Ansätze of codimension one
for the Navier–Stokes field and reduction
of the Navier–Stokes equation

W.I. FUSHCHYCH, R.O. POPOVYCH, G.V. POPOVYCH

Finding exact solutions of the Navier–Stokes equations (NSEs) for an incompressible viscous fluid is an actual problem of mathematical physics and hydrodynamics. There are some ways to solve this problem. One of them is a usage of symmetry analysis [1–8]. In this article we construct a complete set of inequivalent ansätze of codimension one for the Navier–Stokes field. Using them, we reduce the NSEs to systems of partial differential equations in three independent variables and study their symmetry properties.

It is known that the NSEs

\[ \begin{align*}
\frac{du}{dt} + (u \cdot \nabla)u - \Delta u + \nabla p &= 0, \\
\text{div} \, u &= 0
\end{align*} \]  

are invariant under the infinite dimensional algebra \( A(\text{NS}) \) with basic elements

\[ \begin{align*}
\partial_t &= \partial/\partial t, \\
D &= 2t \partial_t + x_a \partial_a - u^a \partial_a - 2p \partial_p, \\
J_{ab} &= x_a \partial_b - x_b \partial_a + u^a \partial_{u^a} - u^b \partial_{u^b}, \\
R(\tilde{m}(t)) &= m^a(t) \partial_a + m^a(t) \partial_{u^a} - m^a_{tt}(t)x_a \partial_p, \\
Z(\chi(t)) &= \chi(t) \partial_p.
\end{align*} \]  

Here and from now on \( \bar{u} = \bar{u}(t, \bar{x}) = \{u^a\} \) is the velocity field of a fluid, \( p = p(t, \bar{x}) \) is the pressure, \( \bar{x} = \{x_a\} \), \( \partial_t = \partial/\partial t, \partial_a = \partial/\partial x_a, \nabla = \{\partial_a\}, \Delta = \nabla \cdot \nabla, m^a = m^a(t), \chi = \chi(t) \) are arbitrary smooth functions of \( t \) (for example, from \( C^\infty((t_0, t_1), \mathbb{R}) \)), \( a, b = 1, 3 \), \( i, j = 1, 2 \), repetition of an index signifies a sum.

The set of operators (2) determines the maximal, in the sense of Lie, invariance algebra of the NSEs [9–11].

**Theorem 1.** A complete set of \( A(\text{NS}) \)-inequivalent one-dimensional subalgebras of \( A(\text{NS}) \) is exhausted by such algebras:

1. \( A_1^1(\kappa) = \langle D + 2\kappa J_{12} \rangle, \quad \kappa \geq 0; \)
2. \( A_2^1(\kappa) = \langle \partial_t + \kappa J_{12} \rangle, \quad \kappa \in \{0; 1\}; \)
3. \( A_3^1(\eta, \chi) = \langle J_{12} + R(0, 0, \eta(t)) + Z(\chi(t)) \rangle, \)

where algebras \( A_3^1(\eta, \chi) \) and \( A_3^1(\tilde{\eta}, \tilde{\chi}) \) are equivalent if \( \exists \varepsilon, \delta \in \mathbb{R}, \exists \lambda \in C^\infty((t_0, t_1), \mathbb{R}) : \)

\[ \tilde{\eta}(t) = (e^{-\varepsilon \eta} e^{2\varepsilon (\chi + \tilde{\lambda} \eta - \tilde{\eta} \lambda))(te^{2\varepsilon} + \delta); \]
where algebras $A_1^1(\vec{m}, \chi)$ and $A_1^1(\vec{m}, \bar{\chi})$ are equivalent if $\exists \varepsilon, \delta \in \mathbb{R}, \exists c \neq 0, \exists B \in O(3), \exists l \in C^\infty((t_0, t_1), \mathbb{R}^3):$

$$
\exists \eta(\vec{m}, \chi)(t) = (ce^{-\varepsilon}B\vec{m}, ce^{2\varepsilon}(\chi + \vec{l} \cdot \vec{m} - \vec{m} \cdot \vec{l}))(te^{2\varepsilon} + \delta). \tag{4}
$$

Theorem 1 is proved by the method described in [12, 13].

