Symmetry analysis and ansatzes for the Schrödinger equations with the logarithmic nonlinearity

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Symmetry properties of the Schrödinger equations with the nonlinearity $u\ln(uu^*)$ are investigated. It is shown that these equations are invariant with respect to various extensions of the Galilei algebra $AG(1,n)$. The conditional symmetry of these nonlinear Schrödinger equations are investigated. Lie, non-Lie dimensional reduction and reduction by number of dependent variables carried out. The exact solutions of these equations are constructed.

1. Introduction. Let us consider the Schödinger equations with the logarithmic nonlinearity:

$$Su \equiv bu\ln(uu^*), \quad b \in \mathbb{R} \tag{1}$$

and

$$Su \equiv (\lambda_1 + i\lambda_2)u\ln(uu^*), \quad \lambda_2 \neq 0, \tag{2}$$

where $S = i\frac{\partial}{\partial x_0} + \lambda\Delta$, $x_0 \equiv t$, $\Delta = \frac{\partial^2}{\partial x_a \partial x_a}$, $a = \overline{1,n}$, $\lambda, \lambda_i \in \mathbb{R}$, n is the number of space variables.

For the case when b is a real constant the equation (1) is equivalent to the equation suggested by I. Bialynicki-Birula and J. Mycielski [1]. The equation (1) is investigated by many authors using different methods (see e.q. [2, 3]). For this case the equation of continuity:

$$\begin{gathered}\frac{\partial\rho}{\partial x_0} + \operatorname{div}\boldsymbol{j} = 0,\\ \rho = (uu^*), \quad \boldsymbol{j} = (j_1, j_2, \ldots, j_n), \quad j_a = -i\lambda\left(u^*\frac{\partial u}{\partial x_a} - u\frac{\partial u^*}{\partial x_a}\right), \quad a = \overline{1,n}\end{gathered} \tag{3}$$

is satisfied.

For the case when $\lambda_2 \neq 0$ the equation of continuity (3) is not satisfied and the formula:

$$\frac{\partial\rho}{\partial x_0} + \operatorname{div}\boldsymbol{j} = \lambda_2\rho\ln\rho$$

can be considered instead of condition (3).

For the equation (2) the conditions:

$$\frac{\partial\rho}{\partial x_0}j_a + \frac{\partial}{\partial x_b}T_{ab} = 0,$$

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where T_{ab} is the stress tensor, $a, b = \overline{1, n}$, are not satisfied (in contrast with the case of the equation (1) [1]).

It will be shown further, that symmetry properties of the equations (1) and (2) are essentially different.

2. Lie symmetry. It is well-known that the equations (1), (2) are invariant under the Galilei algebra $AG(1,n)$ generates by operators:

$$
\begin{gathered}
P_0 = \frac{\partial}{\partial x_0}, \quad P_a = \frac{\partial}{\partial x_a}, \quad J_{ab} = x_a P_b - x_b P_a, \\
Q = i\left(u\frac{\partial}{\partial u} - u^*\frac{\partial}{\partial u^*}\right), \quad G_a = x_0 P_a + \frac{x_a}{2\lambda} Q.
\end{gathered} \tag{4}
$$

However, it appears that the Lie symmetry of the Schrödinger equations with logarithmic nonlinearity are not exhausted by the algebra (4).

Theorem 1. *The equation (1) is invariant with respect to the algebra:*

$$
AG_3(1,n) = \langle AG(1,n), B\rangle, \tag{5}
$$

where $B = I - 2bx_0 Q$*,* $I = u\frac{\partial}{\partial u} + u^*\frac{\partial}{\partial u^*}$.

Theorem 2. *The equation (2) is invariant with respect to the algebra:*

$$
AG_4(1,n) = \langle AG(1,n), C\rangle, \tag{6}
$$

where $C = \exp\{2\lambda_2 x_0\}\left(I - \frac{\lambda_1}{\lambda_2}Q\right)$*, when* $\lambda_2 \neq 0$.

The above theorems can be proved using the Lie algorithm [4, 5].

The operator C generates the following finite transformations [6]:

$$
\begin{gathered}
x_0 \to x_0' = x_0, \quad x_a \to x_a' = x_a, \\
u \to u' = \exp\left\{\theta\left(1 - i\frac{\lambda_1}{\lambda_2}\right)\exp(2\lambda_2 x_0)\right\} u,
\end{gathered} \tag{7}
$$

where θ is a group parameter.

Under transformations (7), the equation (2) becomes:

$$
\exp\left\{-\theta\left(1 - i\frac{\lambda_1}{\lambda_2}\right)\exp(2\lambda_2 x_0)\right\}[Su' - (\lambda_1 + i\lambda_2)u'\ln(u'u'^*)].
$$

This shows that the equation (2) is invariant with respect to the operator C.

