Symmetry reduction of the Navier–Stokes equations to linear two-dimensional systems of equations

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Побудовано повний набір нееквіалентних двовимірних підалгебр максимальної в сенсі Лі (нескінченовимірної) алгебри інваріантноoті рівнянь Навье–Стокса для в'язкої нестискної рідини. Отримано анзаци, що редукують рівняння Навье–Стокса до лінійних систем ДРЧП від двох незалежних змінних. Проведено дослідження симетрійних властивостей редукованих систем та побудовані деякі їх точні розв'язки.

In this article, being continuation of our works [1, 2], we construct ansätze for the Navier–Stokcs (NS) field which reduce the NS equations (NSEs) for an incompressible viscous fluid to linear systems of partial differential equations (PDEs) in two independent variables. To solve this problem we use the method described in [3] and the infinite-dimensional symmetry algebra of the NSEs.

It is known that NSEs

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} - \Delta\boldsymbol{u} + \boldsymbol{\nabla}p = 0, \quad \operatorname{div}\boldsymbol{u} = 0, \tag{1}$$

where $\boldsymbol{u} = \boldsymbol{u}(x) = \{u^1, u^2, u^3\}$ is the velocity field of a fluid, $p = p(x)$ is the pressure, $x = \{t, \boldsymbol{x}\} \in \mathbb{R}^4$, $\boldsymbol{\nabla} = \{\partial/\partial x_a\}$, $a = 1,2,3$, $\Delta = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$, are invariant under the infinite dimensional algebra A^∞ with basis elements

$$\begin{aligned}
&\partial_t = \frac{\partial}{\partial t}, \quad D = 2t\partial_t + x_a\partial_a - u^a\partial u^a - 2p\partial_p,\\
&J_{ab} = x_a\partial_b - x_b\partial_a + u^a\partial u^b - u^b\partial u^a,\\
&R(\boldsymbol{m}) = m^a\partial_a + \dot{m}^a\partial u^a - x_a\ddot{m}^a\partial_p, \quad Z(\alpha) = \alpha(t)\partial_p,
\end{aligned} \tag{2}$$

where $\boldsymbol{m} = \{m^a(t)\}$ and $\alpha(t)$ are arbitrary differentiable function of t; dot means differentiation with respect to t. The set of operators (2) determine the maximal in the sense of Lie invariance algebra of the NSEs [4, 6, 7].

Constructing a complete set of inequivalent two-dimensional subalgebras of A^∞, we choose from it those subalgebras which lie in a linear span of operators J_{ab}, $R(m)$ and $Z(\alpha)$. It is these subalgebras that allow us to construct ansätze which reduce the nonlinear NSEs to linear systems of PDEs in two independent variables.

Theorem 1. *A complete set of A^∞-inequivalent two-dimensional subalgebras of A^∞ is exhausted by such algebras:*

1. $A^1(\boldsymbol{m},\boldsymbol{n}) = \langle R(\boldsymbol{m}), R(\boldsymbol{n})\rangle$, $\ddot{\boldsymbol{m}}\cdot\boldsymbol{n} - \ddot{\boldsymbol{n}}\cdot\boldsymbol{m} = 0$, *and* $\forall\, c_1, c_2 \in \mathbb{R}$ $c_1\boldsymbol{m} + c_2\boldsymbol{n} \neq 0$, *where algebras* $A_1(\boldsymbol{m}^1,\boldsymbol{n}^1)$ *and* $A^1(\boldsymbol{m}^2,\boldsymbol{n}^2)$ *are equivalent if* $\exists\,\{a_{kl}\}_{k,l=1,2}$, $\det\{a_{kl}\} \neq 0$, $B \in O(3)$, $\exists\,\varepsilon,\delta \in \mathbb{R}$:

$$(\boldsymbol{m}^2,\boldsymbol{n}^2)(t) = (B(a_{11}\boldsymbol{m}^1 + a_{12}\boldsymbol{n}^1),\ B(a_{21}\boldsymbol{m}^1 + a_{22}\boldsymbol{n}^1))(te^{2\varepsilon} + \delta); \tag{3}$$

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2. $A^2(\alpha,\beta)=\langle J_{12}+Z(\alpha(t)),R(0,0,\beta(t))\rangle$, $\beta\neq 0$, *where algebras* $A^2(\alpha^1,\beta^1)$ *and* $A^2(\alpha^2,\beta^2)$ *are equivalent if*

$$\exists\, c\neq 0\ \exists\varepsilon,\delta\in\mathbb{R}:\ (\alpha^2,\beta^2)(t)=(e^{2\varepsilon}\alpha^1,c\beta^1)(te^{2\varepsilon}+\delta); \tag{4}$$

3. $A^3(\alpha,\beta)=\langle J_{12}+R(0,0,\beta(t)\int\frac{dt}{(\beta(t))^2}+Z(\alpha(t)),R(0,0,\beta(t))\rangle$, $\beta\neq 0$, *where algebras* $A^3(\alpha^1,\beta^1)$ *and* $A^3(\alpha^2,\beta^2)$ *are equivalent if*

$$\exists\varepsilon,\delta\in\mathbb{R}:\ (\alpha^2,\beta^2)(t)=(e^{2\varepsilon}\alpha^1,e^{-\varepsilon}\beta^1)(te^{2\varepsilon}+\delta); \tag{5}$$

