Some exact solutions of a conformally invariant nonlinear Schrödinger equation

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We consider a nonlinear Schrödinger equation whose symmetry algebra is the conformal algebra. Using some of these symmetries, we construct some ansatzes for solutions of the equation. This equation can be thought of as giving a wave-function description of a classical particle.

1 Introduction

Many authors have proposed nonlinear generalisations of the linear equation of the following type [1, 2, 3, 4, 5, 6, 7]:

$$iu_t + \triangle u = \left(\lambda_1 \frac{\triangle|u|}{|u|} + \lambda_2 \frac{|u|_a|u|_a}{|u|} + \lambda_0 \ln \frac{u}{u^*}\right) u, \tag{1}$$

where $u_t = \frac{\partial u}{\partial t}$, $|u|_a = \frac{\partial u}{\partial x_a}$, $|u| = uu^*$, $a = 1, \ldots, n$, λ_0, λ_1, λ_2 are constants, and we sum over repeated indices. These types of equations were introduced to include effects such as dissipation and diffusion.

The symmetry properties and classification of equations of type (1) are studied in [6, 7]. An important property of all equations of the above type is their admit the Galilei group $G(1,n)$ as symmetries.

In this article we shall consider the following equation belonging to the class (1):

$$iu_t + \triangle u = \frac{\triangle|u|}{|u|} u \tag{2}$$

which has remarkable symmetry properties. Indeed, it has the largest local symmetry algebra of all known nonlinear Schrödinger equations, being invariant under the conformal algebra $AC(1,n+1)$ of $n+2$-dimensional Minkowski space. Thus, since this algebra contains the Poincaré algebras $AP(1,n+1)$, $AP(1,n)$ and so on, equation (2) obeys the principle of Lorentz–Poincaré–Einstein relativity as well as Galilei relativity (see [8] for more details on this effect).

There are other reasons for considering equation (2). First, (2) can be obtained as a reduction of the hyperbolic equation

$$|\Psi|\Box\Psi - \Psi\Box|\Psi| = -\kappa\Psi. \tag{3}$$

Equation (3), with $\kappa = m^2c^2/\hbar^2$ was proposed by Vigier and Gueret [3] and by Guerra and Pusterla [2] as an equation for de Broglie's double solution [1]. Using the following ansatz in (3)

$$\Psi = e^{i(\kappa\tau - (\epsilon x)/2)} u(\tau, \beta x, \delta x),$$

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where $\tau = \alpha x = \alpha_\mu x^\mu$ and ϵ, α, β, δ are constant 4-vectors with $\alpha^2 = \epsilon^2 = 0$, $\beta^2 = \delta^2 = -1$, $\alpha\beta = \alpha\delta = \epsilon\beta = \epsilon\delta = 0$, $\alpha\epsilon = 1$, we obtain the equation

$$iu_\tau + \triangle_2 = \frac{\triangle_2 |u|}{|u|} u$$

with $\triangle_2 = \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}$, $y_1 = \beta x$, $y_2 = \delta x$. This is just equation (2). The ansatz described above is used in reducing nonlinear complex wave equations to nonlinear Schrödinger equations (see [9] for more details).

A second reason for considering (2) is that it arises in connection with the so-called classical limit of quantum mechanics ($\hbar \to 0$). Indeed, writing

$$\psi = A(t, \vec{x}) e^{i\theta(t,\vec{x})/\hbar}$$

in the free Schrödinger equation

$$i\psi_t = -\frac{\hbar}{2m} \triangle\psi$$

we obtain the system

$$\theta_t + \frac{1}{2m}(\nabla\theta)^2 = \frac{\hbar^2}{2m}\frac{\triangle A}{A}, \quad \partial_t(A^2) + \nabla\left(A^2 \frac{\nabla\theta}{m}\right) = 0$$

which, on taking the limit $\hbar \to 0$ gives

$$\theta_t + \frac{1}{2m}(\nabla\theta)^2 = 0, \quad \partial_t(A^2) + \nabla\left(A^2 \frac{\nabla\theta}{m}\right) = 0$$

which is the same system we obtain when we put $u = Ae^{i\theta}$ into (2) (when $m = 1/2$). It is thus possible to think of a classical particle having a wave-function u satisfying (2), but we shall not pursue this interesting question here.