With the algebras $A_1^1-A_4^1$ from theorem 1 and with the algebra $A_4^1$ (if some additional demands are satisfied) one can construct such a set of inequivalent ansätze of codimension one for the Navier–Stokes field:

1. $u_1 = |t|^{-1/2}(v_1 \cos \tau - v_2 \sin \tau) + \frac{1}{2}x_1 t^{-1} - \kappa x_2 t^{-1},$
   
   $$
u_2 = |t|^{-1/2}(v_1 \sin \tau + v_2 \cos \tau) + \frac{1}{2}x_2 t^{-1} + \kappa x_1 t^{-1},$$

   $$u_3 = |t|^{-1/2}v_3 + \frac{1}{2}x_3 t^{-1},$$

   $$p = t^{-1}q + \frac{1}{8}t^{-2}x_a x_a + \frac{1}{2}x^2 t^{-2} \nu^2,$$

where

$$
\tau = \kappa \ln |t|, \quad r = (x_1^2 + x_3^2)^{1/2}, \quad y_1 = |t|^{-1/2}(x_1 \cos \tau + x_3 \sin \tau),
$$

$$y_2 = |t|^{-1/2}(-x_1 \sin \tau + x_3 \cos \tau), \quad y_3 = |t|^{-1/2}x_3.$$ 

here and from now on $\nu^a = \nu^a(y_1, y_2, y_3), \quad q = q(y_1, y_2, y_3)$, numeration of ansätze corresponds to that of algebras in theorem 1.

2. $u_1 = v_1 \cos \kappa t - v_2 \sin \kappa t - \kappa x_2,$

   $$u_2 = v_1 \sin \kappa t + v_2 \cos \kappa t + \kappa x_1,$$

   $$u_3 = v_3, \quad p = q + \frac{1}{2}x^2 \nu^2,$$

where $y_1 = x_1 \cos \kappa t + x_2 \sin \kappa t, \quad y_2 = -x_1 \sin \kappa t + x_2 \cos \kappa t, \quad y_3 = x_3.$

3. $u_1 = x_1 r^{-1}v_1 - x_2 r^{-1}v_2 + x_1 r^{-2},$

   $$u_2 = x_2 r^{-1}v_1 + x_1 r^{-1}v_2 + x_2 r^{-2},$$

   $$u_3 = v_3 + \eta(t)r^{-1}v_2 + \eta(t)\arctg x_2/x_1,$$

   $$p = q - \frac{1}{2}\eta(t)(\eta(t))^{-1}x_3^2 - \frac{1}{2}r^{-2} + \chi(t)\arctg x_2/x_1,$$

where $y_1 = t, \quad y_2 = r, \quad y_3 = x_3 - \eta(t)\arctg x_2/x_1.$

**Remark 1.** The expression for the pressure $p$ from the ansatz (7) is indeterminate in points $t \in \{t_0, t_1\}$, where $\eta(t) = 0$. If there are such points $t$, we will consider the ansatz (7) in intervals $(t_0^n, t_1^n)$ that are contained by the interval $(t_0, t_1)$ and for which one from the conditions

a) $\forall t \in (t_0^n, t_1^n) : \eta(t) \neq 0;$

b) $\eta(t) \equiv 0$ in $(t_0^n, t_1^n)$

is satisfied. In the last case we consider that $\hat{\eta}/\eta := 0$.

4. With the algebra $A_1^1(\vec{m}, \chi)$, an ansatz can be constructed only for such a $t$ wherefor $\vec{m}(t) \neq 0$. If this condition is satisfied, it follows from (2) that the algebra
\( A^1_1(\vec{m}, \chi) \) is equivalent to the algebra \( A^1_1(\vec{m}, 0) \). An ansatz constructed with the algebra \( A^1_1(\vec{m}, 0) \) is

\[
\vec{a} = v^i \vec{m}^i + (\vec{m} \cdot \vec{m})^{-1} v^{i} \vec{m} + (\vec{m} \cdot \vec{x})(\vec{m} \cdot \vec{m})^{-1} \vec{m} - y_i \vec{m}^i,
\]

\[
p = q - \frac{3}{2} (\vec{m} \cdot \vec{m})^{-1} \left( (\vec{m} \cdot \vec{m}) \gamma_i y_i \right)^2 - (\vec{m} \cdot \vec{m})^{-1} (\vec{m} \cdot \vec{x})(\vec{m} \cdot \vec{m})^{-1} (\vec{m} \cdot \vec{x}) + (\vec{m} \cdot \vec{m}) (\vec{m} \cdot \vec{m})^{-2} (\vec{m} \cdot \vec{x})^2,
\]  