4. $A^4=\langle D+2\varkappa J_{12},R(\mu|t|^\sigma\cos(\varkappa\ln|t|),\mu|t|^\sigma\sin(\varkappa\ln|t|),\nu|t|^\sigma)+Z(\varepsilon|t|^{\sigma-3/2})\rangle$, $\varkappa>0$, $\mu\geq 0$, $\nu\geq 0$, $\mu^2+\nu^2=1$, $\varepsilon=0$ *if* $\sigma\neq 1/2$ *and* $\varepsilon\geq 0$ *if* $\sigma=1/2$*;*

5. $A^5=\langle D,R(0,0,|t|^\sigma)+z(\varepsilon|t|^{\sigma-3/2})\rangle$, $\varepsilon=0$ *if* $\sigma\neq 1/2$ *and* $\varepsilon\geq 0$ *if* $\sigma=1/2$*;*

6. $A^6=\langle\partial_t+J_{12},R(\mu e^{\sigma t}\cos t,\mu e^{\sigma t}\sin t,\nu e^{\sigma t}+Z(\varepsilon e^{\sigma t})\rangle$, $\mu\geq 0$, $\nu\geq 0$, $\mu^2+\nu^2=1$, $\varepsilon=0$ *if* $\sigma\neq 0$ *and* $\varepsilon\geq 0$ *if* $\sigma=0$*;*

7. $A^7=\langle\partial_t,R(0,0,e^{\sigma t})+Z(\varepsilon e^{\sigma t})\rangle$, $\sigma\in\{-1;0;1\}$, $\varepsilon=0$ *if* $\sigma\neq 0$ *and* $\varepsilon\in\{0;1\}$ *if* $\sigma=0$*;*

8. $A^8=\langle\partial_t,J_{12}+\theta\partial_3+\varepsilon\partial_p\rangle$, $\theta\in\{0;1\}$, $\varepsilon\geq 0$ *if* $\theta=1$ *and* $\varepsilon\in\{0;1\}$ *if* $\theta=0$*;*

9. $A^9=\langle\partial_t,D+\gamma J_{12}\rangle$, $\gamma\geq 0$*;*

10. $A^{10}=\langle D,J_{12}+R(0,0,\theta|t|^{1/2})+Z(\varepsilon t^{-1})\rangle$, $\theta\geq 0$, $\varepsilon\geq 0$*;*

11. $A^{11}=\langle D+\gamma J_{12},Z(|t|^\varkappa)\rangle$, $\gamma\geq 0$, $\varkappa\in\mathbb{R}$*;*

12. $A^{12}=\langle\partial_t,Z(e^{\sigma t})\rangle$, $\sigma\in\{-1;0;1\}$*;*

13. $A^{13}=\langle\partial_t+J_{12},Z(e^{\sigma t})\rangle$, $\sigma\in\mathbb{R}$*;*

14. $A^{14}(\alpha,\beta)=\langle J_{12}+R(0,0,\beta(t)),Z(\alpha(t))\rangle$, $\alpha\neq 0$, *where algebras* $A^{14}(\alpha^1,\beta^1)$ *and* $A^{14}(\alpha^2,\beta^2)$ *are equivalent if* $\exists\,\varepsilon,\delta\in\mathbb{R}$, $\exists\, c\neq 0$*:* $(\alpha^2,\beta^2)(t)=(c\alpha^1,e^{-\varepsilon}\beta^1)(te^{2\varepsilon}+\delta)$*;*

15. $A^{15}(\alpha,\beta)=\langle J_{12}+Z(\beta(t)),Z(\alpha(t))\rangle$, $\alpha\neq 0$, *where algebras* $A^{15}(\alpha^1,\beta^1)$ *and* $A^{15}(\alpha^2,\beta^2)$ *are equivalent if* $\exists\, c^1\neq 0$, $\exists\,\varepsilon,\delta,c^2\in\mathbb{R}$*:* $(\alpha^2,\beta^2)(t)=(c^1\alpha^1,e^{-\varepsilon}\beta^1+c^2\alpha^1)(te^{2\varepsilon}+\delta)$*;*

16. $A^{16}(\boldsymbol{m},\alpha)=\langle R(\boldsymbol{m}(t)),Z(\alpha(t))\rangle$, $\boldsymbol{m}\neq 0$, $\alpha\neq 0$, *where algebras* $A^{16}(\boldsymbol{m}^1,\alpha^1)$ *and* $A^{16}(\boldsymbol{m}^2,\alpha^2)$ *are equivalent if* $\exists\, c^1\neq 0$, $\exists\, c^2\neq 0$, $\exists\,\varepsilon,\delta\in\mathbb{R}$, $\exists\, B\in O(3)$*:*

$$(\boldsymbol{m}^2,\alpha^2)(t)=(c^1B\boldsymbol{m}^1,c^2\alpha^1)(te^{2\varepsilon}+\delta);$$

17. $A^{17}(\alpha,\beta)=\langle R(0,0,\beta(t)),R\left(0,0,\beta(t)\int\frac{dt}{(\beta(t))^2}\right)+Z(\alpha(t))\rangle$, $\beta\neq 0$, $\alpha\neq 0$, *where algebras* $A^{17}(\alpha^1,\beta^1)$ *and* $A^{17}(\alpha^2,\beta^2)$ *are equivalent if* $\exists\, c\neq 0$, $\exists\,\varepsilon,\delta\in\mathbb{R}$*:* $(\alpha^2,\beta^2)(t)=\left(\frac{e^{2\varepsilon}}{c}\alpha^1,e^{-\varepsilon}c\beta^1\right)(te^{2\varepsilon}+\delta)$*;*

18. $A^{18}(\alpha,\beta)=\langle Z(\alpha(t)),Z(\beta(t))\rangle$, $c^1\alpha+c^2\beta\neq 0\ \forall\, c^1,c^2\in\mathbb{R}$, *where algebras* $A^{18}(\alpha^2,\beta^2)$ *and* $A^{18}(\alpha^2,\beta^2)$ *are equivalent if* $\exists\,\{a_{kl}\}_{k,l=1,2}$, $\det\{a_{kl}\}\neq 0$, $\exists\,\varepsilon,\delta\in\mathbb{R}$*:* $(\alpha^2,\beta^2)(t)=(a_{11}\alpha^1+a_{12}\beta^1,a_{21}\alpha^1+a_{22}\beta^1)(te^{2\varepsilon}+\delta)$.