The main aim of our paper is to exploit the symmetry algebra $AC(1,4)$ to construct exact solutions of equation (2) for $n = 3$. It is not yet possible to give a physical interpretation of the solutions we obtain, but we believe that nontrivial solutions of nonlinear equations are always of interest and give useful information about the possible flows (trajectories, evolutions, bifurcations, asymptotics) of the dynamical system described by (2). Of course, initial and boundary conditions will pick out some special solutions of the equation which can be given a physical interpretation.

In order to construct solutions of (2) in explicit form, it is necessary to know all inequivalent subalgebras of the algebra $AC(1,4)$, and then to construct corresponding ansatzes which reduce (2) to equations in fewer independent variables, even ordinary differential equations. It is not possible to realise this scheme (see [10] for details) in full in this paper: we merely list those subalgebras of the extended Poincaré algebra $A\tilde{P}(1,4) = \langle AP(1,4), D\rangle$ which reduce (2) to ordinary differential equations which we are able to solve in general or find particular solutions for. The solutions of these ordinary differential equations give us exact solutions of (2).

2 Symmetry of (2) in terms of amplitude and phase

To simplify our work, it is convenient to go over to the amplitude-phase representation of the function u:

$$u(t,\vec{x}) = A(t,\vec{x})e^{i\theta(t,\vec{x})} = e^{R(t,\vec{x})+i\theta(t,\vec{x})}$$

in terms of which equation (2) becomes:

$$\theta_t + \nabla\theta\cdot\nabla\theta = 0, \tag{4}$$

$$R_t + \triangle\theta + 2\nabla\theta\cdot\nabla R = 0. \tag{5}$$

Using the standard algorithm for calculating Lie point symmetries (see, for example, [11, 13, 12, 8]) we find the following result:

Theorem 1. *The maximal point-symmetry algebra of the system of equations (4), (5) is algebra with basis vector fields*

$$\begin{aligned}
&P_t = \partial_t, \quad P_a = \partial_a, \quad P_{n+1} = \frac{1}{2\sqrt{2}}(2\partial_t - \partial_\theta), \quad N = \partial_R, \quad J_{ab} = x_a\partial_b - x_b\partial_a,\\
&J_{0,n+1} = t\partial_t - \theta\partial_\theta, \quad J_{0a} = \frac{1}{\sqrt{2}}\left(x_a\partial_t + (t+2\theta)\partial_{x_a} + \frac{1}{2}x_a\partial_\theta\right),\\
&J_{a,n+1} = \frac{1}{\sqrt{2}}\left(-x_a\partial_t + (t-2\theta)\partial_{x_a} + \frac{1}{2}x_a\partial_{x_0}\right),\\
&D = -\left(t\partial_t + x_a\partial_a + \theta\partial_\theta - \frac{n}{2}\partial_R\right),\\
&K_0 = \sqrt{2}\left(\left(t+\frac{\vec{x}^2}{2}\right)\partial_t + (t+2\theta)x_a\partial_{x_a} + \left(\frac{\vec{x}^2}{4}+2\theta^2\right)\partial_\theta - \frac{n}{2}(t+2\theta)\partial_R\right),\\
&K_{n+1} = -\sqrt{2}\left(\left(t-\frac{\vec{x}^2}{2}\right)\partial_t + (t-2\theta)x_a\partial_{x_a} + \left(\frac{\vec{x}^2}{4}-2\theta^2\right)\partial_\theta - \frac{n}{2}(t-2\theta)\partial_R\right),\\
&K_a = 2x_aD - (4t\theta - \vec{x}^2)\partial_{x_a}.
\end{aligned}$$