Note 1. Solutions of the equation (1) can be generated by means of transformations [1]:

$$
x_0 \to x_0' = x_0, \quad x_a \to x_a' = x_a, \quad u \to u' = \exp\{\theta(1 - 2ibx_0)\}u
$$

which are generated by the operator B.

From the commutation relations for the operator C

$$
[C, P_0] = d_1 C, \quad [C, P_a] = [C, J_{ab}] = [C, Q] = [C, G_a] = 0
$$

and for the operator B

$$
[B, P_0] = d_2 Q, \quad [B, P_a] = [B, J_{ab}] = [B, Q] = [B, G_a] = 0, \quad d_1, d_2 \in \mathbb{R}
$$

it follows that the algebras $AG_3(1,n)$ and $AG_4(1,n)$ differ.

3. Lie reduction by number of independent variables. In this paper we systematically use symmetry properties of equations (1) and (2) to find their exact solutions. The method of finding exact solutions of differential equations is based on Lie's ideas of invariant solutions and it is described in full detail in [4, 5].

In this section we describe the some ansatzes of codimension 1 and 2

$$u = f(x_0, \boldsymbol{x})\rho(\omega_1, \omega_2)\exp\{g(x_0, \boldsymbol{x}) + \varphi(\omega_1, \omega_2)\},$$

where the functions f, g and new variables $\omega_i = \omega_i(x_0, \boldsymbol{x})$ are determined by means of operators of subalgebras of $AG_3(1,n)$ and $AG_4(1,n)$.

Let us consider some subalgebras of $AG_3(1,n)$, which reduce the equation (1) to system of differential equations with one and two independent variables.

1) $\langle B + \alpha P_0, J_{ab}\rangle$, $\alpha \neq 0$. The ansatz and corresponding systems of reduced equations has the form:

$$u = \exp\left\{\frac{x_0}{\alpha}\right\}\rho(\omega)\exp\left\{i\left[-\frac{b}{\alpha}x_0^2 + \varphi(\omega)\right]\right\}, \quad \alpha \neq 0, \quad \alpha \in \mathbb{R}, \tag{8}$$

where $\omega = (\boldsymbol{x}^2)^{1/2}$, $\boldsymbol{x}^2 = x_1^2 + \cdots + x_n^2$ and

$$\frac{1}{\alpha}\rho + 2\lambda\dot\rho\dot\varphi + \lambda\rho\ddot\varphi + \lambda\rho\frac{n-1}{\omega}\dot\varphi = 0,$$
$$\lambda\ddot\rho + \lambda(n-1)\omega^{-1}\rho - \lambda\rho\dot\varphi^2 = 2b\rho\ln\rho,$$

where $\dot\rho = \frac{\partial\rho}{\partial\omega}$, $\dot\varphi = \frac{\partial\varphi}{\partial\omega}$, $\ddot\rho = \frac{\partial^2\rho}{\partial\omega^2}$, $\ddot\varphi = \frac{\partial^2\varphi}{\partial\omega^2}$.

2) $\langle B + \alpha P_0, J_{12} + \beta P_3\rangle$, $\alpha, \beta \neq 0$, $\alpha, \beta \in \mathbb{R}$

$$u = \exp\left\{\frac{x_0}{\alpha}\right\}\rho(\omega_1, \omega_2)\exp\left\{i\left[-\frac{b}{\alpha}x_0^2 + \varphi(\omega_1, \omega_2)\right]\right\}, \quad \alpha \neq 0, \quad \alpha \in \mathbb{R}, \tag{9}$$

where

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = \operatorname{arctg}\frac{x_2}{x_1} - \frac{x_3}{\beta}.$$

The system of reduced equations has the form (for the case $n = 3$)

$$\alpha^{-1}\rho + 2\lambda\rho_1\varphi_1 + 2\lambda\rho_2\varphi_2(\omega_1^{-2} + \beta^{-2}) + \lambda\rho\varphi_{11} +$$
$$+ \lambda\rho\varphi_{22}(\omega_1^{-2} + \beta^{-2}) + \lambda\rho\omega_1^{-1}\varphi_1 = 0,$$
$$\lambda\rho_{11} + \lambda\rho_{22}(\omega_1^{-2} + \beta^{-2}) + \lambda\rho\omega_1^{-1} - \lambda\rho\varphi_1^2 + \rho\varphi_2^2(\omega_1^{-2} + \beta^{-2}) = 2b\rho\ln\rho.$$