Theorem 1 is proved with method described in [4, 5].

Lying in a linear span of operators J_{ab}, $R(\boldsymbol{m})$ and $Z(\alpha)$ are algebras 1, 2 and 3. Ansätze constructed with these algebras have the form

$$1.\quad \boldsymbol{u}=\boldsymbol{v}(\omega_1,\omega_2)+\frac{\tilde{\boldsymbol{m}}\cdot\boldsymbol{x}}{\delta}\dot{\boldsymbol{m}}+\frac{\tilde{\boldsymbol{n}}\cdot\boldsymbol{x}}{\delta}\dot{\boldsymbol{n}}-\frac{\boldsymbol{k}\cdot\boldsymbol{x}}{\delta}\dot{\boldsymbol{k}},\quad p=q(\omega_1,\omega_2)+\frac{b_{ij}(t)}{2}x_ix_j,$$

where $\omega_1=t$, $\omega_2=\boldsymbol{k}\cdot\boldsymbol{x}$, $\boldsymbol{k}=\boldsymbol{m}\times\boldsymbol{n}$, $\tilde{\boldsymbol{m}}=\boldsymbol{n}\times\boldsymbol{k}$, $\tilde{\boldsymbol{n}}=\boldsymbol{k}\times\boldsymbol{m}$, $\delta=|\boldsymbol{k}|^2$,

$$b_{ij}=-\frac{1}{\delta}\left[\ddot{m}^i\tilde{m}^i+\ddot{n}^i\tilde{n}^i+\frac{\boldsymbol{k}\cdot\ddot{\boldsymbol{m}}}{\delta}\tilde{m}^ik^j+\frac{\boldsymbol{k}\cdot\ddot{\boldsymbol{n}}}{\delta}\tilde{n}^ik^j\right];$$

$$
\begin{aligned}
2.\quad & u^1 = x_1 v^1(\omega_1,\omega_2) - \frac{x_2}{\omega_2^2}(v^2(\omega_1,\omega_2) - s(t)),\\
& u^2 = x_2 v^1(\omega_1,\omega_2) + \frac{x_1}{\omega_2^2}(v^2(\omega_1,\omega_2) - s(t)),\\
& u^3 = \frac{1}{\beta(t)} v^3(\omega_1,\omega_2) + \frac{\dot\beta(t)}{\beta(t)} x_3,\\
& p = q(\omega_1,\omega_2) - \frac{\ddot\beta(t)}{\beta(t)}\frac{x_3^2}{2} + \alpha(t)\,\mathrm{arctg}\,\frac{x_2}{x_1},
\end{aligned}
$$

where $\omega_1 = t$, $\omega_2 = \sqrt{x_2^1 + x_2^2}$, $s(t) = \int \alpha(t)dt$;

$$
\begin{aligned}
3.\quad & u^1 = x_1 v^1(\omega_1,\omega_2) - \frac{x_2}{\omega_2^2}(v^2(\omega_1,\omega_2) - s(t)),\\
& u^2 = x_2 v^1(\omega_1,\omega_2) + \frac{x_1}{\omega_2^2}(v^2(\omega_1,\omega_2) - s(t)),\\
& u^3 = \frac{1}{\beta(t)} v^3(\omega_1,\omega_2) + \frac{\dot\beta(t)}{\beta(t)} x_3 + \frac{1}{\beta(t)}\,\mathrm{arctg}\,\frac{x_2}{x_1},\\
& p = q(\omega_1,\omega_2) - \frac{\ddot\beta(t)}{\beta(t)}\frac{x_3^2}{2} + \alpha(t)\,\mathrm{arctg}\,\frac{x_2}{x_1},
\end{aligned}
$$

where $\omega_1 = t$, $\omega_2 = \sqrt{x_2^1 + x_2^2}$, $s(t) = \int \alpha(t)dt$;

Here numeration of ansätze correspond to that of algebras in theorem 1. Substituting ansätze 1–3 into the NSEs, we obtain equations reduced

$$1.\ \boldsymbol{v}_1 - \delta \boldsymbol{v}_{22} + q_2\boldsymbol{k} + \frac{\tilde{\boldsymbol{m}}\cdot\boldsymbol{v}}{\delta}\cdot\dot{\boldsymbol{m}} + \frac{\tilde{\boldsymbol{n}}\cdot\boldsymbol{v}}{\delta}\dot{\boldsymbol{n}} + \boldsymbol{e}\cdot\boldsymbol{\omega}_2 = 0, \tag{6}$$

$$\boldsymbol{k}\cdot\boldsymbol{v}_2 = 0, \tag{7}$$

where $\boldsymbol{e} = \boldsymbol{e}(t) = \dfrac{2}{\delta^2}(\dot{\boldsymbol{m}}\cdot\boldsymbol{n} - \dot{\boldsymbol{n}}\cdot\boldsymbol{m})\dot{\boldsymbol{k}}\times\boldsymbol{k} + \dfrac{2\dot{\boldsymbol{k}}\cdot\dot{\boldsymbol{k}} - \ddot{\boldsymbol{k}}\cdot\boldsymbol{k}}{\delta^2}\boldsymbol{k}$;

