Second-order differential invariants of the rotation group $O(n)$ and of its extensions: $E(n)$, $P(1,n)$, $G(1,n)$

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Functional bases of second-order differential invariants of the Euclid, Poincaré, Galilei, conformal, and projective algebras are constructed. The results obtained allow us to describe new classes of nonlinear many-dimensional invariant equations.

0. Introduction

The concept of the invariant is widely used in various domains of mathematics. In this paper, we investigate the differential invariants within the framework of symmetry analysis of differential equations.

Differential invariants and construction of invariant equations were considered by S. Lie [1] and his followers [2, 3]. Tresse [2] had proved the theorem on the existence and finiteness of a functional basis of differential invariants. However, there exist quite a few papers devoted to the construction in explicit form of differential invariants for specific groups involved in mechanics and mathematical physics.

Knowledge of differential invariants of a certain algebra or group facilitates classification of equations invariant with respect to this algebra or group. There are also some general methods for the investigation of differential equations which need tide explicit form of differential invariants for these equations' symmetry groups (see, e.g., [3, 4]).

A brief review of our investigation of second-order differential invariants for the Poincaré and Galilei groups is given in [5, 6]. Our results on functional bases of differential invariants are founded on the Lemma about functionally independent warrants for the proper orthogonal group and two n-dimensional symmetric tensors of the order 2.

We should like to stress that we consider functionally independent invariants of but not irreducible ones, as in the classical theory of invariants.

Bases of irreducible invariants for the group $O(3)$ and three-dimensional symmetric tensors and vectors are adduced in [7].

The definitions of differential invariants differ in various domains of mathematics, e.g. in differential geometry and symmetry analysis of differential equations. Thus, we believe that some preliminary notes are necessary, though these formulae and definitions can be found in [8, 9, 10].

We deal with Lie algebras consisting of the infinitesimal operators

$$X = \xi^i(x,u)\partial_{x_i} + \eta^r(x,u)\partial_{u^r}. \tag{0.1}$$

Here $x = (x^1, x^2, \dots, x^n)$, $u = (u^1, \dots, u^m)$. We usually mean the summation over the repeating indices.

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Definition 1. *The function*

$$F = F(x, u, \underset{1}{u}, \dots, \underset{l}{u}),$$

where $\underset{k}{u}$ is the set of all kth-order partial derivatives of the function u is called a differential invariant for the Lie algebra L with basis elements X_i of the form (0.1) ($L = \langle X_i \rangle$) if it is an invariant of the lth prolongation of this algebra:

$$\overset{l}{X}_s F(x, u, \underset{1}{u}, \dots, \underset{l}{u}) = \lambda_s(x, u, \underset{1}{u}, \dots, \underset{l}{u}) F, \tag{0.2}$$

where the λ_s are some functions; when $\lambda_i = 0$, F is called an absolute invariant; when $\lambda_i \neq 0$, it is a relative invariant.

Further, we deal mostly with absolute differential invariants and when writing ‘differential invariant’ we mean ‘absolute differential invariant’.

Definition 2. *A maximal set of functionally independent invariants of order $r \leq l$ of the Lie algebra L is called a functional basis of the lth-order differential invariants for the algebra L.*

We consider invariants of order 1 and 2 and need the first and second prolongations of the operator X (0.1) (see, e.g., [8–11])

$$\overset{1}{X} = X + \eta^r_i \partial_{u^r_i}, \quad \overset{2}{X} = \overset{1}{X} + \eta^r_{ij} \partial_{u^r_{ij}}$$

the coefficients η^r_i and η^r_{ij} taking the form

$$\eta^r_i = (\partial_{x_i} + u^s_i \partial_{u^s}) \eta^r - u^r_k (\partial_{x_i} + u^s_i \partial_{u^s}) \xi^k,$$
$$\eta^r_{ij} = (\partial_{x_i} + u^s_j \partial_{u^s} + u^s_{jk} \partial_{u^s_k}) \eta^r_i - u^r_{ik} (\partial_{x_j} + u^s_j \partial_{u^s}) \xi^k.$$

While writing out lists of invariants, we shall use the following designations

$$\begin{aligned}
& u_a \equiv \frac{\partial u}{\partial x_a}, \quad u_{ab} \equiv \frac{\partial^2 u}{\partial x_a \partial x_b}, \\
& S_k(u_{ab}) \equiv u_{a_1 a_2} u_{a_2 a_3} \cdots u_{a_{k-1} a_k} u_{a_k a_1}, \\
& S_{jk}(u_{ab}, v_{ab}) \equiv u_{a_1 a_2} \cdots u_{a_{j-1} a_j} v_{a_j a_{j+1}} \cdots v_{a_k a_1}, \\
& R_k(u_a, u_{ab}) \equiv u_{a_1} u_{a_k} u_{a_1 a_2} u_{a_2 a_3} \cdots u_{a_{k-1} a_k} u_{a_k a_1}.
\end{aligned} \tag{0.3}$$

Here and further we mean summation over the repeated indices from 1 to n. In all the lists of invariants, k takes on the values from 1 to n and j takes the values from 0 to k. We shall not discern the upper and lower indices with respect to summation: for all Latin indices

$$x_a x_a \equiv x_a x^a \equiv x^a x_a = x_1^2 + x_2^2 + \cdots + x_n^2.$$

1. Differential invariants for the Euclid algebra

The Euclid algebra $AE(n)$ is defined by basis operators

$$\partial_a \equiv \frac{\partial}{\partial x_a}, \quad J_{ab} = x_a \partial_b - x_b \partial_a. \tag{1.1}$$

Here and further, the letters a, b, c, d, when used as indices, take on the values from 1 to n, n being the number of space variables ($n \geq 3$).

The algebra $AE(n)$ is an invariance algebra for a wide class of many-dimensional scalar equations involved in mathematical physics — the Schrödinger, heat, d'Alembert equations, etc.

In this section, we shall explain in detail how to construct a functional basis of the second-order differential invariants for the algebra $AE(n)$. This basis will be further used to find invariant bases for various algebras containing the Euclid algebra as a subalgebra — the Poincaré, Galilei, conformal, projective algebras, etc.

1.1. The main results. Let us first formulate the main results of the section in the form of theorems.

Theorem 1. *There is a functional basis of second-order differential invariants for the Euclid algebra $AE(n)$ with the basis operators (1.1) for the scalar function $u = u(x_1, \dots, x_n)$ consisting of these $2n + 1$ invariants*

$$u, \quad S_k(u_{ab}), \quad R_k(u_a, u_{ab}). \tag{1.2}$$

Theorem 2. *The second-order differential invariants of the algebra $AE(n)$ (1.1) for the set of scalar functions u^r, $r = 1, \dots, m$, can be represented as functions of the following expressions:*

$$u^r, \quad S_{jk}(u^1_{ab}, u^r_{ab}), \quad R_k(u^r_a, u^1_{ab}). \tag{1.3}$$

1.2. Proofs of the theorems. Absolute differential invariants are obtained as solutions of a linear system of first-order partial differential equations (PDE). Thus, the number of elements of a functional basis is equal to the number of independent integrals of this system. This number is equal to the difference between the number of variables on which the functions being sought depend, and the rank of the corresponding system of PDE (in our case, this rank is equal to the generic rank of the prolonged operator algebra [8, 9].

To prove the fact that N invariants which have been found, $F^i = F^i(x, u, \underset{1}{u}, \dots, \underset{l}{u})$, form a functional basis, it is necessary and sufficient to prove the following statements:

(1) the F^i are invariants;

(2) the F^i are functionally independent;

(3) the set of invariants F^i is complete or N is equal to the difference of the number of variables $(x, u, \underset{1}{u}, \dots, \underset{l}{u})$ and the rank of the system of defining operators.