The above algebra is equivalent to the extended conformal algebra $AC(1,n+1)\oplus\langle N\rangle$. In fact, with new variables

$$x_0 = \frac{1}{\sqrt{2}}(t+2\theta), \quad x_{n+1} = \frac{1}{\sqrt{2}}(t-2\theta) \tag{6}$$

the operators in Theorem 1 can be written as

$$\begin{aligned}
&P_\alpha = \partial_\alpha, \quad J_{\alpha\beta} = x_\alpha\partial_\beta - x_\beta\partial_\alpha, \quad N = \partial_R,\\
&D = -x_\alpha\partial_\alpha + \frac{n}{2}N, \quad K_\alpha = -2x_\alpha D - (x_\mu x^\mu)\partial_\alpha.
\end{aligned} \tag{7}$$

Remark 1. It follows from Theorem 1 that the nonlinear Schrödinger equation (2) is, in $1+3$ time-space, invariant with respect to the Poincaré group $P(1,4)$ of $1+4$ time-space. The basis elements of the algebra $AP(1,4)$ are $\langle P_0, P_1, P_2, P_3, P_4, J_{\alpha\beta}, J_{04}, J_{4a}\rangle$. We also have that the new "time" x_0 and the new coordinate x_4 in (6) depend linearly on the phase function θ and on t, the time.

3 Subalgebras of $AP(1,4) \oplus \langle N \rangle$: ansatzes and solutions

In this section we exploit those subalgebras of the algebra $AP(1,4) \oplus \langle N \rangle$ which reduce the equation (2) in 1+3 time-space dimensions to ordinary differential equations which we are able to solve. In fact, we use the system (4), (5), since we construct ansatzes for the functions θ and R which, when substituted into (4) and (5), yield exact solutions of (2).

Using the methods exposed in [15, 10], we have made a detailed subalgebra analysis of $AP(1,4) \oplus \langle N \rangle$, and we have described all inequivalent subalgebras of rank 3. Here, we give a list of these algebras, the ansatzes and the exact solutions obtained.

$\boldsymbol{A_1} = \langle J_{12} + dN, P_3 + N, P_4 \rangle$ $(d \geq 0)$
Ansatz:

$$\theta = -\frac{1}{2}t + f(\omega), \quad R = x_3 - d\ \arctan\left(\frac{x_2}{x_1}\right) + g(\omega), \quad \omega = x_1^2 + x_2^2.$$

Solution:

$$\theta = -\frac{1}{2}t + \varepsilon\sqrt{\frac{x_1^2 + x_2^2}{2}} + C_1, \quad \varepsilon = \pm 1,$$
$$R = x_3 - d \arctan\left(\frac{x_2}{x_1}\right) - \frac{1}{4}\ln(x_1^2 + x_2^2) + C_2,$$

where C_1, C_2 are constants. These solutions describe processes which have phase linear in time and amplitude constant in time, linear in x_3.

$\boldsymbol{A_2} = \langle J_{04} + dN, P_1 + N, P_2 \rangle$
Ansatz:

$$\theta = \frac{1}{t}f(\omega), \quad R = \ln t + x_1 + g(\omega), \quad \omega = x_3.$$

Solution:

$$\theta = \frac{1}{4t}(x_3 + C_1)^2, \quad R = d\ln t + x_1 - \frac{2d+1}{2}\varepsilon\ln|x_3 + C_1| + C_2, \quad \varepsilon = \pm 1.$$

$\mathbf{A_3} = \langle J_{04} + d_1 N, J_{12} + d_2 N, P_3 + d_3 N \rangle$
Ansatz:

$$\theta = \frac{1}{t}f(\omega), \quad \omega = x_1^2 + x_2^2, \quad R = d_1\ln t - d_2\ \arctan\left(\frac{x_2}{x_1}\right) + d_3 x_3 + g(\omega),$$

where d_1, d_2, d_3 are constants.
Solution:

$$\theta = \frac{(\sqrt{x_1^2 + x_2^2} + C_1)^2}{4t},$$
$$R = d_1\ln t - d_2 \arctan\left(\frac{x_2}{x_1}\right) + d_3 x_3 - \frac{1}{4}\ln\left(x_1^2 + x_2^2\right) -$$
$$- \left(d_1 + \frac{1}{2}\right)\ln\left|\sqrt{x_1^2 + x_2^2} + C_1\right| + C_2.$$

