New nonlinear equations for electromagnetic field having velocity different from c

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Предложены новые нелинейные уравнения для электромагнитного поля, скорость которого в вакууме может быть меньше, чем c. Предложены также нелинейные уравнения для электромагнитного, скалярного и спинорного полей.

1. The Maxwell equations

$$\frac{\partial \vec{D}}{dt} = c \operatorname{rot} \vec{H} - \vec{j}, \quad \frac{\partial B}{\partial t} = -c \operatorname{rot} \vec{E}, \quad \operatorname{div} \vec{D} = \rho, \quad \operatorname{div} \vec{H} = 0. \tag{1}$$

play a basic role in modern electromagnetic theory. When considered in vacuum, Eqs. (1) take the form

$$\frac{\partial \vec{E}}{\partial t} = c \operatorname{rot} \vec{H}, \quad \frac{\partial \vec{H}}{\partial t} = -c \operatorname{rot} \vec{E}, \quad \operatorname{div} \vec{E} = 0, \quad \operatorname{div} \vec{H} = 0. \tag{2}$$

Provided $\vec{D} = \varepsilon \vec{H}$, $\vec{J} = \sigma \vec{E}$, $\vec{B} = \mu \vec{H}$, ε, σ, μ being constants, from (1) it follows that the wave equations hold

$$\varepsilon\mu \frac{\partial^2 \vec{E}}{\partial t^2} - c^2 \Delta \vec{E} + \sigma\mu \frac{\partial^2 \vec{E}}{\partial t} = -\frac{1}{\varepsilon} \vec{\nabla} \rho, \quad \varepsilon\mu \frac{\partial^2 \vec{H}}{\partial t^2} - c^2 \Delta \vec{H} + \sigma\mu \frac{\partial \vec{H}}{\partial t} = \vec{0}. \tag{3}$$

When considered in vacuum ($\varepsilon = \mu = 1$, $\sigma = 0$) Eqs. (3) read

$$\frac{\partial^2 \vec{E}}{\partial t} - c^2 \Delta \vec{E} = \vec{0}, \quad \frac{\partial^2 \vec{H}}{\partial t^2} - c^2 \Delta \vec{H} = \vec{0}. \tag{4}$$

It is a generally accepted axiom of the modern theory of elementary interactions (classical and quantum) that the quantity in (1)–(4) is identified with the velocity of light. That is the fundamental constant.

There are few works devoted to study of nonlinear generalizations of equations (1)–(3) (see, e.g., lists of references in [1–3]).

In the present paper we suggest new nonlinear generalization of Eqs. (1)–(4) based on the following idea: the velocity of light may not coincide with the constant c.

2. Let us admit following Poyting (1884) the standard definition of the energy density and of the electromagnetic flow

$$\rho = \frac{1}{2}(\vec{E}^2 + \vec{H}^2), \quad \rho v_k = c \varepsilon_{kln} E_l H_n, \quad k, l, n = \overline{1, 3}, \tag{5}$$

$\vec{v} = (v_1, v_2, v_3)$ is the velocity of the electromagnetic flow.

It is easy to see of that the formula

$$\vec{v}^{\,2} = c^2 \left\{ 1 - \frac{1}{4} \rho^{-2} (\vec{E}^2 - \vec{H}^2)^2 - \rho^{-2} (\vec{E}\vec{H})^2 \right\} \tag{6}$$

holds.

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From (6) it follows that $\vec{v}^{\,2} \le c^2$ and what is more $\vec{v}^{\,2} = c^2 \;\Leftrightarrow\; \vec{E}^2 - \vec{H}^2 = 0$, $\vec{E}\vec{H} = 0$.