3) The ansatz

$$u = \exp\left\{\frac{x_0}{\alpha}\right\}\rho(\omega_1, \omega_2)\exp\left\{i\left[\frac{x_0 x_1}{\alpha} - \frac{b}{\alpha}x_0^2 - \frac{x_0^3}{6\lambda\alpha} + \varphi(\omega_1, \omega_2)\right]\right\} \tag{10}$$

when $n = 3$ reduces equation (1) to the system:

$$\begin{aligned} &2\lambda\rho_1\varphi_1 + 2\lambda\rho_2\varphi_2 + \alpha^{-1}\rho + \lambda\rho(\omega_2^{-1}\varphi_2 + \varphi_{11} + \varphi_{22}) = 0,\\ &\lambda\rho_{11} + \lambda\rho_{22} + \lambda\omega_2^{-1}\rho_2 - \lambda\rho(\varphi_1^2 + \varphi_2^2) = 2b\rho\ln\rho - (2\lambda\alpha)^{-1}\rho\omega_1, \end{aligned} \tag{11}$$

where

$$\omega_1 = \frac{\lambda x_0^2}{\alpha} - x_1, \quad \omega_2 = (x_2^2 + x_3^2)^{1/2}, \quad \alpha \neq 0,$$
$$\rho_i = \frac{\partial \rho}{\partial \omega_i}, \quad \varphi_i = \frac{\partial \varphi}{\partial \omega_i}, \quad i = 1, 2.$$

Solving the system of reduced equations (11) one can following partial solution of the equation (1)

$$u = \exp\left\{ \frac{x_0^2}{8\lambda\alpha b} + \frac{x_0}{\alpha} - d_1^a x_a + c_1 + \right.$$
$$\left. + i\left[-\frac{x_0}{6\lambda\alpha} - \frac{2b}{\alpha} x_0^2 + \frac{x_0}{\alpha} d_2^a x_a + d_3^a x_a + c_2 \right] \right\}, \tag{12}$$

where $d_k^a, c_i, \alpha \in \mathbb{R}$, $k = 1, 2, 3$, $a = \overline{1, n}$ and d_k^a satisfy the following conditions:

$$d_1^a d_1^a = \frac{1}{8b^2\lambda^2\alpha}, \quad d_1^a d_2^a = \frac{1}{8\lambda^2 b}, \quad d_1^a d_3^a = -\frac{1}{2\lambda\alpha},$$
$$d_2^a d_2^a = \frac{\alpha}{4\lambda^2}, \quad d_2^a d_3^a = -\frac{b}{\lambda}, \quad d_3^a d_3^a = \frac{1}{16\lambda\alpha b^2} - 2bc_1.$$

It is easy to see that the exact solution (12) of the nonlinear equation (1) is non-analytical by b.

Note 2. The ansatzes (7)–(9) follows from the fact that the equation (1) is invariant to the operator B.

Let us adduce some examples of reduction of equation (2).

Example 1. $\langle C + \alpha P_3, J_{12} \rangle$. The ansatz

$$u = \exp\left\{ \frac{1}{\alpha} \exp(2\lambda_2 x_0) x_3 \right\} \rho(\omega_1, \omega_2) \times$$
$$\times \exp\left\{ i\left[-\frac{\lambda_1}{\alpha\lambda_2} \exp(2\lambda_2 x_0) x_3 + \varphi(\omega_1, \omega_2) \right] \right\}, \quad \alpha \neq 0, \tag{13}$$

where $\omega_1 = x_0$, $\omega_2 = (x_1^2 + x_2^2)^{1/2}$ reduces equation (2) (when $n = 3$) to the system:

$$\rho_1 + 2\lambda\rho_2\varphi_2 + \lambda\rho(\omega_2^{-1}\rho_2 + \varphi_{22}) = 2\lambda_2\rho\ln\rho,$$
$$\alpha^{-2}\lambda\exp(4\lambda_2\omega_1)(1 - \lambda_1^2\lambda_2^{-2})\rho + \lambda\rho_{22} + \lambda\omega_2^{-1}\rho_2 - \rho\varphi_1 - \lambda\rho\varphi_2^2 = 2\lambda_1\rho\ln\rho.$$

Example 2. The ansatz

$$u = \exp\left\{ \frac{1}{2\lambda_2\alpha} \exp(2\lambda_2 x_0) \right\} \rho(\omega) \exp\left\{ i\left[-\frac{\lambda_1}{2\alpha\lambda_2^2} \exp(2\lambda_2 x_0) + \varphi(\omega) \right] \right\}, \tag{14}$$

where $\omega = (\boldsymbol{x}^2)^{1/2}$ reduces (2) when $\lambda_2 \neq 0$ to the system of ODE:

$$2\lambda\dot\rho\dot\varphi + \lambda\rho\ddot\varphi + \lambda\rho\omega^{-1}(n-1)\dot\varphi = 2\lambda_2\rho\ln\rho,$$
$$\lambda\ddot\rho + \lambda(n-1)\omega^{-1}\rho + \lambda\rho\dot\varphi^2 = 2\lambda_1\rho\ln\rho.$$