$\boldsymbol{A_4} = \langle J_{04} + dN, J_{23}, P_2, P_3 \rangle$
Ansatz:

$$\theta = \frac{1}{t} f(\omega), \quad R = d \ln |t| + g(\omega), \quad \omega = x_1.$$

Solution:

$$\theta = \frac{(x_1 + C_1)^2}{4t}, \quad R = d \ln |t| - \left(d + \frac{1}{2} \right) \ln |x_1 + C_1| + C_2.$$

$\boldsymbol{A_5} = \langle G_1, J_{04} + d_1 N, P_3 + d_2 N \rangle$ with d_1 arbitrary and $d_2 = 0, 1$
Ansatz:

$$\theta = \frac{x_1^2}{4t} + \frac{1}{t} f(\omega), \quad R = d_1 \ln |t| + d_2 x_3 + g(\omega), \qquad \omega = x_2.$$

Solution:

$$\theta = \frac{x_1^2 + (x_2 + C_1)^2}{4t}, \quad R = d_1 \ln |t| + d_2 x_3 - (d_1 + 1) \ln |x_2 + C_1| + C_2.$$

$\boldsymbol{A_6} = \langle J_{12}, J_{13}, J_{23}, P_4 + dN \rangle$ $(d = 0, 1)$
Ansatz:

$$\theta = -\frac{1}{2} t + f(\omega), \quad R = -d\sqrt{2} t + g(\omega), \quad \omega = x_1^2 + x_2^2 + x_3^2.$$

Solution:

$$\theta = -\frac{1}{2} t + \varepsilon \sqrt{\frac{x_1^2 + x_2^2 + x_3^2}{2}} + C_1, \quad \varepsilon = \pm 1,$$
$$R = -d\sqrt{2} t + d\varepsilon \sqrt{x_1^2 + x_2^2 + x_3^2} - \frac{1}{2} \ln(x_1^2 + x_2^2 + x_3^2) + C_2.$$

$\boldsymbol{A_7} = \langle G_1, G_2, J_{04} + d_1 N, J_{12} + d_2 N \rangle$
Ansatz:

$$\theta = \frac{x_1^2 + x_2^2}{4t} + \frac{1}{t} f(\omega), \quad R = d \ln |t| - d_2 \arctan \frac{x_2}{x_1} + g(\omega), \quad \omega = x_3.$$

Solution:

$$\theta = \frac{x_1^2 + x_2^2 + (x_3 + C_1)^2}{4t},$$
$$R = d_1 \ln |t| - \beta \arctan \frac{x_2}{x_1} - \left(d_2 + \frac{3}{2} \right) \ln |x_3 + C_1| + C_2.$$

$\boldsymbol{A_8} = \langle J_{12}, J_{23}, J_{14}, J_{23}, J_{24}, J_{34} \rangle$
Ansatz:

$$\theta = -\frac{1}{2} t + f(\omega), \quad R = g(\omega), \quad \omega = (t - 2\theta)^2 + 2\left(x_1^2 + x_2^2 + x_3^2 \right).$$

Solution (in implicit form):

$$\theta = -\tfrac{1}{2} t + \tfrac{1}{2} \sqrt{(t - 2\theta)^2 + 2(x_1^2 + x_2^2 + x_3^2)} + C_1,$$
$$R = -\frac{3}{4} \ln \left((t - 2\theta)^2 + 2(x_1^2 + x_2^2 + x_3^2) \right) + C_2.$$

