On unique symmetry of two nonlinear generalizations of the Schrödinger equation

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We prove that two nonlinear generalizations of the nonlinear Schrödinger equation are invariant with respect to a Lie algebra that coincides with the invariance algebra of the Hamilton–Jacobi equation.

Nowadays many authors, who start from various physical considerations, have suggested a wide spectrum of nonlinear equations which can be considered as some nonlinear generalizations of the classical Schrödinger equation. It is necessary to note that some of the suggested equations do not satisfy the Galilean relativistic principle. As a rule this requirement is not used in construction of nonlinear generalizations. Meantime it is well known that the linear Schrödinger equation is compatible with the Galilean relativistic principle and, besides, is invariant with respect to scale and projective symmetries (see, e.g. [1] and references cited therein).

In the [1–6] the construction of nonlinear generalizations of the Schrödinger equation was based on the idea of symmetry and the following problems were solved:

1. Nonlinear Schrödinger equations, which are compatible with the Galilean relativistic principle, are described.

2. All nonlinear equations, which preserve nontrivial $AG_2(1,n)$-symmetry of the linear Schrödinger equation, are constructed.

Let us adduce some nonlinear generalizations of the Schrödinger equation that have $AG_2(1,n)$-symmetry, namely:

$$iU_t + \Delta U = \lambda_1 |U|^{4/n} U, \qquad [1,\ 2] \tag{1}$$

$$iU_t + \Delta U = \lambda_1 \frac{|U|_a |U|_a}{|U|^2} U, \qquad [3,\ 4] \tag{2}$$

$$iU_t + \Delta U = \lambda_1 \frac{\Delta |U|^2}{|U|^2} U, \qquad [6] \tag{3}$$

where $U = U(t,x)$ is an unknown differentiable complex function, $U_t \equiv \frac{\partial U}{\partial t}$, $\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$, $x = (x_1, \ldots, x_n)$, $|U| = \sqrt{UU^*}$, $|U|_a \equiv \frac{\partial |U|}{\partial X_a}$, and $*$ is the sign of complex conjugation.

Consider the generalization of the nonlinear Schrödinger equations (2)–(3) of the following form

$$iU_t + \Delta U = \left(\frac{1}{2}\lambda_0 \frac{\Delta |U|^2}{|U|^2} - \lambda_1 \frac{|U|_a |U|_a}{|U|^2} + \frac{1}{2}\lambda_2 \ln \frac{U}{U^*} \right) U, \tag{4}$$

where $\lambda_k = a_k + ib_k$, a_k and $b_k \in \mathbb{R}$, $k = 0, 1, 2$.

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It is easily seen that some nonlinear equations, which have been suggested by many authors as mathematical models of quantum mechanical, are particular cases of this nonlinear generalization of the Schrödinger equation. Indeed, we obtain from equation (4) (for $\lambda_0 = \lambda_1$ and $\lambda_2 = ib_2$) the following equation

$$iU_t + \Delta U = \left(\lambda_1 \frac{\Delta |U|}{|U|} + ib_2 \ln \left(\frac{U}{U^*} \right)^{1/2} \right) U, \qquad (5)$$

which was proposed in [7] for the stochastic interpretation of quantum mechanical vacuum dissipative effects.

Equation (5) for $b_2 = 0$ reduces to the form

$$iU_t + \Delta U = \lambda_1 \frac{\Delta |U|}{|U|} U, \qquad (6)$$

which was studied in [7–11]. The term on the right hand side of (6) takes into consideration the effect of quantum diffusion. In all these papers the authors, starting from some physical models, assumed that the parameters $\operatorname{Re} \lambda_1$ and b_2 in (5) and (6) are small ($\lambda_1 \neq 0$, $b_2 \neq 0$).

The main purpose of the present paper is to draw attention to equation (5). If we reject the mentioned assumptions as it was done in all mentioned papers [7–11] and put $\lambda_1 = 1$, then the equations

$$iU_t + \Delta U = \frac{\Delta |U|}{|U|} U \qquad (7)$$

and

$$iU_t + \Delta U = \left(\frac{\Delta |U|}{|U|} + ib_2 \ln \left(\frac{U}{U^*} \right)^{1/2} \right) U \qquad (8)$$

have the unique symmetry, which is the same as symmetry as of the Hamilton–Jacobi equation [1].

It means that the nonlinear second-order term $\Delta|U|/|U|$ changes and essentially extends symmetry of the linear Schrödinger equation.

