Q-conditional symmetry of the linear heat equation

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Исследована Q-условная симметрия одномерного линейного уравнения теплопроводности. Получены определяющие уравнения для коэффициентов оператора Q-условной симметрии, изучена их лиевская симметрия, получены некоторые их точные решения. Найдены нелокальные замены, сводящие определяющие уравнения к исходному уравнению теплопроводности. Показано, как можно использовать операторы Q-условной симметрии для линеаризации нелинейных ДУЧП и размножения решений уравнения теплопроводности.

In this article we consider in full detail, as a simple but non-trivial example, how to find and use Q-conditional symmetry of the one-dimensional linear heat equation

$$u_0 = u_{11} \tag{1}$$

$(u = u(x_0, x_1)$, $u_0 = \partial u/\partial x_0$, $u_1 = \partial u/\partial x_1$ and so on.

It is known [1] that the maximal in Lie sense invariance algebra of equation (1) is an algebra with the basis elements

$$\begin{gathered}\partial_0 = \frac{\partial}{\partial x_0}, \quad \partial_1 = \frac{\partial}{\partial x_1}, \quad G = x_0\partial_1 - \frac{1}{2}x_1 u\partial_u, \quad I = u\partial_u, \\ D = 2x_0\partial_0 + x_1\partial_1, \quad \Pi = x_0\left(x_0\partial_0 + x_1\partial_1 - \frac{1}{2}u\partial_u\right) - \frac{x_1^2}{4}u\partial_u.\end{gathered} \tag{2}$$

The problem of finding non-classical symmetry (in our terminology Q-conditional symmetry) was firstly put forward by Bluman and Cole [5]. However, in this important paper the authors did not give explicitly none of operators which would different from those of (2). Below we will present quite complete investigation of this problem. All notions used without explanations are defined in [1–4].

Definition 1 [2, 4]. *A differential equation of order m*

$$S_1(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}) = 0 \tag{3}$$

for a function $u = u(x)$ where $\underset{k}{u}$ denotes all partial derivatives of order k is called conditionally invariant under an operator Q if there is an additional condition of the form

$$S_2(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}) = 0 \tag{4}$$

compatible with (3), that

$$\tilde{Q}S_\alpha\Big|_{\substack{S_1 = 0 \\ S_2 = 0}} = 0, \quad \alpha = 1, 2, \tag{5}$$

In the formula (5) $\tilde{Q}$ is the standard prolongation of Q.

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In that particular case when equation (4) has the form

$$Qu = 0 \tag{6}$$

equation (3) is called Q-conditionally invariant under the operator Q. The notion of Q-conditional invariance coincides with the notion of "non-classical" invariance introduced by Bluman and Cole in the work [5].

The general form of a first-order operator is

$$Q = A(x_0, x_1, u)\partial_0 + B(x_0, x_1, u)\partial_1 + C(x_0, x_1, u)\partial_u, \tag{7}$$

where A, B, C are some differentiable functions of x_0, x_1, u to be determined from the invariance condition (5). It will be noted that because of the imposed condition (6)

$$Qu = 0 \;\Leftrightarrow\; Au_0 + Bu_1 = C \tag{8}$$

there are really only two independent cases of operator (7).

Theorem 1. *The heat equation (1) is Q-conditionally invariant under operator (7) if and only if its coordinates are as follows:*

Case 1.

$$A = 1, \quad B = W^1(x_0, x_1), \quad C = W^2(x_0, x_1)u + W^3(x_0, x_1) \tag{9}$$

and functions $\vec{W} = \vec{W}(x_0, x_1) = \{W^1, W^2, W^3\}$ satisfy equations

$$(\partial_0 + 2W^1_1 - \partial_{11})\vec{W} = \vec{F}, \quad \vec{F} = \{-2W^2_1, 0, 0\}. \tag{10}$$

Case 2.