$\boldsymbol{A_9} = \langle J_{01}, J_{02}, J_{03}, J_{12}, J_{13}, J_{23}\rangle$
Ansatz:

$$\theta = \frac{1}{4t} f(\omega) + \frac{x_1^2 + x_2^2 + x_3^2}{4t}, \quad R = g(\omega), \quad \omega = \theta - \frac{1}{2}t.$$

Solution:

$$\theta = \frac{\vec{x}^2 - 4C_1 t + 8C_1^2}{4t - 8C_1}, \quad R = -\frac{3}{2}\ln\left|\frac{\vec{x}^2 - 2(t - 2C_1)^2}{t - 2C_1}\right| + C_2.$$

$\boldsymbol{A_{10}} = \langle J_{12} + P_0 + d_1 N, P_3 + d_2 N, P_4\rangle$ $(d_1 \geq 0,\ d_2 > 0)$
Ansatz:

$$\theta = f(\omega) - \frac{1}{2}t - \frac{1}{\sqrt{2}}\arctan\frac{x_2}{x_1}, \quad R = g(\omega) - d_1 \arctan\frac{x_2}{x_1} + d_2 x_3,$$
$$\omega = x_1^2 + x_2^2.$$

Solution:

$$\theta = -\frac{1}{2}t - \frac{1}{\sqrt{2}}\arctan\frac{x_2}{x_1} + \frac{\varepsilon}{2}\left(\sqrt{x_1^2 + x_2^2 - 1} - \arctan\sqrt{x_1^2 + x_2^2 - 1}\right) + C_1,$$
$$R = -d_1 \arctan\frac{x_2}{x_1} + d_2 x_3 - \frac{1}{4}\ln|x_1^2 + x_2^2 - 1| -$$
$$- \varepsilon d_1 \arctan\sqrt{x_1^2 + x_2^2 - 1} + C_2, \quad \varepsilon = \pm 1.$$

$\boldsymbol{A_{11}} = \langle G_1 + 2T + dN, P_2 + N, P_3\rangle$
Ansatz:

$$\theta = f(\omega) - \frac{t^3}{6} + \frac{x_1 t}{2}, \quad R = g(\omega) + \frac{d}{\sqrt{2}}t + x_2, \quad \omega = t^2 - 2x_1.$$

Solution:

$$\theta = \frac{\varepsilon}{6}(t^2 - 2x_1)^{3/2} - \frac{t^3}{6} + \frac{x_1 t}{2} + C_1,$$
$$R = -\frac{1}{2}(t^2 - 2x_1) - \frac{\varepsilon d}{\sqrt{2}}(t^2 - 2x_1)^{1/2} + \frac{d}{\sqrt{2}}t + x_2 + C_2, \quad \varepsilon = \pm 1.$$

$\boldsymbol{A_{12}} = \langle G_1 + 2T + dN, J_{23}, P_2, P_3\rangle$
Ansatz:

$$\theta = f(\omega) - \frac{t^3}{6} + \frac{x_1 t}{2}, \quad R = g(\omega) + \frac{d}{\sqrt{2}}t, \quad \omega = t^2 - 2x_1.$$

Solution:

$$\theta = \frac{\varepsilon}{6}(t^2 - 2x_1)^{3/2} - \frac{t^3}{6} + \frac{x_1 t}{2} + C_1,$$
$$R = -\frac{1}{2}(t^2 - 2x_1) + \frac{\varepsilon d}{\sqrt{2}}(t^2 - 2x_1)^{1/2} + \frac{d}{\sqrt{2}}t + C_2, \quad \varepsilon = \pm 1.$$

$\boldsymbol{A_{13}} = \langle G_1 + d_1 N, G_2 + P_2 + d_2 N, P_3 + d_3 N\rangle$ $(d_1, d_2, d_3 \geq 0)$
Ansatz:

$$\theta = f(\omega) + \frac{x_1^2}{4t} + \frac{x_2^2}{2\sqrt{2}(\sqrt{2}t+1)},$$
$$R = g(\omega) + \frac{d_1 x_1}{\sqrt{2}t} + \frac{d_2 x_2}{\sqrt{2}t+1} + d_3 x_3, \quad \omega = t.$$