Let us make in Eqs. (1)–(4) the change

$$c \to v, \quad c^2 \to v^2.$$

This change yields nonlinear equations for electromagnetic field. For example, Eqs. (2) take the form

$$\frac{\partial \vec{E}}{\partial t} = v \operatorname{rot} \vec{H}, \quad \frac{\partial H}{\partial t} = -v \operatorname{rot} \vec{H}, \quad \operatorname{div} \vec{E} = 0, \quad \operatorname{div} \vec{H} = 0. \tag{7}$$

The above equations can be generalized in the following way:

$$\frac{\partial \vec{E}}{\partial t} = \operatorname{rot} (\vec{H} \times \vec{v}), \quad \frac{\partial \vec{H}}{\partial t} = \operatorname{rot} (\vec{v} \times \vec{E}). \tag{8}$$

Eqs. (7), (8) can be interpreted as equations of motion for an electromagnetic field which spreads with velocity $\vec{v}$. Provided $\vec{v}$ is determined by (5), (6), the velocity of electromagnetic field is smaller than c.

One can impose on $\vec{v} = \vec{v}(t, \vec{x})$ equations of hydrodynamics type

$$\Delta\Delta\vec{v} = \vec{0}, \quad \Delta \equiv \frac{\partial}{\partial t} + \lambda v_k \frac{\partial}{\partial v_k} \tag{9}$$

or

$$\Delta\Delta^2\vec{v} = \vec{0}, \tag{10}$$

whence

$$\begin{gathered} \frac{\partial \vec{E}}{\partial t} = \operatorname{rot} (\vec{H} \times \vec{v}), \quad \frac{\partial \vec{H}}{\partial t} = \operatorname{rot} (\vec{v} \times \vec{E}), \\ \Delta\Delta\vec{v} = \vec{0}, \quad \operatorname{div} \vec{E} = 0, \quad \operatorname{div} \vec{H} = 0. \end{gathered} \tag{11}$$

Thus system (11) describes woth the electromagnetic field and its velocity.

Note 1. Eq. (9) possesses unique symmetry properties. Is is invariant under the Poincaré and Galilei groups [3]. That is Eq. (9) satisfies. both the Lorentz–Poincaré–Einstein and Galilei relativity principles.

In addition, we adduce another nonlinear equation

$$\begin{gathered} \frac{\partial \vec{E}}{\partial t} + \lambda_1(\vec{E}^2 - \vec{H}^2, \vec{E}\vec{H}) H_k \frac{\partial \vec{E}}{\partial x_k} + \lambda_2(\vec{E}^2 - \vec{H}^2, \vec{E}\vec{H}) \operatorname{rot} H = \vec{0}, \\ \frac{\partial \vec{E}}{\partial t} + \lambda_3(\vec{E}^2 - \vec{H}^2, \vec{E}\vec{H}) E_k \frac{\partial \vec{H}}{\partial x_k} + \lambda_4(\vec{E}^2 - \vec{H}^2, \vec{E}\vec{H}) \operatorname{rot} E = \vec{0}, \end{gathered}$$

where λ_1, λ_2, λ_3, λ_4 are some smooth functions.

Eqs. (1), (3), (4) are generalized in an analogous way. For example, Eq. (4) is generalized in a way

$$\frac{\partial^2 \vec{E}}{\partial t^2} - v^2 \Delta \vec{E} = \vec{0}, \quad \frac{\partial^2 \vec{H}}{\partial t^2} - v^2 \Delta \vec{H} = \vec{0}. \tag{12}$$

or

$$\frac{\partial^2 \vec{E}}{\partial t^2} - \frac{\partial}{\partial x_l}\left\{ c_{ln}(v^2)\frac{\partial \vec{E}}{\partial x_n}\right\} = \vec{0}, \quad \frac{\partial^2 \vec{H}}{\partial t^2} - \frac{\partial}{\partial x_l}\left\{ c_{ln}(v^2)\frac{\partial \vec{E}}{\partial x_n}\right\} = \vec{0}, \tag{13}$$

or

$$\frac{\partial}{\partial x_\mu}\left\{ c_{\mu\nu}(v^2)\frac{\partial}{\partial x_\nu}\right\} F_{\alpha\beta} = 0, \quad \mu,\nu,\alpha,\beta = \overline{0,3}, \tag{14}$$

where $c_{lm}(v^2)$, $c_{\mu\nu}(v^2)$ are smooth functions on v^2. One can impose on the scalar function v the eikonal equation

$$\left(\frac{\partial v}{\partial t}\right)^2 - \left(\frac{\partial v}{\partial x_k}\right)^2 = \lambda, \quad \lambda = 0, \pm 1. \tag{15}$$

Note 2. In Eqs. (12)–(14) the vector $\vec{v}$ can be defined according to the formula (6).