Let us note that equation (7) for $n = 2$ can be obtained from the nonlinear hyperbolic equation [12]

$$|\psi| \Box \psi - \psi \Box |\psi| = 0,$$

where $\psi = \psi(y_0, y)$, $y = (y_1, y_2, y_3)$, $\Box = \frac{\partial^2}{\partial y_0^2} - \frac{\partial^2}{\partial y_1^2} - \frac{\partial}{\partial y_2^2} - \frac{\partial^2}{\partial y_3^2}$, by means of the ansatz

$$\psi = \varphi(t, x_1, x_2) \exp(a_\mu y_\mu), \quad t = b_\mu y_\mu, \quad x_1 = c_\mu y_\mu, \quad x_2 = d_\mu y_\mu,$$

where the parameters a_μ, b_μ, c_μ, d_μ, $\mu = 0, 1, 2, 3$ satisfy the following conditions:

$$a_\mu b_\mu = 1, \quad b_\mu c_\mu = c_\mu a_\mu = a_\mu d_\mu = d_\mu c_\mu = 0, \quad a_\mu^2 = d_\mu^2 = -1.$$

Now let us formulate theorems which give the complete information about local symmetry properties of equation (4).

Statement 1. *Equation (4) for arbitrary complex constants λ_0, λ_1 and λ_2 is invariant with respect to the Lie algebra with the basic operators*

$$P_t = \frac{\partial}{\partial t}, \quad P_a = \frac{\partial}{\partial x_a}, \quad I = U\frac{\partial}{\partial U} + U^*\frac{\partial}{\partial U^*}, \\ J_{ab} = x_a P_b - x_a P_b, \quad a, b = 1, \ldots, n, \tag{9}$$

$$X = \begin{cases} \left(\dfrac{2a_2}{b_2} I + Q\right) \exp b_2 t, & b_2 \neq 0, \\ 2a_2 t I + Q, & b_2 = 0, \end{cases} \tag{10}$$

where $Q = i\left(U\frac{\partial}{\partial U} - U^*\frac{\partial}{\partial U^*}\right)$.

Statement 2. *Equation (4) for $\lambda_2 = ib_2$ is invariant with respect to the Lie algebra with the basic operators (9) and*

$$\mathcal{G}_a = \exp(b_2 t) P_a + \frac{b_2}{2} x_a Q_1, \quad Q_1 = \frac{1}{2}\exp(b_2 t) Q. \tag{11}$$

Note that the algebra $A\mathcal{G}(1,n)$ with basic operators (9) (without I) and (11) is essentially different from the well-known Galilei algebra $AG(1,n)$ in that it contains commutative relations $[P_t, \mathcal{G}_a] = b_2\mathcal{G}_a$, $[P_t, Q_1] = b_1 Q_1$, since in the $AG(1,n)$ algebra $[P_t, G_a] = P_a$, $[P_t, Q] = 0$.

The operators $\mathcal{G}_a$ generate the following transformations

$$t' = t, \quad x'_a = x_a + v_a \exp(b_2 t), \quad a = 1, \ldots, n, \\ U' = U \exp\left[i\frac{b_2}{2}\exp(b_2 t)\left(x_a v_a + \frac{v_a v_a}{2}\exp(b_2 t)\right)\right], \tag{12}$$

where $v_1, \ldots, v_n$ are arbitrary real group parameters.

Some classes of equations with the $A\mathcal{G}(1,n)$-symmetry were constructed and studied in [4] (see the part II), [13].

Statement 3. *Equation (4) for $\lambda_2 = 0$ is invariant with respect to the Lie algebra with the basic operators (9) and*

$$G_a = tP_a + \frac{x_a}{2} Q, \quad Q, \quad D = 2tP_t + x_a P_a - \frac{n}{2} I, \\ \Pi = t^2 P_t + t x_a P_a + \frac{|x|^2}{4} Q - \frac{nt}{2} I. \tag{13}$$

It is clear that operators (9) and (13) generate the well known generalized Galilei algebra $AG_2(1,n)$ with the additional unit operator I. The linear Schrödinger equation

$$iU_t + \Delta U = 0 \tag{14}$$

is invariant with respect to the $\langle AG_2(1,n), I\rangle$ algebra, too. It is well known that operators G_a, $a = 1, \ldots, n$ generate the Galilean transformations

$$t' = t, \quad x'_a = x_a + v_a t, \quad U' = U \exp\left[\frac{i}{2}\left(x_a v_a + \frac{v_a v_a}{2} t\right)\right] \tag{15}$$

which are essentially different from (12).

So, equation (5) for arbitrary λ_1 and $b_2 \neq 0$, which is a particular case of equation (4), is invariant with respect to the algebra $\langle A\mathcal{G}(1,n), I\rangle$, but in the case $b_2 = 0$ (see equation (6)) it has the $AG_2(1,n)$-symmetry with the additional unit operator I.