$$A = 0, \quad B = 1, \quad C = v(x_0, x_1, u) \tag{11}$$

and function $v = v(x_0, x_1, u)$ satisfies the PDE

$$v_0 = v_{11} + 2vv_{1u} + v^2 v_{uu}. \tag{12}$$

Proof. From the criterion of invariance

$$\underset{2}{Q}(u_0 - u_{11})\Big|_{\substack{u_0 = u_{11},\\ Qu = 0}} = 0, \tag{13}$$

absolutely analogously to the standard Lie's algorithm one finds the defining equations for the coordinates of operator (7) which can be reduced to (9)–(12). It is to be pointed out that unlike Lie's algorithm, in the cases considered above the defining equations (10), (12) are nonlinear ones and it is a typical feature of Q-conditional invariance.

It goes without saying that Q-conditional invariance includes Lie's invariance in particular. So, in our case of the heat equation, we obtain infinitesimals (2) as simplest solutions of (10), (12):

$$\begin{aligned}
&A = 1, \quad \vec{W} = 0 \quad \Rightarrow \quad Q = \partial_0,\\
&A = v = 0, \quad B = 1 \quad \Rightarrow \quad Q = \partial_1,\\
&A = 0, \quad B = 1, \quad v = -\frac{x_1 u}{2x_0} \quad \Rightarrow \quad Q = G,\\
&A = 1, \quad W^1 = \frac{x_1}{2x_0}, \quad W^2 = W^3 = 0 \quad \Rightarrow \quad Q = D,\\
&A = 1, \quad W^1 = \frac{x_1}{x_0}, \quad W^2 = -(2x_0 + x_1^2)/4x_0^2, \quad W^3 = 0 \quad \Rightarrow \quad Q = \Pi.
\end{aligned} \tag{14}$$

Remark 1. System of defining equations (10) was firstly obtained by Bluman and Cole [5]. Further investigation of system (10) was continued in [6], where the question of linearization of the first two equations of (10) had been studied. The general solution of the problem of linearization of equations (10), (12) will be given after a while.

Now let us list some concrete operators (7) of Q-conditional invariance of equation (1) obtained as partial solutions of the defining equations (10), (12). In the following Table we also give corresponding invariant ansätze and the reduced equations.

Of course, operators 1–10 from Table do not exhaust the all possible operators of Q-conditional invariance.

N	Operator Q	Ansatz	Reduced equation
1	$-x_1\partial_0+\partial_1$	$u=\varphi\left(x_0+\frac{x_1^2}{2}\right)$	$\varphi''=0$
2	$-x_1\partial_0+\partial_1+x_1^3\partial_u$	$u=\varphi\left(x_0+\frac{x_1^2}{2}\right)+\frac{x_1^4}{4}$	$\varphi''=-3$
3	$x_1^2\partial_0-3x_1\partial_1-3u\partial_u$	$u=x_1\varphi\left(x_0+\frac{x_1^2}{6}\right)$	$\varphi''=0$
4	$x_1^2\partial_0-3x_1\partial_1-(3u+x_1^5)\partial_u$	$u=x_1\varphi\left(x_0+\frac{x_1^2}{6}\right)+\frac{x_1^5}{12}$	$\varphi''=-15$
5	$x_1\partial_1+u\partial_u$	$u=x_1\varphi(x_0)$	$\varphi'=0$
6	$\operatorname{cth}x_1\partial_1+u\partial_u$	$u=\varphi(x_0)\operatorname{ch}x_1$	$\varphi'-\varphi=0$
7	$-\operatorname{ctg}x_1\partial_1+u\partial_u$	$u=\varphi(x_0)\cos x_1$	$\varphi'+\varphi=0$
8	$\partial_1-u\partial_u-\frac{u}{2x_0-x_1}\partial_u$	$u=(2x_0-x_1)e^{-x_1}\varphi(x_0)$	$\varphi'-\varphi=0$
9	$\partial_1-\sqrt{-2(x_0+u)}\partial_u$	$u=-x_0-\frac{1}{2}[x_1+\varphi(x_0)]^2$	$\varphi'=0$
10	$\left(x_0+\frac{x_1^2}{2}\right)\partial_0-x_1\partial_1$	$u=\varphi\left(x_0x_1+\frac{x_1^3}{3!}\right)$	$\varphi''=0$

Next we study Lie symmetry of the defining equations (10), (12).

