New scale-invariant nonlinear differential equations for a complex scalar field

R.Z. ZHDANOV, W.I. FUSHCHYCH, P.V. MARKO

We describe all complex wave equations of the form $\Box u = F(u, u^*)$ invariant under the extended Poincaré group. As a result, we have obtained the five new classes of $\widetilde{P}(1,3)$-invariant nonlinear partial differential equations for the complex scalar field.

It is well-known that the maximal symmetry group admitted by the nonlinear wave equation

$$\Box u \equiv u_{x_0 x_0} - \triangle_3 u = F(u) \tag{1}$$

with an arbitrary smooth function $F(u)$ is the 10-parameter Poincaré group $P(1,3)$ having the following generators:

$$P_\mu = \partial_\mu, \quad J_{\mu\nu} = g_{\mu\alpha} x_\alpha \partial_\nu - g_{\nu\alpha} x_\alpha \partial_\mu, \tag{2}$$

where $\partial_\mu = \partial/\partial_{x_\mu}$, $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, $\mu,\nu,\alpha = 0,1,2,3$. Hereafter, the summation over the repeated indices from 0 to 3 is understood.

As established in [1] Eq. (1) admits a wider symmetry group only in the two cases:

$$(1)\ F(u) = \lambda u^k, \quad k \neq 1, \tag{3}$$

$$(2)\ F(u) = \lambda e^{ku}, \quad k \neq 0. \tag{4}$$

where λ, k are arbitrary constants.

Eqs. (1) with nonlinearities (3) and (4) admit the one-parameter groups of scale transformations $D(1)$ having the following generators:

$$\begin{aligned} &(1)\ D = x_\mu \partial_\mu + \frac{2}{1-k} u \partial_u, \\ &(2)\ D = x_\mu \partial_\mu - \frac{2}{k} \partial_u. \end{aligned} \tag{5}$$

The 11-parameter transformation group with generators (2) and (5) is called the extended Poincaré group $\widetilde{P}(1,3)$.

The above result admits the following group-theoretical interpretation: on the set of solutions of the nonlinear wave equation (1) two inequivalent representations of the extended Poincaré group are realized. Each representation gives rise to a $\widetilde{P}(1,3)$-nonlinear wave equation with a very specific nonlinearity.

Surprisingly enough, there is no an analogous result for the complex nonlinear wave equation

$$\Box u = F(u, u^*) \tag{6}$$

Physica D, 1996, **95**, P. 158–162.

which is a more realistic model for describing a charged meson field in the modern quantum field theory. Eq. (6) admits the Poincaré group with generators (2) under arbitrary $F(u,u^*)$. It is natural to formulate the following problem: to describe all functions F such that the said equation admits wider symmetry groups. We are interested in those equations of the form (6) which are invariant under the natural extensions of the Poincaré group — the extended Poincaré and the conformal groups.

A usual approach to the description of partial differential equations admitting some Lie transformation group is to fix a representation of the group and then use the infinitesimal Lie method (see, e.g. [2, 3]) to obtain an explicit form of the unknown function F. In this way in the paper [4] two classes of $\widetilde{P}(1,3)$-invariant equations of the form (6) were constructed. But this approach may result in loosing some subclasses of invariant equations (which is the case for the paper mentioned). It means that one should not fix a priori a representation of the group. The only thing to be fixed is the commutational relations of the corresponding Lie algebra. This approach guarantees that all equations admitting a given group will be obtained.

In the paper [5] Rideau and Winternitz study two-dimensional PDEs admitting the extended Poincaré group $\widetilde{P}(1,1)$ using the approach described above. They have classified second-order $\widetilde{P}(1,1)$-invariant equations within the change of independent and dependent variables.

In the present paper we will describe within the affine transformations all equations belonging to the class (6) which are invariant under the 11-parameter extended Poincaré group.

Putting $u = u_1 + iu_2$, $u^* = u_1 - iu_2$ we rewrite the complex equation (6) as a system of two real equations

$$\Box u_j = F_j(u_1,u_2), \quad j = 1,2. \tag{7}$$

Before formulating the principal assertions we make a remark. As a direct check shows, the class of Eqs. (7) is invariant under the linear transformations of dependent variables

$$u_j \to u_j' = \sum_{k=1}^{2} \alpha_{jk} u_k + \beta_j, \tag{8}$$

where α_{jk}, β_j, $j = 1,2$ are arbitrary constants with $\det \|\alpha_{jk}\| \neq 0$.

