to the Schrödinger equation for which the Galilei principle of relativity is true?

In [6, 7], the following generalization of the Schödinger equation was proposed

$$(\lambda_1 S + \lambda_2 S^2 + \cdots + \lambda_n S^n + V)\Psi = 0, \tag{4}$$

$S^2 = SS, \ldots, S^n = S^{n-1}S$, $\lambda_1, \lambda_2, \ldots, \lambda_n$ are arbitrary parameters.

If $V = 0$, equation (4), as well as equation (1), is invariant under the algebra $AG(1,3)$, i.e. this equation is compatible with the Galilei principle of relativity. Is this equation unique among high-order linear equations? In what follows, we get the positive answer for this question.

More precisely, we solve the following problems:

(i) We describe all linear equations of arbitrary order invariant under the algebra $AG(1,3)$.

(ii) We describe the maximal (in Lie sense) symmetry of equation (4) in the two-dimensional space (t,x).

(iii) We describe nonlinear equations of type (4) invariant under the algebra $AG(1,3)$, the extended Galilei algebra $AG_1(1,3) = \langle AG(1,3), D\rangle$, and the full Galilei algebra $AG_2(1,3) = \langle AG_1(1,3), A\rangle$. D and A are the dilation and projective operators, respectively.

(i) For solving the above problems we use the method described in [1, 2, 5, 6, 7].

Theorem 2. *A: Among linear partial differential equations (PDE) of arbitrary even order* $2n$

$$L\Psi = 0,$$
$$L = A + B^{\mu}\partial_{\mu} + C^{\mu\nu}\partial_{\mu\nu} + D^{\mu\nu\sigma}\partial_{\mu\nu\sigma} + \cdots + E^{\overbrace{\mu\nu\sigma\ldots\kappa}^{2n}}\partial_{\underbrace{\mu\nu\sigma\ldots\kappa}_{2n}}, \tag{5}$$

there exists the unique equation

$$(\lambda_1 S + \lambda_2 S^2 + \cdots + \lambda_n S^n)\Psi = \lambda\Psi \tag{6}$$

invariant under the algebra $AG(1,3)$.

B: There are no linear PDE of arbitrary odd order $2n+1$

$$L\Psi = 0,$$
$$L = A + B^{\mu}\partial_{\mu} + C^{\mu\nu}\partial_{\mu\nu} + D^{\mu\nu\sigma}\partial_{\mu\nu\sigma} + \cdots$$
$$\cdots + E^{\overbrace{\mu\nu\sigma\ldots\kappa}^{2n}}\partial_{\underbrace{\mu\nu\sigma\ldots\kappa}_{2n}} + G^{\overbrace{\mu\nu\sigma\ldots\kappa\rho}^{2n+1}}\partial_{\underbrace{\mu\nu\sigma\ldots\kappa\rho}_{2n+1}}, \tag{7}$$

with one non-zero coefficient of the highest derivatives at least, invariant under $AG(1,3)$.

Here, $A, B^{\mu}, C^{\mu\nu}, D^{\mu\nu\sigma}, \ldots, E^{\overbrace{\mu\nu\sigma\ldots\kappa}^{2n}}$, $G^{\overbrace{\mu\nu\sigma\ldots\kappa\rho}^{2n+1}}$ *are arbitrary functions of* t *and* $\boldsymbol{x}$; $\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda$ *are arbitrary constants,* $\lambda_n \neq 0$; $\partial_\mu \equiv \partial/\partial x_\mu$, $\partial_{\mu\nu} \equiv \partial^2/\partial x_\mu \partial x_\nu$, $\ldots$ $(\mu, \nu, \ldots, \rho = \overline{0,3})$.

Proof. The scheme and idea of the proof of the theorem is very simple but the concrete realization is not simple. We describe in more details the proof of part A. Part B is proved in the same way as the first part of the theorem.

According to the Lie method [1, 3, 4], we find the $2n$th prolongations of the operators (2) and consider the system of determining equations

$$\underset{(2n)}{X} L\Psi\Big|_{L\Psi=0} = 0, \quad \forall\, X \in AG(1,3). \tag{8}$$

Writing equations (8) in the explicit form and equating coefficients for equal derivatives, we solve the system of partial differential equations to obtain functions A, B^μ, $C^{\mu\nu}$, $D^{\mu\nu\sigma}$, …, $E^{\overbrace{\mu\nu\sigma\ldots\kappa}^{2n}}$.

Invariance of equation (5) under the operators P_0, P_a results in the fact that functions A, B^μ, $C^{\mu\nu}$, $D^{\mu\nu\sigma}$, …, $E^{\overbrace{\mu\nu\sigma\ldots\kappa}^{2n}}$ do not depend on t and $\boldsymbol{x}$, i.e. these coefficients are arbitrary constants. In other words, our PDE has the form $L\Psi \equiv Q^{(1)}(p_0, p_a)\Psi = 0$, where $Q^{(1)}$ is a polynomial in (p_0, p_a) with constant coefficients.

After taking into account the invariance under the operators J_{ab}, we find that the equation has the form $L\Psi \equiv Q^{(2)}(p_0, p_a^2)\Psi = 0$, where $Q^{(2)}$ is a polynomial in (p_0, p_a^2). After considering the invariance under the Galilei operators G_a, we obtain that the equation has the form $L\Psi \equiv Q^{(3)}(p_0 - \frac{1}{2m}p_a^2)\Psi = 0$, where $Q^{(3)}$ is a polynomial in $(p_0 - \frac{1}{2m}p_a^2)$. In other words, the equation has the form (6). The theorem is proved.