Example 3. The ansatz

$$u = \exp\left\{\operatorname{arctg}\frac{x_2}{x_1}\exp(2\lambda_2 x_0)\right\}\rho(\omega_1,\omega_2)\times$$
$$\times\exp\left\{i\left[\frac{\lambda_1}{\lambda_2}\exp(2\lambda_2 x_0)\operatorname{arctg}\frac{x_2}{x_1}+\frac{x_3^2+\cdots+x_n^2}{4\lambda x_0}+\varphi(\omega_1,\omega_2)\right]\right\}, \tag{15}$$

where $\omega_1 = x_0$, $\omega_2 = (x_1^2+x_2^2)^{1/2}$ reduces the equation (2) (when $n \geq 2$) to the system:

$$\rho_1 + 2\lambda\exp(4\lambda_2\omega_1)\frac{\lambda_1}{\lambda_2}\rho\omega_2^{-2} + \omega_2^2\varphi_2\rho_2 + 2\lambda\rho\omega_2^{-1}\varphi_2 + \lambda\rho\varphi_{22} + \frac{n-2}{2\omega_1}\rho =$$
$$= 2\lambda_2\rho\ln\rho,$$
$$\lambda\exp(4\lambda_2\omega_1)\omega_2^{-2}\rho(1-\lambda_1^2\lambda_2^{-2}) + 2\lambda\rho_2(1+\omega_2^{-1}) - \rho\varphi_1 - \lambda\rho\varphi_2^2 = 2\lambda_1\rho\ln\rho.$$

Example 4. The ansatz

$$u = \exp(\exp(2\lambda_2 x_0)x_1\rho(x_0)\times$$
$$\times\exp\left\{i\left[\frac{\lambda_1}{\lambda_2}x_1\exp(2\lambda_2 x_0)+\frac{x_2^2+\cdots+x_n^2}{4\lambda x_0}+\varphi(x_0)\right]\right\}, \tag{16}$$

reduces the equation (2) when $\lambda_2 \neq 0$ to the system:

$$\dot\rho - 2\lambda\lambda_1\lambda_2^{-1}\rho\exp(4\lambda_2 x_0) + \frac{n-1}{2x_0} = 2\lambda_2\rho\ln\rho,$$
$$\dot\varphi = \lambda\exp(4\lambda_2 x_0) + \lambda\lambda_1\lambda_2^{-1}\exp(2\lambda_2 x_0) - 2\lambda_1\ln\rho. \tag{17}$$

The system of equations (17) by means of the change of variables $\rho = \exp\phi$ is reduced to a linear system of ODE which has the general solution of the form

$$\phi = \frac{\lambda\lambda_1}{\lambda_2^2}\exp(4\lambda_2 x_0) - \exp(2\lambda_2 x_0)\left(d_1 + \frac{n-1}{2}F(2\lambda_2)\right), \quad d_1\in\mathbb{R},$$
$$\varphi = \frac{\lambda}{4\lambda_2}\exp(4\lambda_2 x_0)\left(1-\frac{2\lambda_1^2}{\lambda_2^2}\right) + \frac{\lambda_1}{2\lambda_2}(\lambda+2d_1)\exp(2\lambda_2 x_0) + \tag{18}$$
$$+\lambda_1(n-1)\int F(2\lambda_2)\exp(2\lambda_2 x_0)dx,$$

where

$$F(\theta) = \int\exp(-\theta x_0)\frac{dx_0}{x_0}.$$

The substitution of (18) into the ansatz (16) gives the following solution of the equation (2) when $\lambda_2 \neq 0$ for $n = 1$

$$u = \exp\left\{(x_1-d_1)\exp(2\lambda_2 x_0) + \frac{\lambda\lambda_1}{\lambda_2^2}\exp(4\lambda_2 x_0) + \right.$$
$$\left. + i\left[\left(\frac{\lambda\lambda_1+2\lambda_1\lambda_2 d_1}{2\lambda_2^2} - \frac{\lambda_1}{\lambda_2}x_1\right)\exp(2\lambda_2 x_0) + \frac{\lambda\lambda_2^2-2\lambda\lambda_1^2}{4\lambda_2^3}\exp(2\lambda_2 x_0)\right]\right\}.$$

Note 3. The ansatzes (13)–(16) are obtained from the fact that the equation (2) is invariant with respect to the algebra $AG_4(1,n)$ (as distinct from the equation (1)).

4. Component-wise reduction. The reduction by number of dependent variables of the equations (1), (2) is possible because of invariance these equations respectively to the operators B and C.

1) For reduction of the equation (1) by operator B it is necessary to change or variables:

$$W = F(x_0, \boldsymbol{x}) - i(4bx_0)^{-1}\ln(u/u^*), \quad V = \ln|u| - i(4bx_0)^{-1}\ln(u/u^*), \tag{19}$$

where F is a some real function.