Statement 4. *Equation (5) for $\lambda_1 = 1$ and $b_2 = 0$ (see equation (7)) is invariant with respect to the Lie algebra with the basic operators (9), (13) and*

$$
\begin{aligned}
&G_a^1 = -i\ln\frac{U}{U^*}P_a + x_aP_t, \quad D_1 = -i\ln\frac{U}{U^*}Q + x_aP_a,\\
&\Pi_1 = -\left(\ln\frac{U}{U^*}\right)^2 Q - 2i\ln\frac{U}{U^*}x_aP_a + |x|^2P_t + in\ln\frac{U}{U^*}I,\\
&K_a = tx_aP_t - \left(\frac{|x|^2}{2} + it\ln\frac{U}{U^*}\right)P_a + x_ax_bP_b - \frac{n}{2}x_aI - \frac{ix_a}{2}\ln\frac{U}{U^*}Q.
\end{aligned}
\tag{16}
$$

If we make the substitution $U = \rho\exp iW$, where ρ and W are real functions, then operators (16) are simplified, and we can note that the algebra (9), (13) and (16) is that of the Hamilton–Jacobi equation. So, equation (7) has the same algebra of Lie symmetries as the classical Hamilton–Jacobi equation [1].

Statement 5. *Equation (5) $\lambda_1 = 1$ and $b_2 \neq 0$ (see equation (8)) is invariant with respect to the Lie algebra with the basic operators (9) and*

$$
\begin{aligned}
&G_a = \exp(b_2t)\left(P_a + \frac{b_2}{4}x_aQ\right), \quad D = \exp(-b_2t)(P_t + b_2WQ),\\
&\Pi = \exp(b_2t)\left[\frac{1}{b_2}P_t + x_aP_a + \left(W + \frac{b_2}{4}|x|^2\right)Q - \frac{n}{2}I\right],\\
&G_a^1 = \exp(-b_2t)\left[WP_a + \frac{1}{2}x_aP_t + \frac{b_2}{2}x_aWQ\right], \quad D_1 = 2WQ + x_aP_a,\\
&\Pi_1 = \exp(-b_2t)\left[\left(W + \frac{b_2}{4}|x|^2\right)WQ + Wx_aP_a + \frac{|x|^2}{4}P_t - \frac{n}{2}WI\right],\\
&K_a = \frac{x_a}{b_2}P_t + \left(\frac{2}{b_2}W - \frac{|x|^2}{2}\right)P_a + x_ax_bP_b + 2x_aWQ - \frac{n}{2}x_aI,
\end{aligned}
\tag{17}
$$

where $W = -\frac{i}{2}\ln\frac{U}{U^*}$, the operators Q and I are defined in (9)–(10).

The algebra (9), (13), (16) and one (9), (17) contain the same numbers of basic operators. Moreover, we found the following substitution

$$
|U| = |V|, \quad \frac{U}{U^*} = \left(\frac{V}{V^*}\right)^{\exp(b_2t)}, \quad V = V(\tau, x), \quad \tau = \frac{1}{b_2}\exp(b_2t) \tag{18}
$$

that reduces the algebra (9), (17) to one (9), (13), (16) for the variables $V, \tau, x_1, \ldots, x_n$. It is easily proved that the substitution (18) reduces equation (8) to equation (7) for the function V. So, equation (8) and equation (7) are locally equivalent equations, and are invariant with respect to the algebra of the Hamilton–Jacobi equation.

Note that in [6] the coupled system of Hamilton–Jacobi equations was constructed, which preserves the Lie symmetry of the single Hamilton–Jacobi equation. On the other hand, in [14] generalizations of the Hamilton–Jacobi equations for a complex function were constructed, which are invariant with respect to subalgebras of the algebra of the Hamilton–Jacobi equation.

Finally, we consider the last case, where equation (4) has the nontrivial Lie symmetry. In this case equation (4) has the form

$$iU_t + \Delta U = \left(\frac{\Delta |U|}{|U|} + \frac{1}{2}\lambda_2 \ln \frac{U}{U^*} \right) U. \tag{19}$$

It is easily checked that equation (19) for $\lambda_2 = a_2 + ib_2$ can be reduced with the help of substitution (18) to the same equation but with $\lambda_2 = a_2$. So, we assume that $b_2 = 0$ in equation (19).

Statement 6. *Equation (19) for $\lambda_2 = a_2 \in \mathbb{R}$ is invariant with respect to the Lie algebra with the basic operators (9), (10) at $b_2 = 0$, and*

$$D_1 = 2tP_t + x_a P_a, \quad D_2 = tP_t + \frac{i}{4} \ln \frac{U}{U^*} Q.$$

Note. The substitution

$$U = \rho \exp iW,$$

where $\rho(t, x)$ and $W(t, x)$ are real functions, reduces equation (7) to the following system

$$\frac{\partial \rho}{\partial t} = -\rho \Delta W - 2\frac{\partial \rho}{\partial x_a}\frac{\partial W}{\partial x_a},$$
$$\frac{\partial W}{\partial t} + \frac{\partial W}{\partial x_a}\frac{\partial W}{\partial x_a} = 0,$$

in which the second equation is the Hamilton–Jacobi one.

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