Theorem 2. *The Lie maximal invariance algebra of system (10) is given by the operators*

$$
\begin{aligned}
&\partial_0,\quad \partial_1,\quad G^{(1)}=x_0\partial_1+\partial_{W^1}-\frac{1}{2}W^1\partial_{W^2}-\frac{1}{2}x_1W^3\partial_{W^3},\\
&D^{(1)}=2x_0\partial_0+x_1\partial_1-W^1\partial_{W^1}-2W^2\partial_{W^2},\quad I^{(1)}=W^3\partial_{W^3},\\
&\Pi^{(1)}=x_0\left(x_0\partial_0+x_1\partial_1-W^1\partial_{W^1}-2W^2\partial_{W^2}-\frac{5}{2}W^3\partial_{W^3}\right)+\\
&\qquad+x_1\left(\partial_{W^1}-\frac{1}{2}W^1\partial_{W^2}\right)-\frac{1}{2}\partial_{W^2}-\frac{x_1^2}{4}W^3\partial_{W^3},\\
&X=(f_0+f_1W^1-fW^2)\partial_{W^3}.
\end{aligned}
\tag{15}
$$

where $f=f(x_0,x_1)$ is an arbitrary solution of (1), that is $f_0=f_{11}$.

Theorem 3. *The Lie maximal invariance algebra of equation (12) is given by the operators*

$$
\begin{aligned}
&\partial_0,\quad \partial_1,\quad D^{(2)}=2x_0\partial_0+x_1\partial_1+u\partial_u,\quad D^{(3)}=u\partial_u+v\partial_v,\\
&G^{(2)}=x_0\partial_1-\frac{1}{2}x_1(u\partial_u+v\partial_v)-\frac{1}{2}u\partial_v,\\
&\Pi^{(2)}=x_0\left(x_0\partial_0+x_1\partial_1-\frac{1}{2}u\partial_u-\frac{3}{2}v\partial_v\right)-\frac{x_1^2}{4}(u\partial_u+v\partial_v)-\frac{x_1}{2}u\partial_v,\\
&R=f\partial_u+f_1\partial_v\quad (f_0=f_{11}).
\end{aligned}
\tag{16}
$$

One can get the proofs of these two theorems by means of the standard Lie's algorithm.

Operators (15), (16) can be used to find exact solutions of equations (10), (12). In particular, using the formula of generating solutions at the expense of invariance under $\Pi^{(2)}$

$$
\begin{aligned}
&v^{\mathrm{II}}(x_0,x_1,u)=(1-\theta x_0)^{-3/2}\exp\left\{\frac{\theta x_1^2}{4(1-\theta x_0)}\right\}v^1(x_0',x_1',u')+\frac{\theta}{1-\theta x_0}\frac{x_1u}{2},\\
&x_0'=\frac{x_0}{1-\theta x_0},\quad x_1'=\frac{x_1}{1-\theta x_0},\\
&u'=(1-\theta x_0)^{1/2}\exp\left\{-\frac{1}{4}\frac{\theta x_1^2}{1-\theta x_0}\right\}u\quad(\theta=\mathrm{const})
\end{aligned}
\tag{17}
$$

one can construct new solutions of equations (12) starting from known ones.