Then the change of variables (19) is constructed, the equation (1) has the form:

$$\begin{aligned}
&F_0 - W_0 + V_0 + 4\lambda bx_0(F_a - W_a + V_a)(W_a + F_a) + 2\lambda bx_0(\Delta W - \Delta F) = 0,\\
&\lambda(F_a - W_a + V_a)(F_a - W_a + V_a) + \lambda(\Delta F - \Delta W + \Delta V) -\\
&\qquad - 2bx_0(W_0 - F_0) - 4\lambda b^2x_0^2(W_a - F_a)(W_a - F_a) = 2bV,
\end{aligned}$$

where $F_\mu = \frac{\partial F}{\partial x_\mu}$, $W_\mu = \frac{\partial W}{\partial x_\mu}$, $V_\mu = \frac{\partial V}{\partial x_\mu}$, $\Delta = \frac{\partial^2}{\partial x_a \partial x_a}$ and the operator B has the form $B = \frac{\partial}{\partial W}$.

The reduction of the equation (1) by operator B is equivalent to the condition $W = 0$.

Thus, we can find the solutions of the equation (1) in the form:

$$u = \exp\{V(x_0, \boldsymbol{x}) + (1 - 2ibx_0)F(x_0, \boldsymbol{x})\}, \tag{20}$$

where functions V and F satisfy the system:

$$\begin{aligned}
&F_0 + V_0 - 4\lambda bx_0(F_a + V_a)F_a - 2\lambda bx_0\Delta F = 0,\\
&\lambda(F_a + V_a)(F_a + V_a) + \lambda(\Delta F + V) + 2bx_0(F_0 - 2\lambda bx_0F_aF_a) = 0.
\end{aligned} \tag{21}$$

Case 1. The functions V and F satisfy the conditions:

$$F = f_1(x_0), \quad V = f_2(x_0) + \varphi(\omega), \quad \omega = \omega(\boldsymbol{x}). \tag{22}$$

Substitution of the expression (22) into (21) yields the ODE

$$(\ddot{\varphi} + \dot{\varphi}^2)\theta_1(\omega) + \dot{\varphi}\theta_2(\omega) = 2b\lambda^{-1}\varphi, \tag{23}$$

where

$$\omega_a\omega_a = \theta_1(\omega), \quad \Delta\omega = \theta_2(\omega), \tag{24}$$

and

$$f_1 = c_2 - c_1x_0^{-1}, \quad f_2 = c_1x_0^{-1}, \quad c_1, c_2 \in \mathbb{R}. \tag{25}$$

Note 4. The necessary conditions of compatibility and the general solution of system (24) construct in papers [7, 8].

For the partially case $\omega = \alpha_a x_a$, $\alpha_a\alpha_a = 1$, $\alpha_a \in \mathbb{R}$, $a = \overline{1,n}$, the equation (23) has the form:

$$\ddot{\varphi} + \dot{\varphi}^2 = 2b\lambda^{-1}\varphi. \tag{26}$$

This equation by means of change of variables

$$\dot{\varphi}^2 = \Phi(\varphi)$$

is reduced to a linear equation:

$$\dot{\Phi}(\varphi) + 2\Phi(\varphi) = 4b\lambda^{-1}\varphi.$$

The last equation can be easily integrated and the result is as follows:

$$\int \left[\varphi + c\exp(-2\varphi) - \frac{1}{2}\right]^{-1/2} d\varphi = (2b\lambda^{-1})^{1/2} d\omega, \quad c \in \mathbb{R}. \tag{27}$$

When $c = 0$ we get from (27) the following solution of (26):

$$\varphi = \frac{b}{2\lambda}(\omega + c_3)^2 + \frac{1}{2}, \quad c_3 \in \mathbb{R}. \tag{28}$$

Summarizing results (20), (22), (25), (28) we write down the exact solution of equation (1):

$$u = \exp\left\{\frac{b}{2\lambda}(\alpha_a x_a + c_3)^2 + c_2 + \frac{1}{2} - 2ib(c_2 x_0 - c_1)\right\},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, 3$, $\alpha_a \alpha_a = 1$.