We seek second-order differential invariants in the form

$$F = F(x, u, \underset{1}{u}, \underset{2}{u}).$$

It follows from the condition of invariance with respect to translation operators ∂_a that F does not depend on x_a; evidently, u is an invariant of the operators (1.1). Thus, it is sufficient to seek invariants depending on $\underset{1}{u}$ and $\underset{2}{u}$ only. The criterion of the absolute invariance (0.1) in this case has the form

$$\hat{J}_{ab} F(\underset{1}{u}, \underset{2}{u}) = 0, \tag{1.4}$$

where

$$\hat{J}_{ab} = u^r_a \partial_{u^r_b} - u^r_b \partial_{u^r_a} + 2(u^r_{ac} \partial_{u^r_{bc}} - u^r_{bc} \partial_{u^r_{ac}}), \tag{1.5}$$

the summation over r from 1 to m being implied.

In that way, the problem of finding the second-order differential invariants of the algebra $AE(n)$ is reduced to the construction of a functional basis for the rotational algebra $AO(n)$ with the basis operators (1.5) for m vectors and m symmetric tensors of order 2.

Lemma 1. *The rank of the algebra $AO(n)$ is equal to $(n(n-1))/2$.*

Proof. It is sufficient to prove the lemma for $m = 1$. The basis of the algebra (1.5) consists of $(n(n-l))/2$ operators. According to definition [8], its rank is equal to the generic rank of the coefficient matrix of these operators. Let us put $u_{ab} = 0$ when $a \neq b$ and write down the coefficient columns by $\partial_{u_{ab}}$ of the operators (1.5):

$$\begin{pmatrix} u_{11}-u_{22} & 0 & \cdots & 0 \\ 0 & u_{11}-u_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & u_{n-1,n-1}-u_{nn} \end{pmatrix}. \tag{1.6}$$

When $u_{aa} \neq u_{bb}$ for $a \neq b$ and all $u_{aa} \neq 0$, the determinant of the matrix (1.6) does not vanish, therefore its generic rank (that is, the generic rank the algebra being considered) cannot be less than $(n(n-1))/2$. The lemma is proved. ■

Lemma 2. *The expressions*

$$S_k(u_{ab}), \quad R_k(u_a, u_{ab}) \tag{1.7}$$

are functionally independent.

Proof. To establish independence of expressions (1.7), it is sufficient to consider the case when $u_{ab} = 0$ if $a \neq b$ and $u_{aa} \neq 0$. Let us write down the Jacobian of the invariants

$$\left|\begin{array}{ccc|ccc} 1 & \cdots & 1 & & & \\ 2u_{11} & \cdots & 2u_{nn} & & \mathbf{0} & \\ \cdots & \cdots & \cdots & & & \\ nu_{11}^{n-1} & \cdots & nu_{rr}^{n-1} & & & \\ \hline & & & 2u_1 & \cdots & 2u_n \\ & \cdots & & \cdots & \cdots & \cdots \\ & & & 2u_1u_{11}^{n-1} & \cdots & 2u_nu_{nn}^{n-1} \end{array}\right| \tag{1.8}$$

The Jacobian (1.8) is equal up to a coefficient to the product of two Vandermonde determinants and is not equal to zero if $u_{aa} \neq u_{bb}$ whenever $a \neq b$. Thus, the expressions (1.17) are functionally independent. ■

Proof of Theorem 1. The fact that expressions (1.2) are invariants of $AO(n)$ can be easily proved by direct substitution of these expressions into the invariance conditions. Nevertheless, it is useful to note that $S_k(u_{ab})$ are traces of the symmetric matrix $(u_{ab}) = U$ and its powers, $R_k(u_a, u_{ab})$ are the scalar products of the vector $(u_a) = (u_1, \ldots, u_n)$, the matrix U^{k-1} and the vector $(u_a)^T$.

The invariants for the vector (u_a) and the symmetric tensor (u_{ab}) depend on their $(n(n+3))/2$ elements. Thus, it follows from Lemma 1 that a functional basis of the algebra $AO(n)$ for (u_a) and (u_{ab}) must consist of

$$\frac{n(n+3)}{2} - \frac{n(n-1)}{2} = 2n$$

invariants.

Therefore the set (1.7) is a complete set of functionally independent invariants of the form $F = F(\underset{1}{u}, \underset{2}{u})$ and (1.2) represents a functional basis of the second-order invariants for the algebra $AE(n)$. The theorem is proved. ■

Let us consider the case of two vectors (u_a), (v_a) and two symmetric tensors of the second order (u_{ab}), (v_{ab}). The operators of the rotation algebra have the form (1.5), $u \equiv u^1$, $v \equiv u^2$.

In this case, a functional basis of invariants contains

$$2\left(\frac{n(n-1)}{2} + 2n\right) - \frac{n(n-1)}{2} = \frac{n(n+7)}{2}$$

elements for which we take the following expressions

$$R_k(u_a, u_{ab}), \quad R_k(v_a, u_{ab}), \quad S_{jk}(u_{ab}, v_{ab}). \tag{1.9}$$

The invariance of expressions (1.9) with respect to the operators (1.5) can be easily proved by their direct substitution to (1.4). To establish their functional independence, we shall use the following lemma.

Lemma 3. *Let*

$$U = (u_{ab})_{a,b=1,\dots,n}, \quad V = (v_{ab})_{a,b=1,\dots,n}$$

be symmetric matrices. Then the expressions

$$S_{jk}(u_{ab}, v_{ab}) = \operatorname{tr} U^j V^{k-j}, \quad j = 0, \dots, k; \quad k = 1, \dots, n, \tag{1.10}$$

are functionally independent.

Proof. To prove Lemma 3, it is sufficient to show that the generic rank of the Jacobi matrix of expressions (1.10) is equal to $(n(n+3))/2$ that is the difference between the number of independent elements of U and V and the rank of the operators (1.5). We shall limit ourselves to the case when $u_{ab} = 0$ if $a \neq b$. Then equations (1.10) depend on $(n(n+3))/2$ variables and their independence is equivalent to the nonvanishing of the Jacobian.

Let us write down the elements of the Jacobian which are needed for further reasoning

$$\left|\begin{array}{ccc|ccccccc} 1 & \cdots & 1 & & & & & & & \\ 2u_{11} & \cdots & 2u_n & & & & \mathbf{0} & & & \\ \cdots & \cdots & \cdots & & & & & & & \\ nu_{11}^{n-1} & \cdots & nu_{nn}^{n-1} & & & & & & & \\ \hline & & & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ & \cdots & & 2v_{11} & 4v_{12} & \cdots & 4v_{1n} & 2v_{22} & \cdots & 2v_{nn} \\ & & & & & \cdots & & & & \end{array}\right|. \tag{1.11}$$

Since, in the first n rows, all the elements besides the first n columns are equal to j zero, the Jacobian (1.11) is equal to the product of the Jacobian of the elements $\operatorname{tr} U^k$, $k = 1, \dots, n$, and the Jacobian of all other elements. According to Lemma 2, the expressions $\operatorname{tr} U^k$, $k = 1, \dots, n$, are independent and their Jacobian is not equal to zero; thus, it remains to show the nonvanishing of the Jacobian and the functional independence only for the elements

$$\operatorname{tr} U^j V^{k-j}, \quad j = 0, \ldots, k-1; \quad k = 1, \ldots, n.$$

It follows from (1.11) that it is sufficient to show the nonvanishing of this Jacobian without the $(n+1)$th rows and columns. Thus, to prove the lemma, it is enough to show that the following expressions are independent

$$\operatorname{tr} U^j V^{k-j} V, \quad j = 0, \ldots, k; \quad k = 1, \ldots, n-1. \tag{1.12}$$

The above reasoning allows us to make use of the principle of mathematical induction.

When $n = 1$, u_{11} and v_{11} are independent and the lemma is true. Let us suppose that it is true for $n-1$ and then prove from this that it is valid for n. Let the expressions

$$\operatorname{tr} U^j V^{k-j}, \quad j = 0, \ldots, k; \quad k = 1, \ldots, n-1, \tag{1.13}$$

where U, V are symmetric $(n-1) \times (n-1)$ matrices and are independent. Then, we shall prove the independence of (1.12) for the same matrices. The sets (1.12) and (1.13) coincide with the exception of the following subsets

$$\operatorname{tr} U^j V^{n-j}, \quad j = 0, \ldots, n-1 \tag{1.14}$$

belong only to (1.12) and

$$\operatorname{tr} U^j, \quad j = 1, \ldots, n-1 \tag{1.15}$$

belong only to (1.13).