Solutions of equations (10), (12) can be obtained by the use of reduction on subalgebras of the invariance algebras (15), (16). For example, using the subalgebra $\langle\partial_0+a_i^{(1)}\rangle$ of the algebra (15) we find the following solution of the system (10)

$$
\begin{aligned}
&W^1=\frac{C_1^2-C_3^2}{-C_1\,\mathrm{tg}(C^1x_1+C^2)+C_3\,\mathrm{tg}(C_3x_1+C_4)},\\
&W^2=-C_1C_3\frac{C_1\,\mathrm{tg}(C_3x_1+C_4)-C_3\,\mathrm{tg}(C_1x_1+C_2)}{-C_1\,\mathrm{tg}(C^1x_1+C^2)+C_3\,\mathrm{tg}(C_3x_1+C_4)},\\
&W^3=(\varphi_{11}-W^1\varphi_1-W^2\varphi)e^{ax_0},
\end{aligned}
\tag{18}
$$

where $C_1,\ldots,C_4$ are arbitrary constants, $\varphi=\varphi(x_1)$, $\varphi_{11}=a\varphi$.

Theorem 4. *The system (10) is reduced to the system of disconnected heat equations*

$$
\vec z_0=\vec z_{11}\quad(\vec z=\vec z(x_0,x_1)=\{z^1,z^2,z^3\})
\tag{19}
$$

with the help of the nonlocal transformation

$$
\begin{aligned}
&W^1=-\frac{z_{11}^1z^2-z^1z_{11}^2}{z_1^1z^2-z^1z_1^2},\quad W^2=-\frac{z_{11}^1z_1^2-z_1^1z_{11}^2}{z_1^1z^2-z^1z_1^2},\\
&W^3=z_{11}^3+W^1z_1^3-W^2z^3.
\end{aligned}
\tag{20}
$$

Expressions (20) result in (after using the corresponding operator (7), (9)) the ansatz

$$
u=z^1\varphi(\omega)+z^3,\quad\omega=\frac{z^2}{z^1}
\tag{21}
$$

(z^1, z^2, z^3 are solutions of (19)), and the reduced equation is $\varphi''=0$. This means that

$$
u=C_1z^1+C_2z^2+C_3z^3.
\tag{22}
$$

So, we get just the well-known superposition principle for the heat equation.

Letting $W^2=W^3=0$ we get from (10) the Burger's equation

$$
W_0^1+2W^1W_1^1=W_{11}^1.
\tag{23}
$$

Using Hopf–Cole transformation one obtains solutions of equation (23) in the form

$$W^1 = -\partial_1 \ln f = -\frac{f_1}{f} \quad (f_0 = f_{11}). \tag{24}$$

This result in the operator

$$Q = f\partial_0 - f_1\partial_1. \tag{25}$$

Q-conditional symmetry of equation (1) under the operator (25) lead to the following statement.

Theorem 5. *If function f is an arbitrary solution of the heat equation (1) and u is the general integral of the ODE*

$$f_1 dx_0 + f dx_1 = 0, \tag{26}$$

then u satisfies equation (1).

Proof. We note that equation (26) is a perfect differential equation and therefore its general solution $u(x_0, x_1) = C$ possesses the following property

$$u_0 = f_1, \quad u_1 = f. \tag{27}$$

Having used (27) we obtain

$$u_0 - u_{11} = f_1 - f_1 = 0$$

and the theorem is proved.

Theorem 5 may be considered as another algorithm of generating solutions of equation (1). Indeed, even starting from a rather trivial solution of the heat equation $u = 1$ we get the chain of quite interesting solutions

$$1 \to x_1 \ \to \ x_0 + \frac{x_1^2}{2!} \ \to \ x_0 x_1 + \frac{x_1^3}{3!} \ \to \ \cdots, \tag{28}$$

and among them the solutions

$$\frac{x_1^{2m}}{(2m)!} + \frac{x_0}{1!}\frac{x_1^{2m-2}}{(2m-2)!} + \frac{x_0^2}{2!}\frac{x_1^{2m-4}}{(2m-4)!} + \cdots + \frac{x_0^{m-1}}{(m-1)!}\frac{x_1^2}{2!} + \frac{x_0^m}{m!}, \tag{29}$$