$$2.\ v_1^1 + ((v^1)^2 - \frac{1}{\omega_2^4}(v^2 - s)^2 + \omega_2 v^1 v_2^1 - \left(v_{22}^1 + \frac{3}{\omega_2}v_2^1\right) + \frac{1}{\omega_2}q_2 = 0, \tag{8}$$

$$v_1^2 + \omega_2 v^1 v_2^2 - v_{22}^2 + \frac{1}{\omega_2}v_2^2 = 0, \tag{9}$$

$$v_1^3 + \omega_2 v^1 v_2^3 - v_{22}^3 + \frac{1}{\omega_2}v_2^3 = 0, \tag{10}$$

$$2v^1 + \omega_2 v_2^1 + \frac{\dot\beta}{\beta} = 0, \tag{11}$$

$$3.\ v_1^1 + (v^1)^2 - \frac{1}{\omega_2^4}(v^2 - s)^2 + \omega_2 v^1 v_2^1 - \left(v_{22}^1 + \frac{3}{\omega_2}v_2^1\right) + \frac{1}{\omega_2}q_2 = 0, \tag{12}$$

$$v_1^2 + \omega_2 v^1 v_2^2 - v_{22}^2 + \frac{1}{\omega_2}v_2^2 = 0, \tag{13}$$

$$v^3_1 + \omega_2 v^1 v^3_2 - v^3_{22} - \frac{1}{\omega_2} v^3_2 + \frac{v^2 - s}{\omega_2^2} = 0, \tag{14}$$

$$2v^1 + \omega_2 v^1_2 + \frac{\dot\beta}{\beta} = 0. \tag{15}$$

Here subscripts 1 and 2 mean differentiation by variables ω_1 and ω_2 respectively. Let us show that nonlinear system 1–3 can be transformed to linear PDEs.

Consider system 1 (equations (6)–(7)). After integration of equation (7) by ω_2: $\boldsymbol{k}\cdot\boldsymbol{v} = h(t)$. Further we make the transformation from the symmetry group of the NSEs

$$\tilde{\boldsymbol{u}}(t,\boldsymbol{x}) = \boldsymbol{u}(t,\boldsymbol{x}-\boldsymbol{l}(t)) + \dot{\boldsymbol{l}}(t), \quad \tilde p(t,\boldsymbol{x}) = p(t,\boldsymbol{x}-\boldsymbol{l}(t)) - \ddot{\boldsymbol{l}}(t)\cdot\boldsymbol{x},$$

where $\ddot{\boldsymbol{l}}\cdot\boldsymbol{m} - \boldsymbol{l}\cdot\ddot{\boldsymbol{m}} = \ddot{\boldsymbol{l}}\cdot\boldsymbol{n} - \boldsymbol{l}\cdot\ddot{\boldsymbol{n}} = 0$,

$$\boldsymbol{k}\cdot\left(\dot{\boldsymbol{l}} - \frac{\tilde{\boldsymbol{m}}\cdot\boldsymbol{l}}{\delta}\dot{\boldsymbol{m}} - \frac{\tilde{\boldsymbol{n}}\cdot\boldsymbol{l}}{\delta}\dot{\boldsymbol{n}} + \frac{\boldsymbol{k}\cdot\boldsymbol{l}}{\delta}\dot{\boldsymbol{k}}\right) + h = 0.$$

This transformation does not change ansatz 1 and besides $\boldsymbol{k}\cdot\tilde{\boldsymbol{v}} = 0$, that is $\tilde h(t) \equiv 0$. Therefore, without loss of generality we can assume that $h(t)\equiv 0$.

Let $f = f(\omega_1,\omega_2) = \boldsymbol{m}\cdot\boldsymbol{v}$, $g = g(\omega_1,\omega_2) = \boldsymbol{n}\cdot\boldsymbol{v}$. Since $\ddot{\boldsymbol{m}}\cdot\boldsymbol{n} - \boldsymbol{n}\cdot\ddot{\boldsymbol{m}} = 0$ then $\dot{\boldsymbol{m}}\cdot\boldsymbol{n} - \dot{\boldsymbol{n}}\cdot\boldsymbol{m} = C = \text{const}$. Case $C\neq 0$ is reduced by means of change of the basis of the algebra $A^1(\boldsymbol{m},\boldsymbol{n})$ to case $C = 1$. Let us multiply the scalar equation (6) by $\boldsymbol{m}$, $\boldsymbol{n}$ and $\boldsymbol{k}$. As result we obtain the linear system of PDEs with variable coefficient for functions f, g and q:

$$f_1 - \delta f_{22} + C\left(\frac{\boldsymbol{m}\cdot\boldsymbol{n}}{\delta}f - \frac{\boldsymbol{m}\cdot\boldsymbol{m}}{\delta}g\right) + \frac{2}{\delta^2}C\omega_2(\tilde{\boldsymbol{n}}\cdot\boldsymbol{k}) = 0,$$
$$g_1 - \delta g_{22} + C\left(\frac{\boldsymbol{n}\cdot\boldsymbol{n}}{\delta}f - \frac{\boldsymbol{m}\cdot\boldsymbol{n}}{\delta}g\right) - \frac{2}{\delta^2}C\omega_2(\tilde{\boldsymbol{m}}\cdot\boldsymbol{k}) = 0,$$
$$q_2 = \frac{2}{\delta^2}((\tilde{\boldsymbol{m}}\cdot\dot{\boldsymbol{k}})f + (\tilde{\boldsymbol{n}}\cdot\dot{\boldsymbol{k}})g) + \frac{\omega_2}{\delta^2}(\ddot{\boldsymbol{k}}\cdot\boldsymbol{k} - 2\dot{\boldsymbol{k}}\cdot\dot{\boldsymbol{k}}).$$