The assumption of validity of the lemma for $n-1$ means that for two symmetric tensors of order 2, the set (1.13) is a functional basis of invariants of the rotation algebra. Thus, all the invariants of this algebra can be represented as functions of (1.13). To prove the functional independence of (1.12), it is sufficient to prove the nondegeneracy of the Jacobi matrix of the functions expressing the invariants (1.12) with (1.13). This matrix has the form

$$\left(\begin{array}{cccc|c|c} 1 & & \mathbf{0} & & & \cdots \\ & 1 & & & & \\ & & \ddots & & & \\ & \mathbf{0} & & 1 & & \\ \hline & & & & W & \\ \hline & & \mathbf{0} & & & \frac{\partial(\operatorname{tr} U^j V^{n-j})}{\partial(\operatorname{tr} U^j)} \end{array}\right), \tag{1.16}$$

W being the derivative by $\operatorname{tr} V$ of the expression

$$\operatorname{tr} V^n = F(\operatorname{tr} V^k,\ k = 1, \ldots, n-1).$$

(We know that from the Hamilton–Cayley theorem); $W \neq 0$.

We have only to prove the nonvanishing of the Jacobian of the expressions

$$\operatorname{tr} (U^j V^{n-j}) = F(\operatorname{tr} U^k,\ k = 1, \ldots, n-1, \ldots). \tag{1.17}$$

When $V = E$, the corresponding quadrant of the matrix (1.16) is the unit matrix and its determinant does not vanish identically. This fact proves the nondegeneracy of the matrix (1.16). The expressions (1.17) can be obtained from the Hamilton–Cayley theorem. They are polynomials and, thus, continuous functions of their arguments.

The functional independence of the expressions (1.12) for $(n-1)\times(n-1)$ matrices implies their independence for $n \times n$ matrices. From the above, it follows that the expressions (1.10) are independent, thus Lemma 3 is proved. ■

Proof of Theorem 2. It is easy to see from the structure of the set (1.3) that the invariants involving $(u_a^1), \ldots, (u_a^m)$, $(u_{ab}^2), \ldots, (u_{ab}^m)$ depend on the components of (u_{ab}^1) and of the corresponding vector or tensor, thus it is sufficient to prove the functions independence of each of the following sets:

$$R_k(u_a^r, u_{ab}^1) \qquad \text{for every } r = 1, \ldots, m;$$
$$S_{jk}(u_{ab}^1, u_{ab}^r) \qquad \text{for every } r = 2, \ldots, m;$$

The functional independence of each set of $R_k(u_a^r, u_{ab}^1)$ can be proved similarly to the proof of Lemma 2. The functional independence of the set $S_{jk}(u_{ab}^1, u_{ab}^r)$ easily follows from Lemma 3, u^r are evidently independent of other elements of (1.3).

To make sure that expressions (1.3) are invariants of $AO(n)$, it is sufficient to substitute them into the condition (1.4).

The set (1.3) consists of

$$2mn + m + (m-1)\frac{n(n-1)}{2} = m\left(\frac{n(n+1)}{2} + n + 1\right) - \frac{n(n-1)}{2}$$

elements and, thus, it is complete.

So we have proved that this set forms a basis of invariants for the algebra $AE(1.n)$ (1.1).

1.3. Bases of invariants for the extended Euclid algebra and for the conformal algebra. The extended Euclid algebra $AE_1(n)$ for one scalar function is defined by the basis operators ∂_a, J_{ab} (1.1) and D depending on a parameter λ:

$$D = x_a\partial_a + \lambda u\partial_u \quad (\partial_u = \partial/\partial u). \tag{1.18}$$

The basis of the conformal algebra $AC(n)$ consists of the operators ∂_a, J_{ab} (1.1) and D (1.18) and

$$K_a = 2x_a D - x_a x_b \partial_a. \tag{1.19}$$

Theorem 3. *There is a functional basis for the extended Euclid algebra that has the following form*

(1) when $\lambda \neq 0$*:*

$$\frac{R_k(u_a, u_{ab})}{u^{k(1-2/\lambda)+1}}, \quad \frac{S_k(u_{ab})}{u^{k(1-2/\lambda)}}; \tag{1.20}$$

(2) when $\lambda = 0$*:*

$$u, \quad \frac{R_k(u_a, u_{ab})}{(u_{aa})^k}, \quad \frac{S_k(u_{ab})}{(u_{aa})^k} \quad (k \neq 1); \tag{1.21}$$

a functional basis for the conformal algebra has the following form:

(1) when $\lambda \neq 0$*:*

$$S_k(\theta_{ab})u^{k(2/\lambda-1)}; \tag{1.22}$$

(2) when $\lambda = 0$*:*

$$u, \quad S_k(w_{ab})(u_a u_a)^{-2k} \quad (k \neq n), \tag{1.23}$$

where

$$\begin{aligned} \theta_{ab} &= \lambda u_{ab} + (1-\lambda)\frac{u_a u_b}{u} - \delta_{ab}\frac{u_c u_c}{2u}, \\ w_{ab} &= u_c u_c\left(u_{ab} + \frac{\delta_{ab}}{2-n}u_{dd}\right) - u_c(u_a u_{bc} + u_b u_{ac}), \end{aligned} \tag{1.24}$$

δ_{ab} *being the Kronecker symbol.*

Proof. To find absolute differential invariants of the algebra $AE_1(n)$, it is necessary to add to (1.4) the following condition

$$\overset{2}{D} F \equiv x_a F_{x_a} + \lambda u F_u + (\lambda - 1)u_a F_{u_a} + (\lambda - 2)u_{ab}F_{u_{ab}} = 0. \tag{1.25}$$

Solving equation (1.25) for

$$F = F(u, R_k(u_a, u_{ab}), S_k(u_{ab})),$$

we obtain functional bases (1.20), (1.21) for the extended Euclid algebra.

The second-order differential invariants of the algebra $AC(n)$ are defined by the conditions (1.4), (1.25) and

$$k_a \overset{2}{K}_a F = 0, \tag{1.26}$$

where k_a are arbitrary real numbers, $\overset{2}{K}_a$ are the second prolongations of the operators K_a (1.19):

$$\overset{2}{K}_a = 2x_a \overset{2}{D} + x_b \overset{2}{J}_{ab} + 2\lambda[u\partial_{u_a} + 2u_b\partial_{u_{ab}}] + 2u_a\partial_{u_{cc}} - 4u_b\partial_{u_{ab}}.$$

Solving this system for an arbitrary n requires a lot of cumbersome computations. It is simpler to construct conformally co variant tensors from u, u_a, u_{ab} and then to construct invariants of the rotation algebra.

Definition 3. *Tensors* θ_a *and* θ_{ab} *of order 1 and 2 are called covariant with respect to some algebra* $L = \langle J_{ab}, X_i\rangle$ *if*

$$\begin{aligned} X_i\theta_a &= \sigma^i_{ab}\theta_b + \sigma^i\theta_a, \\ X_i\theta_{ab} &= \rho^i_{ab}\theta_{cb} + \rho^i_{bc}\theta_{ac} + \rho^i\theta_{ab}, \end{aligned} \tag{1.27}$$

X_i *are operators of the form (0.1),* ρ^i, σ^i *are some functions,* σ^i_{ab}, ρ^i_{ab} *are some skew-funmetric tensors.*

It is easy to show that the expressions $S_k(\theta_{ab})$, $R_k(\theta_a, \theta_{ab})$, where θ_a, θ_{ab} are tensors covariant with respect to the algebra L are relative invariants of this algebra.

The fact that θ_{ab} and w_{ab} (1.24) are covariant with respect to the conformal algebra $AC(n)$ can be verified by direct substitution of these tensors into the conditions (1.27) for the operators $\overset{2}{D}$ and $\overset{2}{K}_a$.