$$\begin{gathered} \frac{x_1^{2m+1}}{(2m+1)!} + \frac{x_0}{1!}\frac{x^{2m-1}}{(2m-1)!} + \frac{x_0^2}{2!}\frac{x_1^{2m-3}}{(2m-3)!} + \cdots \\ \cdots + \frac{x_0^{m-1}}{(m-1)!}\frac{x_1^3}{3!} + \frac{x_0^m}{m!}\frac{x_1}{1!}. \end{gathered} \tag{30}$$

It will be also noted that supposing function v in (12) to be independent on x_1 and denoting

$$v = \frac{1}{w(x_0, u)} \tag{31}$$

we get instead of (12) the following remarkable nonlinear heat equation

$$w_0 = \partial_u (w^{-2} w_u). \tag{32}$$

One easily sees that the operator

$$Q = w(x_0, u)\partial_1 + \partial_u \tag{33}$$

sets the connection between equations (32) and (1):

$$\begin{gathered}
w_0 - \partial_u(w^{-2}w_u) = \frac{1}{u_1}\partial_1\left(\frac{u_0 - u_{11}}{u_1}\right), \\
u_0 - u_{11} = \frac{1}{w}\int [w_0 - \partial_u(w^{-2}w_u)]du
\end{gathered} \tag{34}$$

by means of the change of variables

$$w(x_0, u) = \frac{\partial x_1(x_0, u)}{\partial u}, \quad \frac{\partial u(x_0, x_1)}{\partial x_1} = \frac{1}{w(x_0, u)}. \tag{35}$$

This result has been obtained differently in [7, 8].

It suppose v from (12) to have the form

$$v = \varphi(x_0, x_1)u \tag{36}$$

then (12) is reduced to the Burger's equation for φ

$$\varphi_0 = 2\varphi\varphi_1 + \varphi_{11} \tag{37}$$

and one may say that operator

$$Q = \partial_1 + \varphi u\partial_u \tag{38}$$

sets the connection between equation (37) and (1) via the substitution

$$\varphi = f_1/f. \tag{39}$$

Letting

$$v = \varphi(x_0, x_1)u + h(x_0, x_1) \tag{40}$$

and substituting it into (12) one finds the Burger's equation (37) for function φ and the following equation for h

$$h_0 = 2h\varphi_1 + h_{11}. \tag{41}$$

System of equation (37), (41) was also obtained in [6] when considering the system (10). Having made the change of variables

$$\varphi = f_1/f, \quad h = (f_1/f)g - g_1 \tag{42}$$

we reduced (37), (41) to two disconnected heat equations

$$f_0 = f_{11}, \quad g_0 = g_{11}. \tag{43}$$

Now we consider how to linearise the equation (12) in general case. Let us introduce the notations

$$S^1(x_0, x_1, u, v) = v_0 - (v_{11} + 2vv_{1u} + v^2v_{uu}). \tag{44}$$

After changing the variables

$$v = -\frac{z_1}{z_u}, \quad z = z(x_0, x_1, u) \tag{45}$$

we get

$$S^1(x_0, x_1, u, v) = -\frac{1}{z_u}(\partial_1 + v\partial_u)S^2(x_0, x_1, u, z), \tag{46}$$

where

$$S^2(x_0, x_1, u, v) = z_0 - z_{11} + 2\frac{z_1}{z_u}z_{1u} - \frac{z_1^2}{z_u^2}z_{uu}. \tag{47}$$

Having applied the hodograph transformation

$$y_0 = x_0, \quad y_1 = x_1, \quad y_2 = z, \quad R = u \tag{48}$$

we get

$$S^2(x_0, x_1, u, z) = -\frac{1}{R_2}(R_0 - R_{11}), \tag{49}$$

where $R = R(y_0, y_1, y_2)$.

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