Consider two possible cases.

a) Let $C = 0$. Then there are functions $\varphi^i = \varphi^i(\tau,\omega)$, $i = 1,2$, where $\tau = \int\delta(t)dt$, $\omega = \omega_2$, that $f = \varphi^1_\omega$, $g = \varphi^2_\omega$ and $\varphi^i_\tau - \varphi^i_{\omega\omega} = 0$, $i = 1,2$. Therefore

$$\boldsymbol{u} = \left(\varphi^1_\omega(\tau,\omega) + \frac{\dot{\boldsymbol{m}}\cdot\boldsymbol{x}}{\delta}\right)\tilde{\boldsymbol{m}} + \left(\varphi^2_\omega(\tau,\omega) + \frac{\dot{\boldsymbol{n}}\cdot\boldsymbol{x}}{\delta}\right)\tilde{\boldsymbol{n}} - \frac{\dot{\boldsymbol{k}}\cdot\boldsymbol{x}}{\delta}\boldsymbol{k},$$
$$p = \frac{2}{\delta^2} + (\tilde{\boldsymbol{m}}\cdot\dot{\boldsymbol{k}})\varphi^1(\tau,\omega) + \frac{2}{\delta^2}(\tilde{\boldsymbol{n}}\cdot\dot{\boldsymbol{k}})\varphi^2(\tau,\omega) + \frac{1}{2\delta}\left\{\frac{\ddot{\boldsymbol{k}}\cdot\boldsymbol{k} - 2\dot{\boldsymbol{k}}\cdot\dot{\boldsymbol{k}}}{\delta}\omega^2 - \right. \tag{16}$$
$$\left. -(\tilde{\boldsymbol{m}}\cdot\boldsymbol{x})(\ddot{\boldsymbol{m}}\cdot\boldsymbol{x}) - (\tilde{\boldsymbol{n}}\cdot\boldsymbol{x})(\ddot{\boldsymbol{n}}\cdot\boldsymbol{x}) - \frac{\boldsymbol{k}\cdot\ddot{\boldsymbol{m}}}{\delta}(\tilde{\boldsymbol{m}}\cdot\boldsymbol{x})(\boldsymbol{k}\cdot\boldsymbol{x}) - \frac{\boldsymbol{k}\cdot\ddot{\boldsymbol{n}}}{\delta}(\tilde{\boldsymbol{n}}\cdot\boldsymbol{x})(\boldsymbol{k}\cdot\boldsymbol{x})\right\},$$

where $\dot{\boldsymbol{m}}\cdot\boldsymbol{n} - \dot{\boldsymbol{n}}\cdot\boldsymbol{m} = 0$, $\boldsymbol{k} = \boldsymbol{m}\times\boldsymbol{n}$, $\tilde{\boldsymbol{m}} = \boldsymbol{n}\times\boldsymbol{k}$, $\tilde{\boldsymbol{n}} = \boldsymbol{k}\times\boldsymbol{m}$, $\delta = |\boldsymbol{k}|^2$, $\tau = \int\delta(t)dt$, $\omega = \boldsymbol{k}\cdot\boldsymbol{x}$, $\varphi^i_\tau - \varphi_{\omega\omega} = 0$, $i = 1,2$.

b) Let $C = 1$. Then we obtain the following solutions of the NSEs:

$$\boldsymbol{u} = \left(y^i(t)\varphi^i(\tau,\omega) + y^0(t)\omega + \frac{\dot{\boldsymbol{m}}\cdot\boldsymbol{x}}{\delta} - \frac{\tilde{\boldsymbol{n}}\cdot\boldsymbol{x}}{\delta^2}\right)\tilde{\boldsymbol{m}} +$$
$$+\left(z^i(t)\varphi^i(\tau,\omega) + z^0(t)\omega + \frac{\dot{\boldsymbol{n}}\cdot\boldsymbol{x}}{\delta} + \frac{\tilde{\boldsymbol{m}}\cdot\boldsymbol{x}}{\delta^2}\right)\tilde{\boldsymbol{n}} - \frac{\dot{\boldsymbol{k}}\cdot\boldsymbol{x}}{\delta}\boldsymbol{k},$$
$$p = \frac{2}{\delta 2}(\tilde{\boldsymbol{m}}\cdot\boldsymbol{k})\left(y^i(t)\varphi^i(\tau,\omega) + y^0(t)\frac{\omega^2}{2}\right) + \tag{17}$$
$$+\frac{2}{\delta^2}(\tilde{\boldsymbol{n}}\cdot\dot{\boldsymbol{k}})\left(z^i(t)\varphi^i(\tau,\omega) + z^0(t)\frac{\omega^2}{2}\right) + \frac{1}{2\delta}\left\{\frac{\ddot{\boldsymbol{k}}\cdot\boldsymbol{k} - 2\dot{\boldsymbol{k}}\cdot\dot{\boldsymbol{k}}}{\delta}\omega^2 - \right.$$
$$\left.-(\tilde{\boldsymbol{m}}\cdot\boldsymbol{x})(\ddot{\boldsymbol{m}}\cdot\boldsymbol{x}) - (\tilde{\boldsymbol{n}}\cdot\boldsymbol{x})(\ddot{\boldsymbol{n}}\cdot\boldsymbol{x}) - \frac{\boldsymbol{k}\cdot\ddot{\boldsymbol{m}}}{\delta}(\tilde{\boldsymbol{m}}\cdot\boldsymbol{x})(\boldsymbol{k}\cdot\boldsymbol{x}) - \frac{\boldsymbol{k}\cdot\ddot{\boldsymbol{n}}}{\delta}(\tilde{\boldsymbol{n}}\cdot\boldsymbol{x})(\boldsymbol{k}\cdot\boldsymbol{x})\right\},$$