The rank of the second prolongation of the algebra $AC(n)$ is equal to the number of its operators

$$\frac{n(n-1)}{2}+n+n+1=\frac{n(n+3)}{2}+1$$

and, therefore, a functional basis of second-order differential invariants must contain n invariants.

The functional independence of the expressions (1.22) follows from Lemma 2 if we notice that the transformation $u_{ab}\to\theta_{ab}$ is nondegenerated. The same is true for the set (1.23).

The expressions (1.22) and (1.23) satisfy (1.25) and (1.26) for the corresponding λ and they are invariants of the conformal algebra.

All that is stated above leads to the conclusion that (1.22) and (1.23) form functional bases for the conformal algebra $AC(n)$ with $\lambda\neq 0$ and $\lambda=0$, respectively.

Note 1. Using condition (1.26), it is easy to show that when $\lambda\neq 0$ covariant tensors exist for $AC(n)$ of order 2 only; when $\lambda=0$, the tensors w_{ab} (l.24) and u_a are conformally covariant but $S_k(w_{ab})$ and $R_k(u_a,w_{ab})$ are dependent.

Theorem 4. *The second-order differential invariants for a vector function $u=(u^1,\dots,u^m)$ and for the algebra $AE_1(n)=\langle\partial_a,J_{ab},D\rangle$, the operator D having the form*

$$D=x_a\partial_a+\lambda u^r\partial_{u^r} \tag{1.28}$$

with a summation over r from 1 to m, can be represented as the functions of the following expressions:

(1) when $\lambda\neq 0$:

$$\frac{u^r}{u^1}\ (r=2,\dots,m),\quad \frac{S_{jk}(u^1_{ab},u^r_{ab})}{(u^1)^{k(1-2/\lambda)}},\quad \frac{R_k(u^r_a,u^1_{ab})}{(u^1)^{k(1-2/\lambda)+1}};$$

(2) when $\lambda=0$:

$$u^r,\quad R_k(u^r_a,u^1_{ab})(u^1_{aa})^{-k},\quad S_{jk}(u^1_{ab},u^r_{ab})(u^1_{aa})^{-k}$$

(when $r=1$ then $k\neq 1$);

the corresponding basis for the conformal algebra $AC(n)=\langle\partial_a,J_{ab},D,K_a\rangle$ ($K_a=2x_aD-x_bx_b\partial_a$) has the following form:

(1) when $\lambda\neq 0$:

$$\begin{aligned}&S_{jk}(\theta^r_{ab},\theta^1_{ab})(u^1)^{k(2/\lambda-1)},\quad \frac{u^r}{u^1},\\ &R_k(\theta^r_a,\theta^1_{ab})^{k(2/\lambda-1)-1}\quad (r=2,\dots,m);\end{aligned} \tag{1.29a}$$

(2) when $\lambda=0$:

$$\begin{aligned}&u^r\ (r=1,\dots,m),\quad (u^1_du^1_d)^{-2k}S_{jk}(w^1_{ab},w^r_{ab}),\\ &(u^1_du^1_d)^{1-2k}R_k(u^r_a,w^1_{ab})\quad (r=2,\dots,m)\end{aligned} \tag{1.29b}$$

(for the set of invariants $(u^1_du^1_d)^{-2k}S_k(w_{ab})$, k does not take the value n); the tensors θ^r_{ab}, w^r_{ab} are constructed similarly to (1.24) and

$$\theta^r_a=\frac{u^r_a}{u^r}-\frac{u^1_a}{u^1}.$$

Theorem 4 is proved similarly to Theorem 3.

The functional independence of the sets of invariants follows from Lemma 2 and 3 taking into account the fact that transformations $u^r_{ab} \to \theta^r_{ab}$, $u^r_{ab} \to w^r_{ab}$ $(r = 1, \ldots, m)$ and $u^r_a \to \theta^r_a$ $(r = 2, \ldots, m)$ are nondegenerate.

1.4. Differential invariants of the rotation algebra. The rotation algebra is defined by the basis operators J_{ab} (1.1).

The second-order invariants of this algebra for m scalar functions u^r are constructed with x_a, u^r, u^r_a, w^r_{ab} similarly to invariants of the Euclid algebra.

Theorem 5. *There is a functional basis of the second-order differential invariants for the algebra $AO(n)$ that has the form*

$$u^r, \quad S_{jk}(u^1_{ab}, u^r_{ab}), \quad R_k(u^r_a, u^1_{ab}), \quad R_k(x_a, u^1_{ab}), \quad r = 1, \ldots, m;$$

the corresponding basis of invariants for the algebra $\langle J_{ab}, D\rangle$, where D is defined by (1.28), consists of the expressions

$$\frac{u^r}{u^1}\ (r = 2, \ldots, m), \quad \frac{S_{jk}(u^1_{ab}, u^r_{ab})}{(u^1)^{k(1-2/\lambda)}}, \quad R_k(u^r_a, u^1_{ab})(u^1)^{2k/\lambda-1-k},$$
$$R_k(x_a, u^1_{ab})(u^1)^{2/\lambda(k-2)-k+1}, \quad \text{when } \lambda \neq 0;$$
$$u^r, \quad R_k(u^r_a, u^1_{ab})(u^1_{aa})^{-k}, \quad S_{jk}(u^1_{ab}, u^r_{ab})(u^1_{aa})^{-k}\ (k \neq 1 \text{ when } r = 1),$$
$$R_k(x_a, u^1_{ab})(u^1_{aa})^{2-k} \quad \text{when } \lambda = 0.$$

A basis of invariants for the algebra $\langle J_{ab}, D, K_a\rangle$ when $\lambda \neq 0$, consists of the expressions (1.29a) and

$$\frac{R_k(x_a, \theta^1_{ab})}{x^2(u^1)^{(k-1)(1-2/\lambda)}}, \quad k = 2, \ldots, n+1;$$

when $\lambda = 0$ it consists of the expressions (1.29b) and

$$\frac{R_k(x_a, w^1_{ab})}{x^2(w^1_{aa})^{k-1}} \quad (x^2 = x_a x_a).$$

The proof of this theorem is similar to the proofs of Theorems 2 and 3; notice that (x_a) is a co variant tensor with respect to the conformal operators.

2. Differential invariants of the Poincaré and conformal algebra

In this section, we consider differential invariants of the second order for a set of m scalar functions

$$u^r = u^r(x_0, x_1, \ldots, x_n), \quad n \geq 3. \tag{2.1}$$

The Poincaré algebra $AP(1,n)$ is defined by the basis operators

$$p_\mu = ig_{\mu\nu}\frac{\partial}{\partial x_\mu}, \quad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \tag{2.2}$$

where μ, ν take the values $0, 1, \ldots, n$; the summation is implied over the repeated indices (if they are small Greek letters) in the following way:

$$x_\nu x^\nu \equiv x_\nu x^\nu \equiv x^\nu x_\nu = x_0^2 - x_1^2 - \cdots - x_n^2, \quad g_{\mu\nu} = \mathrm{diag}\,(1, -1, \ldots, -1). \tag{2.3}$$

We consider x_ν and x^ν equal with respect to summation not to mix signs of derivatives and numbers of functions.

The quasilinear second-order invariants of the Poincaré algebra were described in [12].

Theorem 6. *There is a functional basis of the second-order differential invariants of the Poincaré algebra $AP(l,n)$ for a set of m scalar functions u^r consisting of*

$$m(2n+3)+(m-1)\frac{n(n+1)}{2}$$

invariants

$$u^r, \quad R_k(u^r_\mu, u^1_{\mu\nu}), \quad S_{jk}(u^r_{\mu\nu}, u^1_{\mu\nu}).$$

In this section, everywhere $k=1,\dots,n+1$; $j=0,\dots,k$; $r=1,\dots,m$.