(8)

where \( y_i = \vec{n}^i \cdot \vec{x}, y_3 = t \),

\[
\vec{n}^i = \vec{n}^i(t), \quad \vec{n}^i \cdot \vec{m} = \vec{n}^1 \cdot \vec{n}^2 = 0, \quad |\vec{n}^i| = 1, \quad \vec{n}^1 \cdot \vec{n}^2 = 0.
\]  

(9)

**Remark 2.** Vector-functions \( \vec{n}^i \) satisfying conditions (9) exist. They can be constructed in such a way: let us fix vector-functions \( k^i = k^i(t) \) for which \( \vec{k}^i \cdot \vec{m} = \vec{k}^1 \cdot \vec{k}^2 = 0, |\vec{k}^i| = 1 \) and set

\[
\vec{n}^1 = \vec{k}^1 \cos \psi(t) - \vec{k}^2 \sin \psi(t), \quad \vec{n}^2 = \vec{k}^1 \sin \psi(t) + \vec{k}^2 \cos \psi(t).
\]  

(10)

Then \( \vec{n}^1 \cdot \vec{n}^2 = \vec{k}^1 \cdot \vec{k}^2 - \dot{\psi} = 0 \) if \( \int (\vec{k}^1 \cdot \vec{k}^2)dt \).

Substituting the ansätze (5), (6) to the NSEs (1), we obtain reduced systems of PDEs that have the same general form

\[
v^a v^1_a - v^1_a + q_1 + \gamma_1 v^2 = 0,
\]

\[
v^a v^2_a - v^2_a + q_2 - \gamma_1 v^1 = 0,
\]

\[
v^a v^3_a - v^3_a + q_3 = 0,
\]

\[
v^a = \gamma_2,
\]  

(11)

where the constant \( \gamma_i \), takes the values

1. \( \gamma_1 = -2 \chi, \quad \gamma_2 = \frac{3}{2}, \) if \( t > 0 \), \( \gamma_1 = 2 \chi, \quad \gamma_2 = \frac{3}{2}, \) if \( t < 0 \).

2. \( \gamma_1 = -2 \chi, \quad \gamma_2 = 0 \).

For the ansätze (7), (8) reduced equations have the form

\[
v^1 + v^1 v^1 + v^3 v^3 - y_2^{-1} v^2 - [v^3_{12} + (1 + \eta y_2^{-2} y_3 v^1_{33} + 2 \eta y_2^{-2} v^2_{33} + q_2 = 0,
\]

\[
v^2 + v^1 v^2 + v^3 v^3 + y_2^{-1} v^1 - [v^2_{12} + (1 + \eta y_2^{-2} y_3 v^1_{33} + 2 \eta y_2^{-2} v^2_{33} = 0,
\]

\[
+ 2 y_2^{-2} v^2 - \eta y_2^{-1} q_3 + \chi y_2^{-1} = 0,
\]

\[
v^3 + v^1 v^3 + v^2 v^2 - [v^3_{12} + (1 + \eta y_2^{-2} y_3 v^1_{33} - 2 \eta y_2^{-2} v^2_{33} + 2 \eta y_2^{-1} v^2 +
\]

\[
+ 2 \eta y_2^{-1} (y_2^{-1} v^2) v^2 + (1 + \eta y_2^{-2} y_3 - \eta y_2^{-1} q_3 - \chi y_2^{-1} = 0,
\]

\[
y_2^{-1} v^1 + v^2 + v^3 = 0.
\]  

(12)

\[
v^i_{1} + v^1 v^1 - i^2 v^1 + q_4 + \rho^i(y_4) v^3 = 0,
\]

\[
v^3 + v^1 v^3 + v^3 + v^3 = 0,
\]

\[
v^1_{1} + \rho^i(y_4) v^3 = 0,
\]  

(13)

where

\[
\rho^1 = \rho^i(y_4) = 2(\vec{m} \cdot \vec{m})^{-1} (\vec{m} \cdot \vec{n}^i), \quad \rho^3 = \rho^3(y_4) = (\vec{m} \cdot \vec{m})^{-1} (\vec{m} \cdot \vec{n}^i).
\]  

(14)
Let us study symmetry properties of the systems (11)–(13). All following results are obtained with the standard Lie algorithm [11, 12]. At first consider the system (11).