where $\dot{\boldsymbol{m}}\cdot\boldsymbol{n} - \dot{\boldsymbol{n}}\cdot\boldsymbol{m} = 1$, $\boldsymbol{k} = \boldsymbol{m}\times\boldsymbol{n}$, $\tilde{\boldsymbol{m}} = \boldsymbol{n}\times\boldsymbol{k}$, $\tilde{\boldsymbol{n}} = \boldsymbol{k}\times\boldsymbol{m}$, $\delta = |\boldsymbol{k}|^2$, $\tau = \int\delta(t)dt$, $\omega = \boldsymbol{k}\cdot\boldsymbol{x}$, $\varphi^i_\tau - \varphi_{\omega\omega} = 0$, $i = 1,2$, $(y^i(t), z^i(t))$, $i = 1,2$ is a fundamental system of solutions for equations

$$\dot{y} + \frac{\boldsymbol{m}\cdot\boldsymbol{n}}{\delta}y - \frac{\boldsymbol{m}\cdot\boldsymbol{m}}{\delta}z = 0, \quad \dot{z} + \frac{\boldsymbol{n}\cdot\boldsymbol{n}}{\delta}y - \frac{\boldsymbol{m}\cdot\boldsymbol{n}}{\delta}z = 0 \tag{18}$$

and $(y^0(t), z^0(t))$ is a particular solutions of the system

$$\dot{y} + \frac{\boldsymbol{n}\cdot\boldsymbol{m}}{\delta}y - \frac{\boldsymbol{m}\cdot\boldsymbol{m}}{\delta}z + \frac{2}{\delta^2}\tilde{\boldsymbol{n}}\cdot\boldsymbol{k} = 0, \quad \dot{z} + \frac{\boldsymbol{n}\cdot\boldsymbol{n}}{\delta}y - \frac{\boldsymbol{m}\cdot\boldsymbol{n}}{\delta}z - \frac{2}{\delta^2}\tilde{\boldsymbol{m}}\cdot\boldsymbol{k} = 0.$$

Remark 1. System (18) can be reduced to one ordinary differential equation of second order. For this aim we introduce new designations:

$$h(t) = \exp\left\{\int\frac{\boldsymbol{n}\cdot\boldsymbol{m}}{\delta}dt\right\}, \quad y(t) = y\cdot h, \quad \tilde{z}(t) = z/h,$$
$$F^1(t) = \frac{\boldsymbol{m}\cdot\boldsymbol{m}}{\delta}h^2, \quad F^2(t) = \frac{\boldsymbol{n}\cdot\boldsymbol{n}}{\delta h^2}.$$

For functions $\tilde{y}$ and $\tilde{z}$ we obtain the system

$$\dot{\tilde{y}} = F^1\cdot\tilde{z}, \quad \dot{\tilde{z}} = -F^2\tilde{y}$$

and hence

$$\left(\frac{\dot{y}}{F^1}\right)^{\cdot} + F^2\tilde{y} = 0. \tag{19}$$

Functions F^1 and F^2 (and vectors $\boldsymbol{m}$ and $\boldsymbol{n}$ respectively) we choose in such manner that fundamental system of solutions of equation (19) should be known. Then solution (17) can be written in closed form.

Remark 2. New solutions can be obtained from solutions (16) and (17) by force of (3) only after transformations generated by operators of type $R(\boldsymbol{m}(t))$ and $Z(\alpha(t))$.

Consider system 2 (equations (8)–(11)). Equations (11) immediately gives

$$v^1 = -\frac{\dot{\beta}}{2\beta} + \frac{h(\omega_1) - 1}{\omega_2^2}, \tag{20}$$

where h is an arbitrary differentiable function of ω_1. Substituting (20) into the remaining equations (8)–(10), we get

$$q_2 = \left(\frac{1}{2}\left(\frac{\dot\beta}{\beta}\right)^{\cdot} - \frac{1}{4}\left(\frac{\dot\beta}{\beta}\right)^2\right)\omega^2 - \frac{\dot h}{\omega_2} + \frac{(h-1)^2}{\omega_2^3} + \frac{(v^2 - s(t))^2}{\omega_2^3}, \tag{21}$$

$$v_1^2 - v_{22}^2 + \left(\frac{h}{\omega_2} - \frac{\dot\beta}{2\beta}\omega_2\right)v_2^2 = 0, \tag{22}$$

$$v_1^3 - v_{22}^3 + \left(\frac{h-2}{\omega_2} - \frac{\dot\beta}{2\beta}\omega_2\right)v_2^3 = 0. \tag{23}$$

After change of independent variables

$$\tau = \int |\beta(t)|dt, \quad \omega = \sqrt{|\beta(t)|}\omega_2 \tag{24}$$

in equations (22) and (23) we obtain the decomposed system of linear equations

$$v_\tau^2 - v_{\omega\omega}^2 + \frac{h(t)}{\omega}v_\omega^2 = 0, \tag{25}$$

$$v_\tau^3 - v_{\omega\omega}^3 + \frac{h(t)-2}{\omega}v_\omega^3 = 0. \tag{26}$$

Remark 3. An arbitrary solution of equation (26) can be written down in the form $v^3 = \tilde v_\omega/\omega$, where $\tilde v^3$ is a solution of (25).