That is why we carry out symmetry classification of Eqs. (7) within the equivalence transformations (8).

Theorem 1. *The system of partial differential equations (7) is invariant under the extended Poincaré group $\widetilde{P}(1,3)$ iff it is equivalent to one of the following systems:*

(i) $$\Box u_1 = u_1^{(a-2)/a}\widetilde{F}_1(\omega),$$
$$\Box u_2 = u_1^{(b-2)/a}\widetilde{F}_2(\omega), \quad \omega = u_1^b u_2^{-a};$$

(ii) $$\Box u_1 = \exp\left((a-2)\frac{u_1}{u_2}\right)\left\{\widetilde{F}_1(\omega) + \frac{u_1}{u_2}\widetilde{F}_2(\omega)\right\},$$
$$\Box u_2 = \exp\left((a-2)\frac{u_1}{u_2}\right)\widetilde{F}_2(\omega), \quad \omega = a\frac{u_1}{u_2} - \ln u_2;$$

$$
\begin{aligned}
&\text{(iii)} \quad \Box u_1 = \exp\left(\frac{a-2}{b}u_2\right)\widetilde{F}_1(\omega),\\
&\qquad \Box u_2 = \exp\left(-\frac{2}{b}u_2\right)\widetilde{F}_2(\omega), \quad \omega = au_2 - b\ln u_1;\\
&\text{(iv)} \quad \Box u_1 = (u_1^2+u_2^2)^{-1/2}\exp\left(\frac{a-2}{b}\arctan\frac{u_1}{u_2}\right)\left\{u_2\widetilde{F}_1(\omega)+u_1\widetilde{F}_2(\omega)\right\},\\
&\qquad \Box u_2 = (u_1^2+u_2^2)^{-1/2}\exp\left(\frac{a-2}{b}\arctan\frac{u_1}{u_2}\right)\left\{u_2\widetilde{F}_2(\omega)-u_1\widetilde{F}_1(\omega)\right\},\\
&\qquad \omega = b\ln(u_1^2+u_2^2) - 2a\arctan\frac{u_1}{u_2};\\
&\text{(v)} \quad \Box u_1 = \exp\left(-\frac{2}{b}u_2\right)\left\{\widetilde{F}_1(\omega)+u_2\widetilde{F}_2(\omega)\right\},\\
&\qquad \Box u_2 = b\exp\left(-\frac{2}{b}u_2\right)\widetilde{F}_2(\omega), \quad \omega = 2bu_1 - u_2^2;\\
&\text{(vi)} \quad \Box u_1 = 0, \quad \Box u_2 = 0;
\end{aligned}
\tag{9}
$$

where $\widetilde{F}_1$, $\widetilde{F}_2$ are arbitrary smooth functions, a, b are arbitrary constants.

And what is more, the basis generators P_μ, $J_{\mu\nu}$ are given by the formulae (2) and the generators of the corresponding groups of scale transformations are given by the following formulae:

$$
\begin{aligned}
&\text{(i)} \quad D = x_\mu\partial_\mu + au_1\partial_{u_1} + bu_2\partial_{u_2}, \quad a\neq 0;\\
&\text{(ii)} \quad D = x_\mu\partial_\mu + a(u_1\partial_{u_1}+u_2\partial_{u_2}) + u_2\partial_{u_1};\\
&\text{(iii)} \quad D = x_\mu\partial_\mu + au_1\partial_{u_1} + b\partial_{u_2}, \quad b\neq 0;\\
&\text{(iv)} \quad D = x_\mu\partial_\mu + a(u_1\partial_{u_1}+u_2\partial_{u_2}) + b(u_2\partial_{u_1}-u_1\partial_{u_2}), \quad b\neq 0;\\
&\text{(v)} \quad D = x_\mu\partial_\mu + u_2\partial_{u_1} + b\partial_{u_2}, \quad b\neq 0;\\
&\text{(vi)} \quad D = x_\mu\partial_\mu.
\end{aligned}
\tag{10}
$$

Theorem 2. *The system of PDE (8) is invariant under the conformal group $C(1,3)$ iff it is equivalent to the following system:*

$$
\Box u_j = u_1^3\widetilde{F}_j\left(\frac{u_1}{u_2}\right), \quad j=1,2.
$$

where F_1, F_2 are arbitrary smooth functions.