For the extended Poincaré algebra $A\tilde{P}(l,n)=\langle p_\mu, J_{\mu\nu}, D\rangle$, where

$$D = x_\mu p_\mu + \lambda u^r p_{u^r} \tag{2.4}$$

($p_{u^r}=i(\partial/\partial u^r)$, the summation over r from 1 to m is implied) the corresponding basis has the following form:

(1) when $\lambda=0$:

$$u^r, \quad S_{jk}(u^r_{\mu\nu}, u^1_{\mu\nu})(u^1_{\alpha\alpha})^{-k}, \quad R_k(u^r_\mu, u^1_{\mu\nu})(u^1_{\alpha\alpha})^{-k};$$

(2) when $\lambda\neq 0$:

$$\frac{u^r}{u^1}, \quad S_{jk}(u^r_{\mu\nu}, u^1_{\mu\nu})(u^1)^{k(2/\lambda-1)}, \quad R_k(u^r_\mu, u^1_{\mu\nu})(u^1)^{2k/\lambda-k-1},$$

where S_{jk}, R_k are defined similarly to (0.3) and the summation over small Greek indices is of the type (2.2).

For the conformal algebra $AC(1,n)=\langle p_\mu, J_{\mu\nu}, D, K_\mu\rangle$, where

$$K_\mu = 2x_\mu D - x_\nu x_\nu p_\mu$$

(D being the dilation operator (2.3)), the corresponding basis consists of the expressions

$$S_{jk}(\theta^r_{\mu\nu}, \theta^1_{\mu\nu})(u^1)^{k(2/\lambda-1)}, \quad \frac{u^r}{u^1}, \quad R_k(\theta^r_\mu, \theta^1_{\mu\nu})(u^1)^{k(2/\lambda-1)-1};$$

when $\lambda\neq 0$; $r=2,\dots,m$, there is no summation over r; the conformally covariant tensors have the form

$$\theta^r_\mu = \frac{u^r_\mu}{u^r} - \frac{u^1_\mu}{u^1}, \quad \theta^r_{\mu\nu} = \lambda u^r_{\mu\nu} + (1-\lambda)\frac{u^r_\mu u^r_\nu}{u^r} - g_{\mu\nu}\frac{u^r_\beta u^r_\beta}{2u^r}.$$

When $\lambda=0$, the corresponding basis of invariants for the conformal algebra has the form

$$u^r, \quad S_{jk}(w_{\mu\nu}, w^1_{\mu\nu})(u^1_\alpha u^1_\alpha)^{-2k}, \quad R_k(u^r_\mu, w^1_{\mu\nu})(u^1_\alpha u^1_\alpha)^{1-2k}, \quad r=2,\dots,m;$$

the tensors $(w^r_{\mu\nu})$,

$$w^r_{\mu\nu} = u^r_\alpha u^r_\alpha\left(u^r_{\mu\nu} - \frac{g_{\mu\nu}}{1-n}u^r_{\beta\beta}\right) - u^r_\beta(u^r_\mu u^r_{\beta\nu} + u^r_\nu u^r_{\beta\mu})$$

are conformally invariant (there is no summation over r).

The proof of Theorem 6 follows from those of Theorems 2, 3 for $x = (x_1, \ldots, x_{n+1})$ if we substitute ix_0 instead of x_{n+1}.

Similarly to the results of Paragraph 1.4, it is possible to construct the invariants of the algebras $\langle J_{\mu\nu}\rangle$, $\langle J_{\mu\nu}, D\rangle$, $\langle J_{\mu\nu}, D, K_\mu\rangle$.

The obtained results allow us to construct new nonlinear many-dimensional equations, e.g. the equation

$$\frac{u_\alpha u_\alpha}{1-n} u_{\nu\nu} - u_\mu u_\nu u_{\mu\nu} = (u_\nu u_\nu)^2 F(u),$$

where F is an arbitrary function, is invariant under the algebra $AC(1,n)$, $\lambda = 0$. The left member of the above equation is equal to $w_{\mu\mu}$.

There is another quasi-linear relativistic equation with rich symmetry properties

$$(1 - u_\alpha u_\alpha) u_{\mu\mu} - u_\alpha u_\mu u_{\alpha\mu} = 0,$$

that is, the Born–Infeld equation. The symmetry and solutions of this equation were investigated in [10, 13]. This equation is invariant under the algebra $AP(1, n+1)$ with the basis operators

$$J_{AB} = x_A p_B - x_B p_A,$$

$A, B = 1, \ldots, n+1$, $x_{n+1} \equiv u$.

Let us consider the class of equations

$$u_{\mu\nu} u_{\mu\nu} = F(u_{\mu\mu}, u_\mu u_\nu u_{\mu\nu}, u_\mu u_\mu, u).$$

It is evident that they are invariant with respect to the Poincaré algebra $AP(1,n)$ out the straightforward search the conformally invariant equations from this class with the standard Lie technique requires a lot of cumbersome calculations. The use of differential invariants turns this problem into one of elementary algebra, e.g. if $\lambda \neq 0$

$$F - u_{\mu\nu} u_{\mu\nu} = -\frac{1}{\lambda} S_2(\theta_{\mu\nu}) + u^{2(1-2/\lambda)} \phi(S_1(\theta_{\mu\nu}) u^{2/\lambda - 1}),$$

where $\theta_{\mu\nu}$ is of the form (1.24) and ϕ is an arbitrary function. Whence

$$F = u^{2(1-2/\lambda)} \phi\left(u^{2/\lambda-1}\left(u_{\mu\mu} - \frac{\lambda+n}{\lambda}\frac{u_\alpha u_\alpha}{u}\right)\right) - \\ - \frac{1}{\lambda^2 u^2}(\lambda^2 + n^2)(u_\alpha u_\alpha)^2 - \frac{2(1-\lambda)}{\lambda u} u_\mu u_\nu u_{\mu\nu} + \frac{2 u_{\mu\mu} u_\alpha u_\alpha}{\lambda u}.$$

It is useful to note that besides the traces of matrix powers (0.3), one can utilize all possible invariants of covariant tensors $\theta^r_{\mu\nu}$, $w^r_{\mu\nu}$ to construct conformally invariant equations.

3. Differential invariants of an infinite-dimensional algebra

It is well-known that the simplest first-order relativistic equation — the eikonal or Hamilton equation

$$u_\alpha u_\alpha \equiv u_0^2 - u_1^2 - \cdots - u_n^2 = 0 \tag{3.1}$$

is invariant under the infinite-dimensional algebra $AP^\infty(1,n)$ generated by the operators [10, 14]

$$X = (b^{\mu\nu} x_\nu + a^\mu)\partial_\mu + \eta(u)\partial_u, \tag{3.2}$$

$-b^{\mu\nu} = b^{\nu\mu}$, a^μ, η being arbitrary differentiate functions on u. Equation (3.1) is widely used in geometrical optics.

In this section, we describe a class of second-order equations invariant under the algebra (3.2).

It is easy to show that the tensor of the rank 2

$$\theta_{\mu\nu} = u_\mu u_{\lambda\nu} u_\lambda + u_\nu u_{\lambda\mu} u_\lambda - 2u_\mu u_\nu u_{\lambda\lambda} \tag{3.3}$$

is covariant under the algebra $AP^\infty(1,n)$ (3.2).

Theorem 7. *The equations of the form*

$$S_k(\theta_{\mu\nu}) = 0, \quad k = 1, 2, \dots, \tag{3.4}$$

S_k being defined as (0.3), are invariant with respect to the algebra $AP^\infty(1,n)$ (3.2).

The problem of the description of all such equations is more difficult and we do not consider it here.

Let us investigate in more detail the quasi-linear second-order equation of the form

$$u_\mu u_{\mu\nu} u_\nu - u_\mu u_\mu u_{\alpha\alpha} = 0. \tag{3.5}$$

Theorem 8. *When $n \geq 2$, equation (3.5) is invariant with respect to the algebra $A\tilde{P}^\infty(1,n)$ with generators of the form*

$$X + d(u) x_\mu \partial_\mu,$$

X is of the form (3.2), $d(u)$ is an arbitrary function on u.