From equation (21)

$$q = \frac{\omega_2^2}{4}\left(\left(\frac{\dot\beta}{\beta}\right)^{\cdot} - \frac{1}{2}\left(\frac{\dot\beta}{\beta}\right)^2\right) - \dot h \ln\omega_2 - \frac{(h-1)^2}{2\omega_2^2} + \int \frac{(v^2(\tau,\omega) - s(t))^2}{\omega_2^3}d\omega_2. \tag{27}$$

Formulas (20), (24)–(27) and anzats 2 give a solution of the NSEs.

Remark 4. New solutions can be obtained from this solution by force of (4) only alter transformations generated by operators of type J_{ab}, $R(\boldsymbol{m}(t))$ and $Z(\alpha(t))$.

Let us to investigate the symmetry properties of the equation

$$f_t + \frac{h(t)}{r}f_r - f_{rr} = 0 \tag{28}$$

and to construct some its exact solutions.

Theorem 2. *The maximal, in the sense of Lie, invariance algebra of equation (28) is the algebra*

1. $\mathcal{A}_1 = \langle f\partial_f, g(t,r)\partial_f\rangle$ *if* $h(t) \neq \mathrm{const}$;
2. $\mathcal{A}_2 = \langle \partial_t, D, \Pi, f\partial_f, g(t,r)\partial_f\rangle$ *if* $h = \mathrm{const},\ h \notin \{0; -2\}$;
3. $\mathcal{A}_3 = \langle \partial_t, D, \Pi, f\partial_f, \partial_r + \frac{h}{2r}f\partial_f, G = 2t\partial_t - \left(r - \frac{ht}{r}\right)f\partial_f, g(t,r)\partial_f\rangle$ *if* $h \in \{0; -2\}$.

Here $D = 2t\partial_t + r\partial_r$, $\Pi = 4t^2\partial_t + 4tr\partial_r - (r^2 + 2(1-h)t)f\partial_f$, $g(t,r)$ *is an arbitrary solution of (28).*

Theorem 2 is proved by the standard Lie algorithm [4].

Consider case $h = \mathrm{const}$ in detail.

Theorem 3. *If* $h = -2n$, $n \in \mathbb{N}$, *then any solution of (28) have the form* $f = \left(\frac{1}{r}\partial_r\right)^n \tilde{f}$, *where* $\tilde{f}$ *is a solution of the one-dimensional linear heat equation:* $\tilde{f}_t = \tilde{f}_{rr}$.

To prove the theorem 3 one should use the remark 3.

Reducing equation (28) by inequivalent one-dimensional subalgebras of $\mathcal{A}_2$ we construct the following solutions:
by subalgebra $\langle \partial_t + au\partial_u \rangle$, where $a \in \{-1; 0; 1\}$:

$$f = e^{-t} r^{\nu} (C_1 J_\nu(r) + C_2 Y_\nu(r)) \quad \text{if } a = -1,$$
$$f = e^{t} r^{\nu} (C_1 I_\nu(r) + C_2 K_\nu(r)) \quad \text{if } a = 1,$$
$$f = C_1 r^{h+1} + C_2 \quad \text{if } h \neq -1 \text{ and } a = 0,$$
$$f = C_1 \ln r + C_2 \quad \text{if } h = -1 \text{ and } a = 0;$$

here J_ν, Y_ν is the Bessel functions of real variable, I_ν, K_ν is the Bessel functions of complex variable, $\nu = (h+1)/2$;
by subalgebra $\langle D + 2au\partial_u \rangle$, where $a \in \mathbb{R}$:

$$f = t^a e^{-\omega/8} \omega^{\frac{h-1}{4}} W\left(\frac{h-1}{4} - a, \frac{h+1}{4}, \frac{\omega}{4}\right),$$

where $\omega = r^2/t$ and $W(k,m,x)$ is the general solution of the Whittaker equation

$$4x^2 y'' = (x^2 - 4kx + 4m^2 - 1)y;$$

by the subalgebra $\langle \partial_t + \Pi + au\partial_u \rangle$, where $a \in \mathbb{R}$:

$$f = (4t^2+1)^{\frac{h-1}{4}} \exp\left\{-t\omega + \frac{a}{2}\operatorname{arctg} 2t\right\} \varphi(\omega),$$

where $\omega = \frac{x^2}{4t^2+1}$ and φ is a solution of the equation

$$4\omega\varphi'' + 2(1-h)\varphi' + (\omega - a)\varphi = 0;$$

if $a = 0$ then

$$\varphi(\omega) = \omega^\mu \left(C_1 J_\mu\left(\frac{\omega}{2}\right) + C_2 Y_\mu\left(\frac{\omega}{2}\right)\right), \quad \mu = \frac{1+h}{4}.$$

Consider equation (28) when $h(t)$ is an arbitrary function of time.

Theorem 4. *Equation (28) is Q-conditionally invariant under the operators*

1. $Q = \partial_t + A(t,r)\partial_r + (B^1(t,r)u + B^2(t,r))\partial_u$, *where*
$$A_t - \frac{h}{r}A_r + \frac{h}{r^2}A - A_{rr} + 2A_rA - \frac{h'}{r} + 2B^1_r = 0,$$
$$B^i_t + \frac{h}{r}B^i_r - B^i_{rr} + 2A_rB^i = 0, \quad i = 1,2;$$

2. $Q = \partial_r + B(t,r,u)\partial_u$, *where*
$$B_t - \frac{h}{r^2}B + \frac{h}{r}B_r - B_{rr} + 2BB_{ur} - B^2B_{uu} = 0.$$

Equation (28) does not have other operators of Q-conditional symmetry.