3. Let us turn to the generalization of the linear d'Alembert–Klein–Gordon–Fock equation

$$-\hbar^2\frac{\partial^2 u}{\partial t^2} = (-\hbar^2 c^2 \Delta + m^2 c^4)u. \tag{16}$$

After the change $c \to v(t,\vec{x})$ it takes the form

$$-\hbar^2\frac{\partial^2 u}{\partial t^2} = (-\hbar^2 v^2 \Delta + m^2 c^4)u, \tag{17}$$

where v is determined by (6), functions $\vec{E}$, $\vec{H}$ satisfying Eqs. (1)–(4) or Eqs. (7)–(8).

Note 3. The vector of velocity of spread of the scalar field u can be defined in the following way:

$$v_k = \lambda(|u|)\left(u^*\frac{\partial u}{\partial x_k} + u\frac{\partial u^*}{\partial x_k}\right) \tag{18}$$

or

$$v_\mu = \lambda(|u|)\left(u^*\frac{\partial u}{\partial x_\mu} + u\frac{\partial u^*}{\partial x_\mu}\right), \tag{19}$$

where $\lambda(|u|)$ is an arbitrary smooth function.

4. The Dirac equation for the spinor field

$$(-i\hbar\gamma_\mu\partial_\mu + mc)\psi = 0 \tag{20}$$

is rewritten in the form of nonlinear system

$$(-i\hbar\gamma_\mu\partial_\mu + mv)\psi = 0, \quad v = \left(v_1^2 + v_2^2 + v_3^2\right)^{1/2}, \tag{21}$$

$$\Delta\Delta v_k = 0 \quad \text{or} \quad v_a\frac{\partial v_\mu}{\partial x_a} = 0. \tag{22}$$

Note 4. The vector of velocity of spread of the spinor field can be defined by the formula (6). In this case, $\vec{E}$, $\vec{H}$ are vectors characterizing elecromagnetic field which is generated by the spinor field

$$F_{\mu\nu} = \lambda_1 \bar{\psi}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)\psi, \tag{23}$$

λ_1 is some small parameter.

Note 5. The vector of velocity of the spinor field can be defined as follows

$$v_k = \lambda_2 \bar{\psi} \gamma_k \psi$$

or

$$v_\mu = \lambda_3 \bar{\psi} \gamma_\mu \psi.$$

One can demand that the above vectors have to satisfy conditions (22).

Detailed symmetry analysis and construction of exact solutions of the above suggested equations will be carried out in future paper.

Note 6. The classical wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0$$

in our approach is generalised in the following way

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x_k} \left\{ a_{kl}(\bar{v}) \frac{\partial u}{\partial x_l} \right\} = 0, \quad a_{kl} = \lambda_1(v^2) v_k v_l + \lambda_2(v^2)\delta_{ik},$$
$$\Delta\Delta v_i = 0 \quad \text{or} \quad \Delta\Delta^2 v_i = 0.$$

In one dimensional space the wave equation has the form

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x_k} \left(v^2 \frac{\partial u}{\partial x} \right) = 0, \quad \Delta\Delta v + \lambda_3 \frac{\partial^2 v}{\partial x^2} = 0,$$

λ_1, λ_2, λ_3 are smooth functions of v^2.

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3. Fushchych W.I., Shtelen W.M., Serov N.I., Symmetry analysis and exact solutions of nonlinear equations of mathematical physics, Kiev, Naukova Dumka, 1989, 336 p.