Solution:

$$\theta = \frac{x_1^2}{4t} + \frac{x_2^2}{2\sqrt{2}(\sqrt{2}t+1)} + C_1,$$
$$R = \frac{d_1 x_1}{\sqrt{2}t} + \frac{d_2 x_2}{\sqrt{2}t+1} + d_3 x_3 - \frac{1}{2}\ln|t| - \frac{1}{2}|\sqrt{2}t+1| + C_2.$$

4 The extended subalgebras of the extended Poincaré algebra $A\tilde{P}(1,4)$

If we add the dilatation operator D to the algebra $AP(1,4)\oplus N$, we obtain the algebra $A\tilde{P}(1,4)\oplus N = \langle P_\mu, J_{\mu\nu}, N, D\rangle$. In this section we give a list of the subalgebras of $A\tilde{P}(1,4)\oplus N$ which are not equivalent to subalgebras of $AP(1,4)\oplus N$, as well as the corresponding ansatzes and solutions of (4), (5).

$\boldsymbol{A_{14}} = \langle J_{04} + a_1 N, D + a_2 N, P_3\rangle$ $(a_1 \geq 0,\ a_2$ arbitrary$)$
Ansatz:

$$\theta = \frac{x_1^2}{t} f(\omega), \quad R = g(\omega) + a_1 \ln|t| - \left(a_1 + a_2 + \frac{3}{2}\right)\ln|x_1|, \quad \omega = \frac{x_1}{x_2}.$$

Solution:

$$\theta = \frac{x_1^2}{4t}, \quad R = a_1 \ln|t| + \left(a_2 - a_1 + \frac{1}{2}\right)\ln|x_1| - 2(a_2+1)\ln|x_2| + C.$$

$\boldsymbol{A_{15}} = \langle J_{12} + a_1 J_{04} + a_2 N, D + a_3 N, P_3\rangle$ $(a_1 > 0)$
Ansatz:

$$\theta = \frac{x_1^2 + x_2^2}{t} f(\omega), \quad R = g(\omega) + \left(\frac{3}{4} + \frac{a_3}{2}\right)\ln\left(x_1^2 + x_2^2\right) - a_2 \arctan\frac{x_2}{x_1},$$
$$\omega = 2\ln|t| - \ln(x_1^2 + x_2^2) + 2a_1 \arctan\frac{x_2}{x_1}.$$

Solution:

$$\theta = \frac{x_1^2 + x_2^2}{4t},$$
$$R = g\left(2\ln|t| - \ln\left(x_1^2 + x_2^2\right) + 2a_1 \arctan\frac{x_2}{x_1}\right) - $$
$$- a_2 \arctan\frac{x_2}{x_1} - \frac{1}{2}\ln\left(x_1^2 + x_2^2\right),$$

where g is an arbitrary function of one variable.

$\boldsymbol{A_{16}} = \langle J_{04} + a_1 N, J_{12} + a_2 N, D + a_3 N\rangle$ ($a_1, a_2 \geq 0$; a_3 arbitrary)
Ansatz:

$$\theta = \frac{x_1^2 + x_2^2}{t} f(\omega),$$
$$R = g(\omega) + a_1 \ln|t| - a_2 \arctan\frac{x_2}{x_1} - \frac{2a_1 + 2a_3 + 3}{4}\ln(x_1^2 + x_2^2),$$
$$\omega = \frac{x_1^2 + x_2^2}{x_3^2}.$$

Solution:

$$\theta = \frac{x_1^2 + x_2^2}{4t},$$
$$R = a_1 \ln|t| - a_2 \arctan\frac{x_2}{x_1} - \frac{a_1 + 1}{2}\ln(x_1^2 + x_2^2) - \frac{1 + 2a_3}{2}\ln|x_3| + C.$$