This theorem is proved by method described in [3].

Therefore, unlike Lie symmetry, Q-conditional symmetry, theorem 4 (28) for arbitrary $h(t)$ is rather wide. Thus, in particular, theorem 4 give rise to that equation (28) is Q-conditional invariant under the operators ∂_r and $G = (2t + C)\partial_r - rf\partial_f$. By means of reduction of equation (28) using the operator G we obtain the following solution

$$f = \frac{c_1}{\sqrt{4t + 2C}} \exp\left\{ -\frac{r^2}{4t + 2C} + 2\int \frac{h(t)}{4t + 2C} dt \right\},$$

and generalizing this we can construct solutions of the form

$$f = \sum_{k=0}^{N} T_k(t) r^{2k} \exp\left\{ -\frac{r^2}{4t + 2C} \right\}.$$

Other class of solutions of (28) is given with formula

$$f = \sum_{k=0}^{N} T_k(t) r^{2k}.$$

For example, if $N = 1$ then $f = C_1\left(r^2 + 2t - 2\int h(t)dt\right) + C_2$. We here do not present results for arbitrary N as they arc very cumbersome.

Consider system 3 (equations (12)–(15)). In this case we get

$$v^1 = -\frac{\dot\beta}{2\beta} + \frac{h(t) - 1}{\omega_2^2}, \tag{29}$$

$$v^2_\tau - v^2_{\omega\omega} + \frac{h(t)}{\omega} v^2_\omega = 0, \tag{30}$$

$$v^3_\tau - v^3_{\omega\omega} + \frac{h(t) - 2}{\omega} v^3_\omega + \frac{v^2 - s(t)}{\omega^2} = 0, \tag{31}$$

$$q = \frac{\omega_2^2}{4}\left(\left(\frac{\dot\beta}{\beta}\right)^{\cdot} - \frac{1}{2}\left(\frac{\dot\beta}{\beta}\right)^2 \right) - \dot h \ln\omega_2 - \frac{(h-1)^2}{2\omega_2^2} - \int \frac{v^2(\tau,\omega) - s(t))^2}{\omega_2^3} d\omega_2, \tag{32}$$

where $\tau = \int |\beta(t)| dt$, $\omega = \sqrt{|\beta(t)|}\omega_2$, $s(t) = \int a(t)dt$. Formulas (29)–(32) and ansatz 3 give a solution of the NSEs.

Remark 5. New solutions can be obtained from this solution by force of (5) only after transformations generated by operators of type J_{ab}, $R(\boldsymbol{m}(t))$ and $Z(a(t))$.

Let us to write down system (30)–(31) in the form

$$f_\tau - f_{\omega\omega} + \frac{\tilde h(\tau)}{\omega} f_\omega = 0, \tag{33}$$

$$g_\tau - g_{\omega\omega} + \frac{\tilde h(\tau) - 2}{\omega} g_\omega + \frac{\tilde f - \tilde s(\tau)}{\omega^2} = 0. \tag{34}$$

If (f, g) is a solution of (33)–(34) then $(f, g + g^0)$, where function g^0 satisfies the equation

$$g^0_\tau - g^0_{\omega\omega} + \frac{\tilde{h}(\tau) - 2}{\omega} g^0_\omega = 0 \qquad (35)$$

is also a solution of (33)–(34).

System (33)–(34) has for some $\tilde{s}(\tau)$ particular solutions of the form

$$f = \sum_{k=0}^{N} T_k(\tau)\omega^{2k}, \quad g = \sum_{k=0}^{N-1} S^k(\tau)\omega^{2k},$$

where $T^0(\tau) = \tilde{s}(\tau)$. For example, if $\tilde{s}(\tau) = -2C_1 \int (\tilde{h}(\tau) - 1)d\tau + C_2$, $N = 1$ then $f = C_1 \left(\omega^2 - 2\int (h(\tau) - 1)d\tau\right) + C_2$, $g = -C_1\tau$.

Let $\tilde{s}(\tau) = 0$.

Theorem 5. *The maximal, in the sense of Lie, invariance algebra of system (33)–(34) when* $\tilde{s}(\tau) = 0$ *is the algebra*

1. $\langle f\partial_f + g\partial_g, \tilde{f}(\tau,\omega)\partial_f + \tilde{g}(\tau,\omega)\partial_g\rangle$ *if* $\tilde{h} \neq \mathrm{const}$;
2. $\langle 2\tau\partial_\tau + \omega\partial_\omega, \partial_\tau, f\partial_f + g\partial_g, \tilde{f}(\tau,\omega)\partial_f + \tilde{g}(\tau,\omega)\partial_g\rangle$ *if* $\tilde{h} \neq \mathrm{const}$ *and* $\tilde{h} \neq 0$;
3. $\langle 2\tau\partial_\tau + \omega\partial_\omega, \partial_\tau, \dfrac{f}{\omega}\partial_g, f\partial_f + g\partial_g, \tilde{f}(\tau,\omega)\partial_f + \tilde{g}(\tau,\omega)\partial_g\rangle$ *if* $\tilde{h} = 0$.

Here $(\tilde{f}, \tilde{g})$ *is an arbitrary solution of (33)–(34) when* $\tilde{s}(\tau) = 0$.

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