Case 2. $V = 0$ and F satisfy the overdetermined system:

$$F_0 - 4\lambda b x_0 F_a F_a - 2\lambda b x_0 \Delta F = 0,$$
$$\lambda F_a F_a + \lambda \Delta F + 2b x_0 F_0 - 4\lambda b^2 x_0^2 F_a F_a = 0.$$

For this case the ansatz (2) has the form:

$$u = \exp\{1 - 2ibx_0)F(x_0, \boldsymbol{x})\}. \tag{29}$$

Consequence. *The ansatz (29) gives the solutions of the equation (1) if the real function F satisfy:*

$$F_t - \lambda b F_a F_a = 0, \quad F_t + \lambda b \Delta F = 0, \quad t = x_0^2. \tag{30}$$

The system (30) have non-trivial symmetry properties:

Theorem 3. *The overdetermined system (30) is invariant with respect to the extended Galilei algebra having basis elements:*

$$P_t = \frac{\partial}{\partial t}, \quad t = x_0^2, \quad P_a, \quad J_{ab}, \quad P_{n+1} = \frac{\partial}{\partial F},$$
$$G_a^{(1)} = FP_a - x_a(2\lambda b)^{-1} P_t, \quad D^{(1)} = 2t\partial_t + x_a P_a.$$

Note 5. The operator $G_a^{(1)}$ generates the transformation:

$$t \to t' = t - (2\lambda b)^{-1}\theta_a x_a - (4\lambda b)^{-1}\theta_a^2, \quad x_b \to x_b' = x_b,$$
$$x_a \to x_a' = x_a + \theta_a F, \quad F \to F' = F,$$

where θ_a is a group parameter.

2) For reduction of the equation (2) ($\lambda_1 \neq 0$) by operator C it is necessary to change of variables:

$$W = F(x_0, \boldsymbol{x}, \omega), \quad \omega = \frac{1}{2}\exp(-2\lambda_2 x_0)(\ln|u| - (2i\lambda_1)^{-1}\lambda_2 \ln(u/u^*)),$$
$$V = \lambda_1 \ln|u| - i\frac{1}{2}\lambda_2 \ln(u/u^*). \tag{31}$$

Substituting (31) for the partially case $F_\omega = 1$ into the equation (2) we get:

$$
\begin{aligned}
&(2\lambda_1)^{-1}V_0 - \exp(2\lambda_2 x_0)(F_0 - W_0) + 2\lambda[(2\lambda)^{-1}V_a - \\
&\quad - \exp(2\lambda_2 x_0)(F_a - W_a)((2\lambda_2)^{-1}V_a + \lambda_1(\lambda_2)^{-1}\exp(2\lambda_2 x_0)(F_a - W_a)] + \\
&\quad + (2\lambda_2)^{-1}\lambda[\Delta V + 2\lambda_1 \exp(2\lambda_2 x_0)(\Delta F - \Delta W)] - (\lambda_1)^{-1}\lambda_2 V = 0, \\
&\lambda[(2\lambda_1)^{-1}V_a - \exp(2\lambda_2 x_0)(F_a - W_a)(2\lambda_1)^{-1}V_a - \exp(2\lambda_2 x_0)(F_a - W_a)] + \\
&\quad + \lambda[(2\lambda_1)^{-1}\Delta V - \exp(2\lambda_2 x_0)(\Delta F - \Delta W)] - (2\lambda_2)^{-1}[V_0 + \\
&\quad + 2\lambda_1 \exp(2\lambda_2 x_0)(F_0 - W_0)] - \lambda(4\lambda_2^2)^{-1}[V_a + 2\lambda_1 \exp(2\lambda_2 x_0)(F_a - W_a)] \times \\
&\quad \times [V_a + 2\lambda_2 \exp(2\lambda_2 x_0)(F_a - W_a)] - V = 0,
\end{aligned}
$$

and the operator C has the form:

$$
C = \frac{\partial}{\partial W}. \tag{32}
$$

From (31), (32) follows that the solutions of the equation (2) (with $\lambda_1, \lambda_2 \neq 0$) we can find in the form:

$$
u = \exp\left\{(2\lambda_1\lambda_2)^{-1}V(\lambda_2 + i\lambda_1) - (\lambda_1)^{-1}F\exp(2\lambda_2 x_0)(\lambda_2 - i\lambda_1)\right\},
$$

where the real functions V and F satisfy the system:

$$
\begin{aligned}
&(2\lambda_1)^{-1}V_0 - \exp(2\lambda_2 x_0)F_0 + 2\lambda[(2\lambda_1)^{-1}V_a - \exp(2\lambda_2 x_0)(F_a - (2\lambda_2)^{-1}V_a) + \\
&\quad + \lambda_1(\lambda_2)^{-1}\exp(2\lambda_2 x_0)F_a] + \\
&\quad + (2\lambda_2)^{-1}\lambda[\Delta V + 2\lambda_1\exp(2\lambda_2 x_0)\Delta F] - (\lambda_1)^{-1}\lambda_2 V = 0, \\
&\lambda[(2\lambda_1)^{-1}V_a - \exp(2\lambda_2 x_0)F_a] + \lambda[(2\lambda_1)^{-1}\Delta V - \exp(2\lambda_2 x_0)\Delta F - \\
&\quad - (2\lambda_2)^{-1}[V_0 + 2\lambda_1\exp(2\lambda_2 x_0)F_0] - \\
&\quad - \lambda(4\lambda_2^2)^{-1}[V_a + 2\lambda_1\exp(2\lambda_2 x_0)F_a]^2 - V = 0.
\end{aligned} \tag{33}
$$