The proofs of Theorems 7 and 8 can be easily obtained with the Lie technique using the criterion of invariance

$$\overset{2}{X} S_k(\theta_{\mu\nu})\Big|_{S_k(\theta_{\mu\nu})=0} = 0,$$

where $\overset{2}{X}$ is the second prolongation of the operator X [8–10].

4. Differential invariants of the Galilei algebra

4.1. It is well-known that the heat equation

$$\begin{gathered} 2\mu u_t + \Delta u = 0, \quad \Delta u \equiv u_{aa}, \\ u = u(t, \boldsymbol{x}), \quad \boldsymbol{x} = (x_1, \dots, x_n), \quad n \geq 3 \end{gathered} \tag{4.1}$$

is invariant under the generalized Galilei algebra $AG_2^I(1,n)$ with the basis operators

$$\begin{gathered} \partial_t = \frac{\partial}{\partial t}, \quad \partial_a = \frac{\partial}{\partial x_a}, \quad J_{ab} = x_a\partial_b - x_b\partial_a, \\ G_a = t\partial_a + \mu x_a u \partial_u \quad \left(\partial_u = \frac{\partial}{\partial u}\right), \quad u\partial_u, \quad D = 2t\partial_t + x_a\partial_a + \lambda u \partial_u, \\ A = tD - t^2\partial_t + \frac{\mu \boldsymbol{x}^2}{2} u\partial_u \quad \left(\lambda = -\frac{n}{2}\right). \end{gathered} \tag{4.2}$$

The Schrödinger equation

$$2im\psi_t + \psi_{aa} = 0, \tag{4.3}$$

$\psi = \psi(t, \boldsymbol{x})$ being a complex-valued function, is also invariant [16] under the generalized Galilei algebra with the basis operators

$$\begin{gathered} p_0 = i\frac{\partial}{\partial t}, \quad p_a = -i\frac{\partial}{\partial x_a}, \quad J_{ab} = x_a p_b - x_b p_a, \quad J = i(\psi\partial_\psi - \psi^*\partial_{\psi^*}), \\ G_a = tp_a - mx_a J, \quad D = 2tp_0 - x_a p_a + \lambda I \quad (I = \psi\partial_\psi + \psi^*\partial_{\psi^*}), \\ A = t^2 p_0 - tx_a p_a + \lambda t I + \frac{m\boldsymbol{x}^2}{2}J \quad \left(\lambda = -\frac{n}{2}\right). \end{gathered} \tag{4.4}$$

The asterisk means the complex conjugation.

We shall designate the algebra (4.4) with the symbol $AG_2^{II}(1,n)$. Besides,

$$AG^I(1,n) = \langle \partial_t, \partial_a, u\partial_u, G_a, J_{ab} \rangle,$$

the operators being of the form (4.2). A basis of the algebra $AG_1^I(1,n)$ consists of the basis operators or $AG^I(1,n)$ and of the operator D. Furthermore $AG^{II}(1,n) = \langle p_0, p_a, J, J_{ab}, G_a \rangle$ (4.4). A basis of the algebra $AG_1^{II}(1,n)$ consists of the previous operators and also D (4.4).

To simplify the form of invariants, we introduce the following change of dependent variables:

$$u = \exp\varphi, \quad \psi = \exp\phi \quad \left(\operatorname{Im}\phi = \operatorname{arctg}\frac{\operatorname{Im}\psi}{\operatorname{Re}\psi}\right). \tag{4.5}$$

All the indices k in the expressions of the type (0.3) here will take on values from 1 to n, the indices j will take on values from 0 to k.

We seek invariants of the algebra $AG_2^I(1,n)$ in the form

$$F = F(\varphi_t, \varphi_a, \varphi_{tt}, \varphi_{at}, \varphi_{ab}). \tag{4.6}$$

Obviously, they do not include φ, x_a, and t because the basis (4.2) contains operators ∂_φ, ∂_a, ∂_t.

Using the definition of an absolute differential invariant (0.2) we get the following conditions on the function F (4.6):

$$\overset{2}{J}_{ab}F = \varphi_a F_{\varphi_b} - \varphi_b F_{\varphi_a} + F_{\varphi_{bt}}\varphi_{at} - \varphi_{bt}F_{\varphi_{at}} + 2\varphi_{ac}F_{\varphi_{bc}} - 2\varphi_{bc}F_{\varphi_{ac}} = 0, \tag{4.7}$$

$$\overset{2}{G}_a F = -\varphi_a F_{\varphi_t} + \mu F_{\varphi_a} - 2\varphi_{at}F_{\varphi_{tt}} - \varphi_{ab}F_{\varphi_{bt}} = 0, \tag{4.8}$$

$$\overset{2}{D} F = -2\varphi_t F_{\varphi_t} - \varphi_a F_{\varphi_a} - 4\varphi_{tt}F_{\varphi_{tt}} - 3\varphi_{at}F_{\varphi_{at}} - 2\varphi_{ab}F_{\varphi_{ab}} = 0, \tag{4.9}$$

$$\overset{2}{A} F = t\overset{2}{D} F + x_a \overset{2}{G}_a F - \lambda F_{\varphi_t} - 2\varphi_t F_{\varphi_{tt}} - \varphi_a F_{\varphi_{at}} + \mu\delta_{ab}F_{\varphi_{ab}} = 0. \tag{4.10}$$

From equations (4.8), we can see that the tensors

$$\theta_a = \mu\varphi_{at} + \varphi_b\varphi_{ab}, \quad \varphi_{ab} \tag{4.11}$$

are covariant with respect to the algebra $AG^I(1,n)$ $(\mu \neq 0)$.

Theorem 9. *There is a functional basis of absolute differential invariants for the algebra $AG^I(1,n)$, when $\mu \neq 0$, consisting of these $2n+2$ invariants:*

$$\begin{gathered} M_1 = 2\mu\varphi_t + \varphi_a\varphi_a, \quad M_2 = \mu^2\varphi_{tt} + 2\mu\varphi_a\varphi_{at} + \varphi_a\varphi_b\varphi_{ab}, \\ R_k = R_k(\theta_a, \theta_{ab}), \quad S_k = S_k(\varphi_{ab}). \end{gathered} \tag{4.12}$$

For the algebra $AG_1^I(1,n)$ $(\mu\neq 0)$ *such a basis has the form*

$$\frac{M_2}{M_1^2},\quad \frac{R_k}{M_1^{2+k}},\quad \frac{S_k}{M_1^k}. \tag{4.13}$$

For the algebra $AG_2^I(1,n)$ $(\mu\neq 0)$*, there is a basis of the form*

$$\frac{N_2}{N_1^2},\quad \frac{\hat R_k}{N_1^{2+k}},\quad \frac{\hat S_k}{N_1^k}\quad (k=2,\dots,n), \tag{4.14}$$

where

$$\begin{aligned}
&N_1=2\mu\varphi_t+\varphi_a\varphi_a+\varphi_{aa},\\
&N_2=\mu^2\varphi_{tt}+2\mu\left(\frac{1}{n}\varphi_t\varphi_{aa}+\varphi_a\varphi_{at}\right)+\varphi_a\varphi_b\varphi_{ab}+\frac{1}{n}\varphi_a\varphi_a\varphi_{bb}+\frac{1}{n}\varphi_{bb}^2,\\
&\hat R_k=\sum_{l=0}^{k}R_l(\varphi_{aa})^{k-1}\frac{(-n)^l k!}{l!(k-l)!},\\
&\hat S_k=\sum_{l=0}^{k}\frac{(-n)^l(k-1)!(k+1)}{(l+1)!(k-l)!}S_l(\varphi_{aa})^{k-l},
\end{aligned} \tag{4.15}$$

S_k, R_k *are defined by (4.12) and* θ_a *has the form (4.11).*

The proof of this theorem is similar to the proof of Theorems 2 and 3. We shall present here only some hints to the proof.