**Theorem 2.** The maximal, in the sense of Lie, invariance algebra of (11) is the algebra

\[ a) \ \langle \partial_\alpha, \partial_\eta, J^{1}_{12} \rangle \quad \text{if} \quad \gamma_1 \neq 0; \]

\[ b) \ \langle \partial_\alpha, \partial_\eta, J^{1}_{ab} \rangle \quad \text{if} \quad \gamma_1 = 0, \ \gamma_2 \neq 0; \]

\[ c) \ \langle \partial_\alpha, \partial_\eta, J^{1}_{ab}, D^1 \rangle \quad \text{if} \quad \gamma_1 = \gamma_2 = 0. \]

Here

\[ J^{1}_{ab} = y_0 \partial_0 - y_0 \partial_a + v^a \partial_v - v^b \partial_v, \quad D^1 = y_0 \partial_a - v^a \partial_v - 2q \partial_q. \]

All Lie symmetry operators of (11) are induced by operators from A(NS). Namely, the operators \( J^{1}_{ab}, D^1 \) are induced by \( J_{ab}, D \) and the operators \( c_a \partial_a \) \( (c_a = \text{const}) \), \( \partial_q \) is done by

\[ R(|t|^{1/2}(c_1 \cos \tau - c_2 \sin \tau, c_1 \sin \tau + c_2 \cos \tau, c_3)), \quad Z(|t|^{-1}) \]

for the anzatz (5) and by

\[ R(c_1 \cos \omega t - c_2 \sin \omega t, c_1 \sin \omega t + c_2 \cos \omega t, c_3), \quad Z(1) \]

for the anzatz (6) respectively. Therefore, Lie reduction of the system (11) gives only solutions that can be obtained by reducing the NSEs with two- and three-dimensional subalgebras of A(NS). Let us proceed to the system (12). Let \( A^{\text{max}} \) be the maximal, in the sense of Lie, invariance algebra of (12). Studying symmetry properties of (12), one has to consider the following cases.

A. \( \eta, \chi \equiv 0 \). Then

\[ A^{\text{max}} = \langle \partial_1, D^1_2, R_1(\psi(y_1)), Z^1(\lambda(y_1)) \rangle, \]

where \( D^1_2 = 2y_1 \partial_1 + y_2 \partial_2 + y_3 \partial_3 - v^a \partial_v - 2q \partial_q, \quad Z^1(\lambda(y_1)) = \lambda(y_1) \partial_q, \quad R_1(\psi(y_1)) = \psi \partial_1 + \psi_1 \partial_2 - \psi_1 y_3 \partial_2; \) here and from now on \( \psi = \psi(y_1), \lambda = \lambda(y_1) \) are arbitrary smooth functions of \( y_1 = t \).

B. \( \eta \equiv 0, \chi \neq 0 \). In this case expansion of \( A^{\text{max}} \) is for \( \chi = (C_1y_1 + C_2)^{-1}, \) where \( C_1, C_2 = \text{const} \). Let \( C_1 \neq 0 \). It can be done with the equivalence transformation (3) so that the constant \( C_2 \) will vanish, i.e. \( \chi = Cy^{-1} \) where \( C = \text{const} \). Then

\[ A^{\text{max}} = \langle D^1_2, R_1(\psi(y_1)), Z^1(\lambda(y_1)) \rangle. \]

If \( C_1 = 0, \chi = C = \text{const} \) and

\[ A^{\text{max}} = \langle \partial_1, R_1(\psi(y_1)), Z^1(\lambda(y_1)) \rangle. \]