$\boldsymbol{A_{17}} = \langle J_{04} + a_1 D + a_2 N, J_{12} + a_3 D + a_4 N, P_3\rangle$ ($a_1^2 + a_3^2 \neq 0$).
Ansatz:

$$\theta = \frac{x_1^2 + x_2^2}{t} f(\omega),$$
$$R = g(\omega) + \frac{3a_1 + 2a_2}{2}\ln|t| - \frac{3a_3 + 2a_4}{2}\arctan\frac{x_2}{x_1} - $$
$$- \frac{3a_1 + 2a_2}{4}\ln\left(x_1^2 + x_2^2\right),$$
$$\omega = a_1 \ln|\theta| - a_1 \ln|t| + 2a_3 \arctan\frac{x_2}{x_1} - \ln\left(x_1^2 + x_2^2\right).$$

Solution:

$$\theta = \frac{x_1^2 + x_2^2}{4t},$$
$$R = \frac{a_1 + 2a_2}{2}\ln|t| - \frac{a_1 + 2a_2 + 2}{4}\ln(x_1^2 + x_2^2) - \frac{a_3 + 2a_4}{2}\arctan\frac{x_2}{x_1} + C.$$

$\boldsymbol{A_{18}} = \langle J_{04} + D + aN, J_{23}, P_2, P_3\rangle$
Ansatz:

$$\theta = x_1^2 f(\omega), \quad R = g(\omega) - \left(a + \frac{3}{2}\right)\ln|x_1|, \quad \omega = t.$$

Solution:

$$\theta = \frac{x_1^2}{4t + C_1}, \quad R = (a+1)\ln|4t + C_1| - \left(a + \frac{3}{2}\right)\ln|x_1| + C_2.$$

$\boldsymbol{A_{19}} = \langle G_1, J_{04} + a_1 D + a_2 N, P_3\rangle$ ($a_1 \neq 0$, a_2 arbitrary)
Ansatz:

$$\theta = \frac{x_2^2}{4t} f(\omega) + \frac{x_1^2}{4t}, \quad R = g(\omega) - \frac{3a_1 + 2a_2}{2a_1}\ln x_2, \quad \omega = t x_2^{(1-a_1)/a_1}.$$

Solution:

$$\theta = \frac{x_1^2 + x_2^2}{4t}, \quad R = \frac{a_1 + 2a_2}{2}\ln|t| - \frac{a_1 + 2a_2 + 2}{2}\ln x_2 + C.$$

$\boldsymbol{A_{20}} = \langle J_{04} - D + M + aN, J_{23}, P_2, P_3\rangle$
Ansatz:

$$\theta = f(\omega) + \frac{1}{2\sqrt{2}}\ln|t|, \quad R = g(\omega) + \left(a - \frac{3}{2}\right)\ln|x_1|, \quad \omega = \frac{x_1^2}{t}.$$

Solution:

$$\theta = \frac{1}{2\sqrt{2}}\ln|t| + \frac{x_1^2}{t}(|x_1| + \varepsilon\sqrt{x_1^2 - 4\sqrt{2}t}) + \frac{\varepsilon}{2\sqrt{2}}\ln\left|\frac{\sqrt{x_1^2 - 4\sqrt{2}t} - |x_1|}{\sqrt{x_1^2 - 4\sqrt{2}t} + |x_1|}\right| + C_1,$$

$$R = \frac{a-1}{2}\ln|t| - \frac{1}{4}\ln|x_1^2 - 4\sqrt{2}t| + \frac{\varepsilon(2+a)}{2}\ln\left|\frac{\sqrt{x_1^2 - 4\sqrt{2}t} - |x_1|}{\sqrt{x_1^2 - 4\sqrt{2}t} + |x_1|}\right| + C_2.$$

$\boldsymbol{A_{21}} = \langle J_{12}, J_{13}, J_{23}, J_{04} - D + M + aN\rangle$
Ansatz:

$$\theta = f(\omega) + \frac{1}{2\sqrt{2}}\ln|t|, \quad R = g(\omega) + \frac{2a-3}{4}\ln\vec{x}^2, \quad \omega = \frac{\vec{x}^2}{t}.$$