It is evident that the function F must depend on the invariants of the Euclid algebra

$$F=F(\varphi_t,\varphi_{tt},R_k(\varphi_a,\varphi_{ab}),R_k(\varphi_{at},\varphi_{ab}),S_{\varphi_{ab}})).$$

First we construct two invariants of $AG^I(1,n)$ M_1 and M_2 (4.12) which depend on φ_t and φ_{tt} respectively. The other invariants of the adduced basis (4.12) do not depend on φ_t or φ_{tt} and the sets $\{M_1,M_2\}$ and $\{R_k,S_k\}$ are independent. The invariants R_k, S_k are constructed with the covariant tensors θ_a, φ_{ab} (4.11) similarly to invariants of the conformal algebra investigated above, and it is easy to see that they are independent.

The generic ranks of the prolonged algebras $AG^I(1,n)$, $AG_1^I(1,n)$, $AG_2^I(1,n)$ are equal to the numbers of their operators and from this fact we can compute the number of elements in the bases for these algebras.

Adding to (4.7) and (4.8) the condition (4.9), we obtain from the invariants (4.12) the basis (4.13) for the algebra $AG_1^I(1,n)$.

Relative invariants $\hat R_k$, $\hat S_k$ (4.15) of the algebra $AG_2^I(1,n)$ were found from the equation

$$\lambda F_{\varphi_t}-2\varphi_t F_{\varphi_{tt}}-\varphi_a F_{\varphi_{at}}+\mu\delta_{ab}F_{\varphi_{ab}}=0,$$

$F=F(R_k,S_k)$, and then we constructed absolute invariants using (4.9). Besides, it is possible to construct analogues to $\hat R_k$, $\hat S_k$ with $AG_2^I(1,n)$-covariant tensors θ_a (4.11) and

$$\theta_{ab}=\varphi_{ab}-\frac{2\delta_{ab}}{n}(\varphi_c\varphi_c+\mu\varphi_t).$$

Considering (φ_{at}), (φ_a), (φ_{ab}) as independent vectors and tensors and putting $\varphi_{ab} = 0$ whenever $a \neq b$, $\varphi_a = 0$, we see from Lemma 2 that the adduced sets of invariants are independent.

Note 2. A basis of invariants for the Galilei algebra without translations contains expressions (4.12) and

$$R_k(h_a, \phi_{ab}), \quad \frac{1}{2}\mu \boldsymbol{x}^2 - \varphi t,$$

the Galilei-covariant vector h_a having the form

$$h_a = \mu x_a - t\varphi_a.$$

Let us also adduce an A-covariant tensor

$$\hat{h}_a = \frac{\mu x_a}{t} - \varphi_a$$

depending on x_a, and a relative invariant of the operators A and D (4.2)

$$\exp\left\{\varphi - \frac{\mu \boldsymbol{x}^2}{2t}\right\}$$

with which it is possible to construct a basis of invariants for the algebra $\langle G_a, J_{ab}, D, A\rangle$.

We have presented a method to find the bases of invariants for Lie algebras for which J_{ab} (1.1) are basis operators. Further, we shall adduce functional bases for the algebras $AG_2^I(1,n)$ where $\mu = 0$ and $AG_2^{II}(1,n)$ where $\mu = 0$ or $\mu \neq 0$. We omit proofs because they are similar to proofs of the previous theorems.

It is evident from the conditions (4.7)–(4.10) that the case $\mu = 0$ for the algebra $AG_2^I(1,n)$ has to be specially considered. The tensors (φ_a) and (φ_{ab}) are covariant with respect to this algebra; the tensor (θ_a) involved in invariants is defined by an implicit correlation

$$\varphi_{bt} = \theta_a \varphi_{ab}. \tag{4.16}$$

Theorem 10. *There is a functional basis of the second-order differential invariants for the algebra $AG^I(1,n)$, where $\mu = 0$, that has the form*

$$\begin{aligned} &M_1 = \varphi_t - \varphi_a\theta_a, \quad M_2 = \varphi_{tt} - \varphi_{at}\theta_a, \\ &R_k = R_k(\varphi_a, \varphi_{ab}), \quad S_k = S_k(\varphi_{ab}). \end{aligned} \tag{4.17}$$

The corresponding basis for the algebra $AG_1^I(1,n)$, where $\mu = 0$ has the form

$$\frac{M_1^2}{M_2}, \quad \frac{R_k}{M_1^k}, \quad \frac{S_k}{M_1^k};$$

for the algebra $AG_2^I(1,n)$, when $\mu = 0$, it has the form

$$\frac{R_k}{M^{1/2k}}, \quad \frac{S_k}{M^{1/2k}},$$

where R_k, S_k are defined by (4.17) and

$$M = (\varphi_t - \theta_a\varphi_a)^2 + (\varphi_{tt} - \varphi_{at}\theta_a)(\lambda + \varphi_a\varphi_b r_{ab}).$$

Here, the matrix $\{r_{ab}\} = \{\varphi_{ab}\}^{-1}$; $\theta_a = r_{ab}\varphi_{bt}$ are the same as in (4.16).

Note 3. It is possible to use, instead of M_1, M_2, the invariants

$$\hat{M}_1 = \begin{vmatrix} \varphi_t & \varphi_1 & \cdots & \varphi_n \\ \varphi_{1t} & \varphi_{11} & \cdots & \varphi_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_{nt} & \varphi_{n1} & \cdots & \varphi_{nn} \end{vmatrix}, \qquad \hat{M}_2 = \begin{vmatrix} \varphi_{tt} & \varphi_{1t} & \cdots & \varphi_{nt} \\ \varphi_{1t} & \varphi_{11} & \cdots & \varphi_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_{nt} & \varphi_{n1} & \cdots & \varphi_{nn} \end{vmatrix},$$

which have been found in [17] as the solution of the problem of finding the equations invariant under the Galilei algebra when $\mu = 0$.

Note 4. The invariants for the algebra $\langle J_{ab}, G_a, J, D, A\rangle$ (4.2), where $\mu = 0$, which depend on x_a, t, can be constructed with φ_a, φ_{ab} and the following covariant vector

$$\hat{h}_a = \frac{h_a}{t} + \frac{2}{n} t\varphi_a\varphi_t + \frac{4}{n}\frac{x_b\varphi_b\varphi_a}{t},$$

where $h_a = x_b\varphi_{ab} + t\varphi_{at}$ is covariant with respect to the operators G_a when $\mu = 0$.

4.2. Let us proceed to describe the basis of the invariants for the algebra $AG_2^{II}(1,n)$.

Theorem 11. *Any absolute differential invariant of order ≤ 2 for the algebras listed below is a function of the following expressions:*

(1) $AG^{II}(1,n)$, $m \ne 0$:

$$\begin{aligned} &\phi + \phi^*, \quad M_1 = 2im\phi_t + \phi_a\phi_a, \quad M_1^*, \\ &M_2 = -m^2\phi_{tt} + 2im\phi_a\phi_{at} + \phi_a\phi_b\phi_{ab}, \quad M_2^*, \\ &S_{jk} = S_{jk}(\phi_{ab}, \phi^*_{ab}), \quad R_k^1 = R_k(\theta_a, \phi_{ab}), \\ &R_k^2 = R_k(\theta_a^*, \phi_{ab}), \quad R_k^3 = R_k(\phi_a + \phi_a^*, \phi_{ab}), \end{aligned}$$

the covariant tensors being $\theta_a = -im\phi_{at} + \phi_b\phi_{ab}$;

(2) $AG_1^{II}(1,n)$, $m \ne 0$:

$$\begin{aligned} &\frac{M_1^*}{M_1}, \quad \frac{M_2}{M_1^2}, \quad \frac{M_2^*}{M_1^2}, \quad \frac{R_k^l}{M_1^{2+k}} \ (l = 1,2), \quad \frac{R_k^3}{M_1^k}, \quad \frac{S_{jk}}{M_1^k}, \\ &\phi + \phi^* \quad \text{when } \lambda = 0, \quad M_1 e^{(2/\lambda)(\phi+\phi^*)} \quad \text{when } \lambda \ne 0; \end{aligned}$$