For other values of \( \chi, \) i.e. when \( \chi_{11} \chi \neq \chi_1 \chi_1 \),

\[ A^{\text{max}} = \langle R_1(\psi(y_1)), Z^1(\lambda(y_1)) \rangle. \]

C. \( \eta \neq 0 \). With the equivalence transformation (3), we do \( \chi = 0 \). In this case expansion of \( A^{\text{max}} \) is for \( \eta = \pm(C_1y_1 + C_2)^{1/2}, \) where \( C_1, C_2 = \text{const} \). Let \( C_1 \neq 0 \).
It can be done with the equivalence transformation (3) so that the constant \( C_2 \) will vanish, i.e. \( \eta = C|y_1|^{1/2} \), where \( C = \text{const} \). Then

\[
A^{\text{max}} = \langle D_2^1, Z^1(\lambda(y_1)), R_2(|y_1|^{1/2}), R_2(|y_1|^{1/2} \ln |y_1|) \rangle,
\]

where \( R_2(\psi(y_1)) = \psi \partial_3 + \psi_1 \partial_{v^a} \). If \( C_1 = 0 \), i.e. \( \eta = C = \text{const} \),

\[
A^{\text{max}} = \langle \partial_1, Z^1(\lambda(y_1)), \partial_3, y_1 \partial_3 + \partial_{v^a} \rangle.
\]

For other values of \( \eta \), i.e. when \( \langle \eta^2 \rangle_{11} \neq 0 \),

\[
A^{\text{max}} = \langle Z^1(\lambda(y_1)), R_2(\eta(y_1)), R_2(\eta(y_1) \int (\eta(y_1))^{-2} dy_1) \rangle.
\]

In all cases considered above, Lie symmetry operators of (12) are induced by operators from A(NS). Namely, the operators \( \partial_1, D^2_2, Z^1(\lambda(y_1)) \) are induced by \( \partial_1, D, Z(\lambda(t)) \) respectively. In case \( \eta \equiv 0 \) the operator \( R_1(\psi(y_1)) \) and in case \( \eta \neq 0 \) the operator \( R_1(\psi(y_1)) \) where \( \psi_\eta - \psi_\bar{\eta} = 0 \) are done by \( R(0,0,\psi(t)) \). Therefore, Lie reduction of the system (12) gives only solutions that can be obtained by reducing the NSEs with two- and three-dimensional subalgebras of A(NS).

When \( \eta = \chi = 0 \) the system (12) describes axisymmetric motion of a fluid and can be transformed into a system of two equations for a stream function \( \Psi^1 \) and a function \( \Psi^2 \) that are determined by

\[
\Psi^1_3 = y_2 v^1, \quad \Psi^2_1 = -y_2 v^3, \quad \Psi^2 = y_2 v^2.
\]

The transformed system has been studied by L.V. Kapitanskiy [8].

Consider the system (13). Let us introduce the notations

\[
t = y_3, \quad \rho = \int \rho^3(t) dt, \quad R_3(\psi^1(t), \psi^2(t)) = \psi^i \partial_i + \psi^i_1 \partial_{v^a} - \psi^i_{1a} \partial_{y_1} y_i \partial_{y_1},
\]

\[
Z^1(\lambda(t)) = \lambda(t) \partial_y, \quad S = \partial_{v^a} - \rho^1(t) y_i \partial_{y_i},
\]

\[
E(\chi(t)) = 2 \chi \partial_3 + \chi t y_i \partial_{y_i} + (\chi \partial_3 - \chi v^3) \partial_{v^3} - \left(2 \chi q + \frac{1}{2} \chi y_i y_j \right) \partial_{y_i},
\]

\[
J_{12} = y_i \partial_2 - y_2 \partial_i + v^1 \partial_{v^3} - v^2 \partial_{v^1}.
\]

**Theorem 3.** The maximal, in the sense of Lie, invariance algebra of (13) is the algebra

1)  \[ \langle R_3(\psi^1(t), \psi^2(t)), Z^1(\lambda(t)), S, E(\chi(t)), E(\chi^2(t)), v^3 \partial_{v^a}, J_{12} \rangle, \]

where \( \chi^1 = e^{-\hat{\rho}(t)} \int e^{\hat{\rho}(t)} d\rho, \chi^2 = e^{-\hat{\rho}(t)} \), if \( \rho^1 = \rho^2 = 0, \)