Solution:

$$\theta = \frac{1}{2\sqrt{2}}\ln|t| + \frac{|\vec{x}|}{8t}\left(|\vec{x}| + \varepsilon\sqrt{\vec{x}^2 - 4\sqrt{2}t}\right) + $$
$$+ \frac{\varepsilon}{2\sqrt{2}}\ln\left|\frac{\sqrt{\vec{x}^2 - 4\sqrt{2}t} - |\vec{x}|}{\sqrt{\vec{x}^2 - 4\sqrt{2}t} + |\vec{x}|}\right| + C_1, \quad \varepsilon = \pm 1,$$
$$R = \frac{4a-1}{2}\ln|t| - \frac{2a+1}{4}\ln\vec{x}^2 + \frac{\varepsilon a}{2}\ln\left|\frac{\sqrt{\vec{x}^2 - 4\sqrt{2}t} - |\vec{x}|}{\sqrt{\vec{x}^2 - 4\sqrt{2}t} + |\vec{x}|}\right| -$$
$$- \frac{1}{4}\ln|\vec{x}^2 - 4\sqrt{2}t| + C_2.$$

5 Structure of the solutions

Most of the solutions we have obtained can be put into six classes, as follows.

Class 1: The phase and amplitude depend linearly on t and have the following structure:

$$\theta = \theta_{11}t + \theta_{12}(\vec{x}), \quad R = R_{11}t + R_{12}(\vec{x})$$

with θ_{11} and R_{11} being constants.

Class 2: The phase and amplitude have the structure

$$\theta = \frac{\theta_{21}}{t} + \theta_{22}(\vec{x}), \quad R = R_{11}\ln|t| + R_{22}(\vec{x})$$

with θ_{21} and R_{21} being constants.

Class 3: The phase and amplitude depend logarithmically on t:

$$\theta = \frac{\theta_{31}}{t} + \theta_{32}(w, \vec{x}), \quad w = \frac{\vec{x}^2}{t}, \quad R = R_{31}\ln|t| + R_{32}(w, \vec{x})$$

with θ_{31} and R_{31} being constants.

Class 4: The phase depends on t inversely, and the amplitude depends on t inversely and logarithmically

$$\theta = \frac{\theta_{41}(\vec{x})}{t} + \frac{\theta_{42}(\vec{x})}{t+a} + \theta_{43}(\vec{x}),$$
$$R = \frac{R_{41}(\vec{x})}{t} + \frac{R_{42}(\vec{x})}{t+a} + R_{43}(\vec{x})\ln|t| + R_{44}(\vec{x})\ln|t| + a.$$

Class 5: The amplitude is an implicit function of the phase:

$$\theta = \theta_{51}t + \theta_{52}(w), \quad w = (t-2\theta)^2 + 2\vec{x}^2, \quad R = R_{51}(w)$$

with θ_{51} a constant.

Class 6: The amplitude and phase depend on two invariants w and $\vec{x}^2$:

$$\theta = \frac{\theta_{61}(w)}{t}, \quad w = \theta - \frac{1}{2}t, \quad R = R_{61}(w).$$

Since equation (2) is invariant under the conformal group $C(1,4)$, with the infinitesimal operators of the conformal algebra given in Theorem 1, we can act on the solutions we have obtained with group elements (see [8] for the formulas giving this action explicitly) and obtain families of solutions of equation (2). These families of solutions, or orbits of the group passing through a given exact solution, are what Petiau called *guided waves* [16].

We leave open the question of the physical interpretation of equation (2) and its solutions. However, we note that, in as much as the system (5), (6) does not contain Planck's constant $\hbar$, the nonlinear Schrödinger equation (2) does not describe a quantal system in the standard sense of this term. The system (5), (6) is also obtained when $\psi = Ae^{i\theta/\hbar}$ is substituted into (...).

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