(3) $AG_2^{II}(1,n)$, $m \ne 0$, $\lambda = -\frac{n}{2}$:

$$N_1 e^{(-4/n)(\phi+\phi^*)}, \quad \frac{N_1}{N_1^*}, \quad \frac{N_2}{N_1^2}, \quad \frac{N_2^*}{N_1^2}, \quad \frac{\hat{R}_k^l}{N_1^{2+k}} \ (l = 1,2), \quad \frac{\hat{R}_k^3}{N_1^k}, \quad \frac{\hat{S}_{jk}}{N_1^k},$$

where

$$\begin{aligned} &N_1 = 2im\phi_t + \phi_{aa} + \phi_a\phi_a, \\ &N_2 = -m^2\phi_{tt} + 2im\left(\phi_a\phi_{at} + \frac{1}{n}\phi_t\phi_{aa}\right) + \phi_a\phi_b\phi_{ab} + \frac{1}{n}\phi_a\phi_a\phi_{bb} + \frac{1}{n}\phi_{aa}^2, \\ &\hat{S}_{jk} = \sum_{l=0}^{k}\sum_{r=0}^{j} S_{rl}(-n)^l C_j^r C_k^{l+1-r}(\phi_{aa})^{j-r}(\phi^*_{aa})^{k-l-j+r} + k(\phi_{aa})^j(\phi^*_{aa})^{k-j-1}, \\ &\hat{R}_k^l = \sum_{j=0}^{k} R_j^l(\phi_{aa})^{k-j}\frac{(-n)^j k!}{j!(k-j)!} \quad (l = 1,2,3). \end{aligned}$$

The invariants for the algebras $AG^{II}(1,n)$, $AG_1^{II}(1,n)$ $(m=0)$ can be constructed similarly to the case of real function. Let us adduce a functional basis for the algebra $AG_2^{II}(1,n)$.

(1) when $\lambda=0$, then there is a basis consisting of the following expressions:

$$\phi+\phi^*,\quad \frac{N_1^2}{N_2^2},\quad \frac{N_1^{*2}}{N_2},\quad \frac{(S_{jk})^2}{N_1^k},\quad (R_k^l)^2N_1^{-k-1}\ (l=1,2,4);$$

(2) $\lambda\neq 0$:

$$N_1e^{(4/\lambda)(\phi+\phi^*)},\quad \frac{N_1^*}{N_1},\quad N_3e^{(3/\lambda)(\phi+\phi^*)},\quad \frac{(R_k^l)^2}{N_1^k}\ (l=1,2,3),\quad \frac{(S_{jk})^2}{N_1^k},$$

where

$$\begin{aligned}
&N_1=(\phi_t-\theta_a\phi_a)^2+(\phi_{tt}-\theta_a\phi_{at})(\lambda+\phi_a\phi_{ab}r_{ab})\\
&\qquad (with\ \{r_{ab}\}=\{\phi_{ab}\}^{-1}\ and\ \theta_a=r_{ab}\phi_{bt}),\\
&N_2=(\phi_t-\phi_c\theta_c)\phi_a^*\phi_b^*r_{ab}^*-(\phi_t^*-\phi_c^*\theta_c^*)\phi_a\phi_b r_{ab},\\
&N_3=(\phi_t-\phi_t^*)-\tau_a(\phi_a-\phi_a^*)\ \ (\tau_a(\lambda\phi_{ab}+\phi_a\phi_b)=\phi_b\phi_t+\lambda\phi_{bt}),\\
&R_k^1=R_k(\phi_a,\phi_{ab}),\quad R_k^2=R_k(\phi_a^*,\phi_{ab}),\quad R_k^3=R_k(\theta_a-\theta_a^*,\phi_{ab}),\\
&R_k^4=R_k(\rho_a,\phi_{ab})\ \ (\rho_a=(\phi_t-\theta_b\phi_b)(\phi_c^*r_{ac}-\phi_c r_{ac}^*)-\phi_b\phi_d r_{bd}(\theta_a-\theta_a^*)).
\end{aligned}$$

The proof of this theorem will be easier if we notice that by putting $\mu=im$ in (4.4), we obtain operators similar to the operators (4.2).

The change of variables (4.5) in the adduced invariants allows us to obtain bases for the algebras AG_2^I and AG_2^{II} in the representations (4.2) and (4.4). These results can also be generalized for the case of several scalar functions.

4.3. Let us present some examples of new invariant equations

$$\begin{aligned}
&\phi_{tt}+\frac{1}{\mu^2}\left\{2\mu\left(\frac{1}{n}\phi_t\phi_{aa}+\phi_a\phi_t\right)+\phi_a\phi_b\phi_{ab}+\frac{1}{n}\phi_a\phi_a\phi_{bb}+\frac{1}{n}\phi_{bb}^2\right\}=\\
&\qquad=(2\mu\phi_t+\phi_a\phi_a+\phi_{aa})^2F,
\end{aligned}\tag{4.18}$$

$$\begin{aligned}
&-m^2\phi_{tt}+2im\left(\phi_a\phi_{at}+\frac{1}{n}\phi_t\phi_{aa}\right)+\phi_a\phi_b\phi_{ab}+\frac{1}{n}\phi_a\phi_a\phi_{bb}+\frac{1}{n}\phi_{aa}^2=\\
&\qquad=(2im\phi_t+\phi_a\phi_a+\phi_{aa})^2F.
\end{aligned}\tag{4.19}$$

Equations (4.18) and (4.19) are invariant, respectively, under the algebras $AG_2^I(1,n)$, $\mu\neq 0$ (4.2), and $AG_2^{II}(1,n)$, $m\neq 0$ (4.4). The F's are arbitrary functions of the invariants for corresponding algebras.

Evidently, wide classes of invariant equations can be constructed with the adduced invariants.

5. Conclusion

It is well-known that a mathematical model of physical or some other phenomena must obey one of the relativity principles of Galilei or Poincaré. Speaking the language of mathematics, it means that the equations of the model must be invariant under the Galilei or the Poincaré groups. Having bases of differential invariants for these groups (or for the corresponding algebras), we can describe all the invariant scalar equations, or sort the invariant ones out of a set of equations.

The construction of differential invariants for vector and spinor fields presents more complicated problems. The first-order invariants for a four-dimensional vector potential had been found in [18]. The cases of spinor and many-dimensional vector Poincaré-invariant equations and corresponding bases of invariants are still to be investigated.

Note 5. After having prepared the present paper, we became acquainted with the article [19] where realizations of the Poincaré group $P(1,1)$ and the corresponding conformal group were investigated, and all second-order scalar differential equations invariant under these groups were obtained. Reference [19] contains bases of absolute differential invariants of the order 2 for the Poincaré, the similitude, and the conformal groups in $(1+1)$-dimensional Minkowski space for various realizations of the corresponding Lie algebras.

Note 6. It was noticed by the referee that an essential misunderstanding arose in the calculation of second prolongations for differential operators, e.g. in formulae (1.5) and (1.25).

When we calculate such prolongations with the usual Lie technique (see, e.g., [8]), we imply that action of an operator of the form $X^{ab}\partial_{u_{ab}}$, where X^{ab} are some functions, is as follows

$$X^{ab}\partial_{u_{ab}}(u_{cd}u_{cd}) = 2X^{ab}u_{ab}, \quad \partial_{u_{ab}}u_{cd} = \delta_{ac}\delta_{bd}.$$

With this assumption, $\partial_{u_{ab}}u_{ba} = 0$, $a \neq b$.

Otherwise, the second prolongation of the operator J_{ab} (1.1) will be of the form

$$\overset{2}{J}_{ab} = J_{ab} + \hat{J}_{ab},$$
$$\hat{J}_{ab} = u_a\partial_{u_b} - u_b\partial_{u_a} + u_{ac}\partial_{u_{bc}} - u_{bc}\partial_{u_{ac}} + u_{ab}(\partial_{u_{bb}} - \partial_{u_{aa}}).$$

Note 7. The equations which are conditionally invariant with respect to the Poincaré and Galilei algebras were investigated in [20, 21].

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