Case 1: $V = 0$. For this case the ansatz

$$
u = \exp\left\{-(\lambda_1)^{-1}F\exp(2\lambda_2 x_0)(\lambda_2 - i\lambda_1)\right\}
$$

reduces the equation (2) when $\lambda_1 \neq 0$ to the system:

$$
\begin{aligned}
&F_0 + \lambda\lambda_1(\lambda_2)^{-1}\Delta F = 0, \\
&F_0 + \lambda\lambda_1(\lambda_2)^{-1}\exp(2\lambda_2 x_0)F_a F_a = 0.
\end{aligned}
$$

Case 2: $F = 0$. For this case the ansatz

$$
u = \exp\left\{(2\lambda_1\lambda_2)^{-1}V(\lambda_2 + i\lambda_1)\right\}
$$

reduces the equation (2) with $\lambda_1 \neq 0$ to the overdetermined system:

$$
\begin{aligned}
&V_0 + \lambda(\lambda_2)^{-1}V_a V_a + \lambda\lambda_1(\lambda_2)^{-1}\Delta V - 2\lambda_2 V = 0, \\
&V_0 + \lambda(\lambda_1^2 - \lambda_2^2)(2\lambda_1\lambda_2)^{-1}V_a V_a - \lambda\lambda_2(\lambda_1)^{-1}\Delta V + 2\lambda_2 V = 0.
\end{aligned} \tag{34}
$$

For the partially case $\lambda_1^2 = \lambda_2^2$ the system (34) has the form:

$$
V_0 + \lambda(2\lambda_2)^{-1}V_a V_a = 0, \quad V_0 + \lambda\Delta V \mp 2\lambda_2 V = 0,
$$

and for the partially case $3\lambda_1^2 = \lambda_2^2$ this system has the form:

$$V_0 + \lambda(2\lambda_2)^{-1}V_aV_a - \lambda_2 V = 0, \quad \sqrt{3}V_0 \mp \lambda\Delta V = 0.$$

5. Conditional symmetry. The symmetry of the equations (1), (2) can be extended essentially, if we put a certain additional condition on its solutions (see [4, 9, 10]). As to Schrödinger equations with the logarithmic nonlinearity one of such additional conditions is vanishing of the interior potential [11] that is equivalent to the following condition:

$$\Delta|u| = 0, \quad |u| = (uu^*)^{1/2}. \tag{35}$$

Theorem 4. *The equation (1) is conditionally invariant with respect to the following algebras:*

1) $AG_5(1,n) = \langle AG_3(1,n), Q_1\rangle$,

where

$$Q_1 = x_0P_0 + x_aP_a - \frac{i}{2}\ln(uu^{*-1})Q$$

with additional condition (35);

2) $AG_5(1,n) = \langle AG(1,n), Q_2\rangle$,

where the operator Q_2 is of the form [9]:

$$Q_2 = \frac{i}{2}\ln(uu^{*-1})Q + x_0P_0$$

if the module of the function u satisfies the condition

$$\lambda\Delta|u| = 2b|u|\ln|u|. \tag{36}$$

Note 6. The operator Q_1 generates the following finite transformations:

$$x_0 \to x_0' = \theta_1 x_0, \quad x_a \to x_a' = \theta_1 x_a, \quad u \to u' = |u|(uu^{*-1})^{1/2\theta_1},$$

and the operator Q_2 generates the following transformations:

$$x_0 \to x_0' = \theta_2 x_0, \quad x_a \to x_a' = x_a, \quad u \to u' = |u|(uu^{*-1})^{-1/2\theta_2},$$

where θ_1 and θ_2 are group parameters.

Theorem 5. *The equation (2) is conditional invariant with respect to the algebra:*

$$AG_7(1,n) = \langle AG_4(1,n), Q_3\rangle,$$

where

$$Q_3 = Q_1 - Q_2 = x_aP_a - i\ln(uu^{*-1})Q,$$

and the operator C is of the form $C = \exp(2\lambda_2 x_0)I$. The additional condition has the form (34).

Note 7. The operator Q_3 generates the transformations:

$$x_0 \to x_0' = x_0, \quad x_a \to x_a' = \theta_3 x_a, \quad u \to u' = |u|(uu^{*-1})^{\theta_3}, \quad a = \overline{1,n}.$$

The following theorems can be proved by means of conditional invariance algorithm (see e.g. [5, 10]).

So we can see that the additional conditions (34) and (35) expand the symmetry of the equations (1), (2).

6. Applications: non-Lie reduction. In this section we consider some non-Lie ansatzes for the equations (1), (2) which cannot, be obtained by means of classical Lie approach. The examples of non-Lie reduction of the Schrödinger equations with degree nonlinearity are adduced in [12, 13].