Consequence. *Among fourth-order linear PDE there exists the unique equation invariant under the algebra $AG(1,3)$ with basic operators (2). This equation has the form*

$$(\lambda_1 S + \lambda_2 S^2)\Psi = \lambda\Psi,$$

where $\lambda_2 \neq 0$.

(ii) Now, we consider equation (4) in two dimensions t, x and carry out symmetry classification of potentials $V = V(x)$ of this equation, i.e., we find all functions $V = V(x)$ admitting an extension of symmetry of (4). The following statement is true.

Theorem 3. *Two-dimensional equation (4) with $\lambda_n \neq 0$, $n \neq 1$ is invariant under the following algebras:*

(1) $\langle P_0, I\rangle$, *iff $V(x)$ is an arbitrary differentiable function;*

(2) $AG(1,1) = \langle P_0, P_1, G, I\rangle$, *iff* $V = \text{const}$*;*

(3) $AG_2(1,1) = \langle \tilde{P}_0, P_1, G, D, A, I\rangle$, *iff* $V = V_1 = \text{const}$ *the following equalities are true:*

$$\frac{\lambda_k}{\lambda_n} = \binom{n}{k}\left(\frac{V_1}{\lambda_n}\right)^{(n-k)/n}, \quad k = 1, \ldots, n-1; \tag{9}$$

(4) $\langle \tilde{P}_0, D, A, I\rangle$, *iff $V = V_1 + C/x^{2n}$, V_1, C are constants and (9) are true; $\binom{n}{k}$ are the binomial coefficients.*

The operators in Theorem 3 have the following representation:

$$\begin{gathered} P_0 = p_0, \quad P_1 = p_1, \quad G = tp_1 - mx, \quad \tilde{P}_0 = \tilde{p_0} = P_0 + \sqrt[n]{V_1/\lambda_n}, \\ D = 2t\tilde{p_0} - xp_1 - (i/2)(2n-3), \quad A = t^2\tilde{p_0} - tD - (1/2)mx^2, \end{gathered} \tag{10}$$

I is the unit operator.

Consequence. *The 2nth-order PDE*

$$(S^n + V(x))\Psi = 0$$

is invariant under the following algebras:

(1) $\langle P_0, I\rangle$, *iff* $V(x)$ *is an arbitrary differentiable function;*
(2) $AG(1,1) = \langle P_0, P_1, G, I\rangle$, *iff* $V = \text{const}$;
(3) $AG_2(1,1) = \langle P_0, P_1, G, D, A, I\rangle$, *iff* $V = 0$;
(4) $\langle P_0, D, A, I\rangle$, *iff* $V = C/x^{2n}$, *where* C *is an arbitrary constant.*

The above operators have representation (10) with $V_1 = 0$.

Note that symmetry classification of potentials for the fourth-order PDE of the form

$$(\lambda_1 S + \lambda_2 S^2 + V(x))\Psi = 0$$

was carried out in [8]. In this case, symmetry operators have representation (10) with $V_1 = \frac{\lambda_1^2}{4\lambda_2}$ and $n = 2$.

(iii) Now, let us consider nonlinear PDE of type (4) in $(r+1)$-dimensional space:

$$S^n\Psi + F(\Psi\Psi^*)\Psi = 0, \tag{11}$$

where Ψ^* is complex conjugated function, n is an arbitrary integer power, F is an arbitrary complex function of $\Psi\Psi^*$.

We study symmetry classification of (11), i.e. we find all functions $F(\Psi\Psi^*)$ which admit an extension of symmetry of equation (11).

Theorem 4. *Equation (11) is invariant under the following algebras:*

(1) $\langle P_0, P_a, J_{ab}, G_a, Q_1\rangle$, *iff* F *is an arbitrary differentiable function;*
(2) $\langle P_0, P_a, J_{ab}, G_a, Q_1, Q_2\rangle$, *iff* $F = \text{const} \neq 0$;
(3) $\langle P_0, P_a, J_{ab}, G_a, Q_1, \tilde{D}\rangle$, *iff* $F = C(\Psi\Psi^*)^k$, $k \neq 0$;
(4) $\langle P_0, P_a, J_{ab}, G_a, Q_1, D, A\rangle$, *iff* $F = C(\Psi\Psi^*)^{(2n)/(r+2-2n)}$;
(5) $\langle P_0, P_a, J_{ab}, G_a, Q_1, Q_2, D, A\rangle$, *iff* $F = 0$.

Here, indices a, b *are from* 1 *to* r, $a \neq b$, k *is an arbitrary number* ($k \neq 0$), *and the above operators have the following representation:*

$$P_0 = p_0, \quad P_a = p_a, \quad J_{ab} = x_a p_b - x_b p_a, \; G_a = t\partial_{x_a} + imx_a Q_1,$$
$$Q_1 = \Psi\partial_\Psi - \Psi^*\partial_{\Psi^*}, \quad Q_2 = \Psi\partial_\Psi + \Psi^*\partial_{\Psi^*},$$
$$\tilde{D} = 2t\partial_t + x^c\partial_{x_c} - (n/k)Q_2, \quad D = 2t\partial_t + x^c\partial_{x_c} - \frac{r+2-2n}{2}Q_2,$$
$$A = t^2\partial_t + tx^c\partial_{x_c} + (i/2)mx^c x_c Q_1 - \frac{r+2-2n}{2}tQ_2,$$

where summation from 1 *to* r *over the repeated indices* c *is understood.*

Thus, in the present paper, we have described the unique linear PDE of arbitrary even order which is invariant under the Galilei group. We have investigated the exhaustive symmetry classification of potentials $V(x)$ of (4) and functions $F(\Psi\Psi^*)$ of the nonlinear equation (11), i.e. we have pointed out all functions admitting an extension of the invariance algebra.

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