Proofs of the Theorems 1, 2 are carried out with the use of infinitesimal algorithm by Lie [2, 3]. Here we present the proof of the Theorem 1 only.

Within the framework of the Lie's approach a symmetry operator for the system of PDE (7) is looked for in the form

$$
X = \xi_\mu(x,u)\partial_\mu + \eta_1(x,u)\partial_{u_1} + \eta_2(x,u)\partial_{u_2}, \tag{11}
$$

where $\xi_\mu(x,u)$, $\eta_j(x,u)$ are some smooth functions.

Necessary and sufficient condition for the system of PDE (7) to be invariant under the group having the infinitesimal operator (11) reads

$$
\widetilde{X}(\Box u_j - F_j)\Big|_{\substack{\Box u_1 - F_1 = 0\\ \Box u_2 - F_2 = 0}} = 0, \quad j=1,2, \tag{12}
$$

where $\widetilde{X}$ stands for the second prolongation of the operator X.

Splitting relations (12) by independent variables we get a Killing type system of PDE for ξ_μ, η_k. Integrating it we have:

$$\begin{aligned} &\xi_\mu = 2x_\mu g_{\alpha\beta} x_\alpha k_\beta - k_\mu g_{\alpha\beta} x_\alpha x_\beta + c_{\mu\alpha} g_{\alpha\beta} x_\beta + d x_\mu + e_\mu, \quad \mu = \overline{0,3}, \\ &\eta_k = \sum_{j=1}^{2} a_{kj} u_j + b_k(x) - 2 g_{\alpha\beta} k_\alpha x_\beta u_k, \quad k = 1,2, \end{aligned} \tag{13}$$

where k_α, $c_{\mu\nu} = -c_{\nu\mu}$, d, e_μ, a_{kj} are arbitrary constants, $b_k(x)$ are arbitrary functions satisfying the following relations:

$$\begin{aligned} &\sum_{k=1}^{2} \left(\sum_{l=1}^{2} a_{kl} u_l + b_k(x) - 2 g_{\alpha\beta} k_\alpha x_\beta u_k \right) F_{j u_k} + \Box b_j(x) + \\ &\qquad + 2(d + 3 g_{\alpha\beta} k_\alpha x_\beta) F_j - \sum_{l=1}^{2} a_{jl} F_l = 0, \quad j = 1,2. \end{aligned} \tag{14}$$

From (13) and (14) it follows that the system of PDE (7) is invariant under the Poincaré group $P(1,3)$ having the generators (2) with arbitrary F_1, F_2. To describe all functions F_1, F_2 such that system (7) admits the extended Poincaré group $\widetilde{P}(1,3)$ one has to solve the following two problems:

• to describe all operators D of the form (11), (13) which together with the operators (2) satisfy the commutational relations of the Lie algebra of the group $\widetilde{P}(1,3)$:

$$\begin{aligned} &[P_\alpha, P_\beta] = 0, \quad [P_\alpha, J_{\beta\gamma}] = g_{\alpha\beta} P_\gamma - g_{\alpha\gamma} P_\beta, \\ &[J_{\alpha\beta}, J_{\mu\nu}] = g_{\alpha\nu} J_{\beta\mu} + g_{\beta\mu} J_{\alpha\nu} - g_{\alpha\mu} J_{\beta\nu} - g_{\beta\nu} J_{\alpha\mu}, \\ &[D, J_{\alpha\beta}] = 0, \quad [P_\alpha, D] = P_\alpha, \quad \alpha,\beta,\gamma,\mu,\nu = \overline{0,3}; \end{aligned}$$

• to solve system of PDE (14) for each operator D obtained.

Substituting the operator $D \equiv X$ with ξ_μ, η_k of the form (11) and (13) into the above commutational relations and computing the coefficients of the linearly-independent operators ∂_{x_μ} we arrive at the following relations:

$$\begin{aligned} &k_\alpha = 0, \quad c_{\mu\nu} = 0, \quad \alpha,\ \mu,\ \nu = 0,\ldots,3, \\ &\frac{\partial b_k(x)}{\partial x_\mu} = 0, \quad k = 1,2, \quad \mu = 0,\ldots,3. \end{aligned}$$

Consequently, the generator of the one-parameter scale transformation group D admitted by the PDE (7) necessarily takes the form

$$D = x_\mu \partial_\mu + \sum_{i=1}^{2} \left(\sum_{j=1}^{2} A_{ij} u_j + B_i \right) \partial_{u_i}, \tag{15}$$

where A_{ij}, B_i are some constants.