1) The ansatz

$$u = x_0^2\rho(\omega_1,\omega_2)\exp\{i[\alpha_a x_a - 4bx_0\ln x_0 + x_0\varphi(\omega_1,\omega_2)]\},\\ \omega_1 = \frac{x_1}{x_2},\quad \omega_2 = \frac{x_2}{x_0},\quad \alpha_a\in\mathbb{R},\quad a=\overline{1,n} \tag{37}$$

reduces the equation (1) to the system:

$$2\rho - \omega_1\rho_1 - \omega_2\rho_2 + 2\lambda\rho_1\varphi_1 + 2\lambda\rho_2\varphi_2 + \lambda\rho(\varphi_{11}+\varphi_{22}) = 0,\\ \rho_{11}+\rho_{22} = 0,\\ \lambda\varphi_1^2 + \lambda\varphi_2^2 - \omega_1\varphi_1 - \omega_2\varphi_2 = 4b - \lambda\alpha_a\alpha_a - 2b\ln\rho,\quad a=3,\ldots,n,\quad \alpha_a\in\mathbb{R}. \tag{38}$$

2) The ansatz

$$u = x_0^2\rho(\omega_1,\omega_2)\exp\left\{i\left[\frac{x_1^2}{4\lambda x_0} - 4bx_0\ln x_0 + x_0\varphi(\omega_1,\omega_2)\right]\right\}, \tag{39}$$

where

$$\omega_1 = \frac{x_1}{x_0} - \operatorname{arctg}\frac{x_3}{x_2},\quad \omega_2 = \frac{x_2^2+x_3^2}{x_0}$$

reduces the equation (1) (when $n=3$) to the system:

$$2\rho - \omega_1\rho_1(1+\omega_2^{-2}) - \omega_2\rho_2 + \rho_2\varphi_2 + \rho\omega_2\varphi_2 + \rho\varphi_{11}(1+\omega_2^{-2}) + \rho\varphi_{22} = 0,\\ \rho_{11}(1+\omega_2^{-2}) + \omega_2^2\rho_{22} + \omega_2\rho_2 = 0,\\ \lambda(1+\omega_2^{-2})\varphi_1^2 + \lambda\varphi_2^2 - \omega_2\varphi + \varphi - 4b + 2b\ln\rho = 0. \tag{40}$$

3) The ansatz

$$u = x_0^2\rho(\omega_1,\omega_2)\exp\left\{i\left[\frac{x_1^2}{4\lambda x_0} - 4bx_0\ln x_0 + x_0\varphi(\omega_1,\omega_2)\right]\right\},\\ \omega_1 = \frac{x_2}{x_0},\quad \omega_2 = \frac{x_3}{x_0} \tag{41}$$

reduces the equation (1) to the system (when $n=3$):

$$2\rho - \omega_1\rho_1 - \omega_2\rho_2 + 2\lambda\rho_1\varphi_1 + 2\lambda\rho_2\varphi_2 + \frac{1}{2}\rho + 2\lambda\rho(\varphi_{11}+\varphi_{22}) = 0,\\ \rho_{11}+\rho_{22} = 0,\\ \lambda\varphi_1^2 + \lambda\varphi_2^2 - \omega_1\varphi_1 - \omega_2\varphi_2 + \varphi - 4b + 2b\ln\rho = 0. \tag{42}$$

Note 8. The ansatzes (37), (39), (41) are obtained as a consequence of conditional invariance of the equation (1) respect to the algebra $AG_5(1,n)$.

4) The ansatz

$$u = \exp\left\{\frac{2}{\alpha}\exp(2\lambda_2 x_0)\right\}\rho(\omega)\exp\left\{i\left[\exp\left(\frac{2x_0}{\alpha}\right)\varphi(\omega)\right]\right\},$$
$$\omega = (\boldsymbol{x}^2)^{1/2}\exp\left\{-\frac{x_0}{\alpha}\right\}, \quad \alpha \neq 0, \tag{43}$$

reduces equation (2) with $\lambda_1 = 0$, $\lambda_2 \neq 0$ to the system ODE:

$$\rho\ddot{\varphi} + \dot{\rho}\dot{\varphi} + (n-1)\omega^{-1}\rho\dot{\varphi} + \alpha^{-1}\omega\dot{\rho} = 2\lambda_2\rho\ln\rho,$$
$$\ddot{\rho} + (n-1)\omega^{-1}\dot{\rho} = 0, \tag{44}$$
$$\lambda\alpha\dot{\varphi}^2 - \omega\dot{\varphi} + 2\varphi = 0.$$

The systems of reduced equations (38), (40), (42), (44) are overdetermined. Therefore it is necessary to consider their compatibility.

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