2)  \[ \langle R_3(\psi^1(t), \psi^2(t)), Z^1(\lambda(t)), S, E(\chi(t)) + 2 a_1 v^3 \partial_{v^3} + 2 a_2 J_{12} \rangle, \]

where \( a_1, a_2, a_3 \) are fixed constants, \( \chi = e^{-\hat{\rho}(t)} \int e^{\hat{\rho}(t)} d\rho + a_3 \) if

\[
\rho^1 = e^{2 \hat{\rho}(t)} (\rho(t))^{-\frac{1}{2} - a_1} (C_1 \cos(a_2 \ln \rho(t)) - C_2 \sin(a_2 \ln \rho(t))),
\]

\[
\rho^2 = e^{2 \hat{\rho}(t)} (\rho(t))^{-\frac{1}{2} - a_1} (C_1 \sin(a_2 \ln \rho(t)) + C_2 \cos(a_2 \ln \rho(t))),
\]

where \( \hat{\rho}(t) = \int e^{\hat{\rho}(t)} d\rho + a_3, C_1, C_2 = \text{const}, (C_1, C_2) \neq (0,0); \)

3)  \[ \langle R_3(\psi^1(t), \psi^2(t)), Z^1(\lambda(t)), S, E(\chi(t)) + 2 a_1 v^3 \partial_{v^3} + 2 a_2 J_{12} \rangle, \]

\[
(15)
\]
where \( \alpha_1, \alpha_2 \) are fixed constants, \( \chi = e^{-\tilde{\rho}(t)} \) if
\[
\rho^1 = e^{\frac{3}{2} \tilde{\rho}(t) - \alpha_1 \tilde{\rho}(t)}(C \sin(\alpha_2 \tilde{\rho}(t)) - C_2 \sin(\alpha_2 \tilde{\rho}(t))),
\]
\[
\rho^2 = e^{\frac{3}{2} \tilde{\rho}(t) - \alpha_1 \tilde{\rho}(t)}(C \sin(\alpha_2 \tilde{\rho}(t)) + C_2 \sin(\alpha_2 \tilde{\rho}(t))),
\]
where \( \tilde{\rho}(t) = \int e^{\tilde{\rho}(t)} dt, \) \( C_1, C_2 = \text{const}, \) \( (C_1, C_2) \neq (0, 0) \).

4) \( \langle R_3(\psi^1(t), \psi^2(t)), Z^1(\lambda(t)), S \rangle \) in all other cases.

Here \( \psi^3 = \psi^3(t), \lambda = \lambda(t) \) are arbitrary smooth functions of \( t = y_3 \).

**Remark 3.** If functions \( \rho^b = \rho^b(t) \) are determined by \( (14), \) \( e^{\tilde{\rho}(t)} = C|\bar{m}(t)|, \) where \( C = \text{const} \) and it follows from the condition \( \rho^1 = \rho^2 = 0 = \) that \( \bar{m} = |\bar{m}(t)|\hat{e}, \) where \( |\hat{e}| = 1, \hat{e} = \text{const}. \)

**Remark 4.** Vector-functions \( \bar{n}^i \) from remark 2 are determined up to the transformation
\[
\bar{n}^1 = \bar{n}^1 \cos \delta - \bar{n}^2 \sin \delta, \quad \bar{n}^2 = \bar{n}^1 \sin \delta + \bar{n}^2 \cos \delta,
\]
where \( \delta = \text{const}. \) Therefore, choosing \( \delta, \) we can do so that \( C_2 = 0 \) (then \( C_1 \neq 0 \)).