Before integrating the determining Eqs. (14) we simplify the operator D using the equivalence relation (8). Making in (15) the change of variables (8) with $\beta_j = 0$ (which does not alter the form of the operators P_μ, $J_{\mu\nu}$) we have

$$D' = x_\mu \partial_\mu + \sum_{i=1}^{2} \left(\sum_{j=1}^{2} \widetilde{A}_{ij} u'_j + \widetilde{B}_i \right) \partial_{u'_i},$$

where

$$\widetilde{A}_{ij} = \sum_{k,l=1}^{2} \alpha_{ik}\, A_{kl}\, \alpha_{lj}^{-1}, \quad \widetilde{B}_i = \sum_{k=1}^{2} \alpha_{ik} B_k, \quad i = 1, 2. \tag{16}$$

Here α_{lj}^{-1} are elements of the (2×2)-matrix inverse to the matrix $\|\alpha_{ij}\|$.

Since an arbitrary (2×2)-matrix can be reduced to the Jordan form by the transformation (16) we may assume, without loss of generality, that the matrix $\|\widetilde{A}_{ij}\|$ is in Jordan form. The further simplification of the form of operator (15) is achieved at the expense of the transformation (8) with $\alpha_{ik} = 0$.

As a result, the set of operators (15) is divided into the six equivalence classes whose representatives are adduced in (10).

Next, integrating corresponding system of PDE (14) we get $\widetilde{P}(1,3)$-invariant systems of equations (9).

Note 1. When proving the Theorem 1 we solve the classical problem of representation theory: the description of inequivalent representations of the extended Poincaré group which are realized on the set of solutions of the system of nonlinear PDE (7). The representation space (i.e. the set of solutions of system (7)) is not a linear vector space, whereas in the standard representation theory it is always the case. This fact makes impossible a direct application of the standard methods of linear representation theory (for more detail, see [5, 6]).

Note 2. If one put in the formulae (1) and (3) from (6) $a = k_1$, $b = k_2$ and $a = k_1$, $b = 0$ respectively, then we get $\widetilde{P}(1,3)$-invariant systems of PDE constructed in [4].

Further, if we make in (6) the change of variables

$$u_1 = \frac{1}{2}\,(u + u^*)\,, \quad u_2 = \frac{1}{2i}\,(u - u^*)\,,$$

then we get the six classes of inequivalent PDE for complex field invariant under the extended Poincaré group.

Equations of the form (3) are widely used in the quantum field theory to describe at the classical level spinless charged mesons [7]. But PDE (3) with arbitrary F_1, F_2 is "two general" to be used as a reasonable mathematical model of a real physical process. The nonlinearities F_1, F_2 should be restricted in some way. To our minds the symmetry selection principle is the most natural way of achieving this target. Furthermore, the wide symmetry of the equation under study makes it possible to apply the symmetry reduction procedure to obtain its exact solutions. Since all connected subgroups of the extended Poincaré group are known [8–10] one can apply the said procedure to reduce and to construct particular solutions of the PDE (9). This problem is now under consideration and will be a topic of our future paper.

Acknowledgments. One of the authors (R.Zh.) is supported by the Alexander von Humboldt Foundation. The authors are thankful to the referee for useful suggestions.

1. Fushchych W.I., Serov N.I., *J. Phys. A*, 1983, **16**, 3645.
2. Olver P., Applications of Lie groups to differential equations, New York, Springer, 1986.
3. Ovsjannikov L.V., Group analysis of differential equations, Moscow, Nauka, 1978.
4. Fushchych W.I., Yegorchenko I.A., *J. Phys. A*, 1989, **22**, 2643.
5. Rideau G., Winternitz P., *J. Math. Phys.*, 1990, **31**, 1095.
6. Fushchych W.I., Zhdanov R.Z., Lagno V.I., *J. Nonlinear Math. Phys.*, 1994, **1**, 295.
7. Bogoljubov N.N., Shirkov D.V., Introduction into theory of quantized fields, Moscow, Nauka, 1984.
8. Patera J., Winternitz P., Zassenhaus J., *J. Math. Phys.*, 1975, **16**, 1597.
9. Patera J., Winternitz P., Zassenhaus J., *J. Math. Phys.*, 1976, **17**, 717.
10. Patera J., Sharp R.T., Winternitz P., Zassenhaus J., *J. Math. Phys.*, 1976, **17**, 977.