The operators \( R_3(\psi^1, \psi^2) + \alpha S, Z^1(\lambda) \) are induced by \( R(\bar{I}) + Z(\chi), Z(\lambda) \) respectively, where \( \bar{I} = \psi^3 \bar{n}^3 + \psi^3 \bar{m}, \psi^3(\bar{n}^3 \cdot \bar{m}) + 2\psi^3(\bar{n}^3 \cdot \bar{m}) = \alpha, \chi = \frac{3}{2}(\bar{m} \cdot \bar{m})^{-1}(-\psi^3(\bar{n}^3 \cdot \bar{m})), \)
\[
\text{if } \bar{m} = |\bar{m}(t)|\hat{e}, \text{ where } \hat{e} = \text{const}, |\hat{e}| = 1, \text{ the operator } J^1_{12} \text{ is induced by } e^{-J_{23}} + eJ_{31} + eJ_{12}. \)

For
\[
\bar{m} = \beta_3 e^{a_1 t}(\beta_2 \cos \tau, \beta_2 \sin \tau, \beta_1)^T, \quad \beta_1^2 + \beta_2^2 = 1, \quad \tau = \alpha t + \delta,
\]
the operator \( \partial_t + \alpha J_{12} \) induces the operator \( \partial_{y_3} - \beta_1 \alpha J_{12} + \sigma \bar{v}^3 \partial_{x^2} \) if such vector-functions \( \bar{m} \) are chosen:
\[
\bar{n}^1 = \bar{k}^1 \cos \beta_1 \tau + \bar{k}^2 \sin \beta_1 \tau, \quad \bar{n}^2 = -\bar{k}^1 \sin \beta_1 \tau + \bar{k}^2 \cos \beta_1 \tau,
\]
where \( \bar{k}^1 = (- \sin \tau, \cos \tau, 0)^T, \bar{k}^2 = (\beta_1 \cos \tau, \beta_1 \sin \tau, -\beta_2)^T. \) For
\[
\bar{m} = \beta_3 |t + \beta_4|^{|+1/2}(\beta_2 \cos \tau, \beta_2 \sin \tau, \beta_1)^T, \quad \beta_1^2 + \beta_2^2 = 1
\]
\[
\tau = \alpha \ln |t + \beta_4| + \delta,
\]
the operator \( D + 2\beta_4 \partial_t + 2\alpha J_{12} \) induces the operator \( D^1_{12} + 2\beta_4 \partial_{y_3} - 2\beta_1 \alpha J_{12} + 2\sigma \bar{v}^3 \partial_{x^2} \) if vector-functions \( \bar{n}^i \) are chosen in the form \( (15). \) In all other cases the basis elements of the maximal, in the sense of Lie, invariance algebra of \( (13) \) are not induced by operators from \( A(\text{NS}). \)

**Remark 5.** The invariance algebra of a system of the form \( (13) \) with a parameter-function \( \rho^3 = \rho^3(t) \) is like one with a different parameter-function \( \tilde{\rho}^3 = \rho^3(t). \) It suggest an idea that there is a local transformation of variables with which one can make \( \rho^3 \) to vanish. Indeed, let us transform variables in the way
\[
\bar{y}_i = y_i e^{\frac{1}{2} \tilde{\rho}(t)}, \quad \bar{y}_3 = \int e^{\tilde{\rho}(t)} dt, \quad \bar{\psi}^i = \left( \psi^i + \frac{1}{2} y_i \rho^3(t) \right) e^{-\frac{1}{2} \tilde{\rho}(t)}, \quad \bar{\psi}^3 = \psi^3,
\]
\[
\bar{q} = q e^{-\tilde{\rho}(t)} + \frac{1}{8} y_i \bar{y}_i (\rho^3(t))^2 - 2 \rho^3(t) e^{-\tilde{\rho}(t)}.
\]
As a result, we obtain the system
\[
\begin{align*}
\tilde{v}_3^i + \tilde{v}_j^i \tilde{v}_{jj}^i + \tilde{q}_i + \tilde{\rho}^i(\tilde{y}_3)\tilde{v}^3 = 0, \\
\tilde{v}_3^3 + \tilde{v}_j^3 - \tilde{v}_{jj}^3 = 0, \\
\tilde{v}_j^3 = 0,
\end{align*}
\]
for functions \(\tilde{v}^a = \tilde{v}^a(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3), \tilde{q} = \tilde{q}(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)\), where \(\tilde{\rho}^i(\tilde{y}_3) = \rho^i(t)e^{-\frac{2}{3}\tilde{\rho}(t)}\), subscripts 1, 2, 3 mean differentiation with respect to \(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3\) accordingly.