On the new constants of motion for two- and three-particle equations

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The new constants of motion are found for a number of relativistic and quasirelativistic two-particle equations of the Dirac–Breit and the Bethe–Salpeter type and for the Krolikowski three-particle equation.

It was first noted by Dirac [1] that the Hamiltonian of a relativistic particle of spin-$\frac{1}{2}$ in a spherically symmetric field

$$H = \gamma_0\gamma_a p_a - \gamma_0 m + V(x^2), \qquad p_a = -i\frac{\partial}{\partial x_a}$$

(where γ_0, γ_a are the Dirac matrices, $a = 1,2,3$, $x^2 = x_1^2 + x_2^2 + x_3^2$) commuted with the operator

$$Q = \gamma_0(2S_aJ_a - 1/2) \tag{1}$$

where $J_a = \varepsilon_{abc}x_bp_c$, $S_a = \frac{1}{4}i\varepsilon_{abc}\gamma_b\gamma_c$. In other words, besides the obvious motion constant and angular momentum J_a, there is the additional constant of motion (1) for the Dirac equation with spherical potential.

The Dirac motion constant plays an important role in the solution of the Dirac equation by separation of variables. It causes the decomposition of the radial equations onto non-coupled subsystems corresponding to the fixed eigenvalues of operator (1).

In this letter it is demonstrated that the additional constants of motion exist for a number of relativistic and quasirelativistic two and three-particle equations and the explicit form of these motion constants is found.

Consider the generalised Breit equation in the CM frame

$$i\frac{\partial}{\partial t}\psi = (H^{(1)} + H^{(2)} + V)\psi \tag{2}$$

where $H^{(1)}$ and $H^{(2)}$ are the single-particle Hamiltonians,

$$H^{(\alpha)} = \gamma_0^{(\alpha)}\gamma_a^{(\alpha)}p_a - \gamma_0^{(\alpha)}m_{(\alpha)}, \qquad \alpha = 1,2,$$

$\{\gamma_0^{(1)}, \gamma_a^{(1)}\}$ and $\{\gamma_0^{(2)}, \gamma_a^{(2)}\}$ are the commuting sets of the 16×16 matrices defined by the relations

$$[\gamma_\mu^{(1)}, \gamma_\nu^{(2)}]_- \equiv \gamma_\mu^{(1)}\gamma_\nu^{(2)} - \gamma_\nu^{(2)}\gamma_\mu^{(1)} = 0, \qquad \mu,\nu = 0,1,2,3,$$
$$[\gamma_\mu^{(\alpha)}, \gamma_\nu^{(\alpha)}]_+ \equiv \gamma_\mu^{(\alpha)}\gamma_\nu^{(\alpha)} + \gamma_\nu^{(\alpha)}\gamma_\mu^{(\alpha)} = 2g_{\mu\nu}.$$

V is the interaction potential of the following general form

$$\begin{aligned} V = V_1 - \gamma_0^{(1)}\gamma_0^{(2)}[V_2\gamma_a^{(1)}\gamma_a^{(2)} + V_3\gamma_a^{(1)}x_a\gamma_b^{(2)}x_b + V_4] + \\ + V_5\gamma_a^{(1)}\gamma_a^{(2)} + V_6\gamma_a^{(1)}x_a\gamma_b^{(2)}x_b, \quad V_k = V_k(x^2), \quad k = 1,2,\ldots,6, \end{aligned} \tag{3}$$

where x_a and p_a are the internal coordinates and momenta.

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For $V_1 = V_2 = x^2 V_3 = 1/x$ and $V_4 = V_5 = V_6 = 0$ formula (3) defines the Breit potential [2, 3]. If V_k are arbitrary functions of x^2 this formula gives the generalised potential of two-particle interaction including various potentials of quark models of mesons [4–7].

The obvious motion constant of equation (2) is the angular momentum operator $\hat{J}_a$

$$\hat{J}_a = \varepsilon_{abc} x_b p_c + \hat{S}_a, \quad \hat{S}_a = S_a^{(1)} + S_a^{(2)}, \quad S_a^{(\alpha)} = \frac{1}{4} i\varepsilon_{abc}\gamma_b^{(\alpha)}\gamma_c^{(\alpha)}. \tag{4}$$

It happens, however, that as in the case of Dirac equation with spherically symmetric potential one can show the additional constant of motion for the generalised Breit equation. This motion constant has the form

$$\hat{Q} = \gamma_0^{(1)}\gamma_0^{(2)}[(\hat{S}_a\hat{J}_a)^2 - \hat{S}_a\hat{J}_a - \hat{J}_a\hat{J}_a], \tag{5}$$

where J_a and $\hat{S}_a$ are given in (4). Actually one can make sure by direct verification that the operator (5) commutes with the Hamiltonian (2). For this purpose it is convenient to represent $\hat{Q}$ in the form

$$\hat{Q} = [Q^{(1)}, Q^{(2)}]_+ - \frac{1}{2}\gamma_0^{(1)}\gamma_0^{(2)},$$

where $Q^{(\alpha)}$ are the operators obtained from (1) by the substitution $\gamma_0 \to \gamma_0^{(\alpha)}$, $S_a \to S_a^{(\alpha)}$, $J_a \to \hat{J}_a$, These operators satisfy the conditions

$$\begin{aligned}
&[Q^{(\alpha)}, S_a^{(\alpha)} p_a]_+ = \gamma_0^{(\alpha)} S_a^{(\alpha')} p_a, \\
&[Q^{(\alpha)}, S_a^{(\alpha)} x_a]_+ = \gamma_0^{(\alpha)} S_a^{(\alpha')} x_a, \\
&[Q^{(\alpha)}, S_a^{(\alpha')} p_a]_- = [Q^{(\alpha)}, S_a^{(\alpha')} x_a]_- = 0, \quad \alpha' \neq \alpha.
\end{aligned}$$

So we have found a new constant of motion for the generalised Breit equation in the form (5). This motion is also admitted by the Bethe–Salpeter equation in first approximation by e^2 and by the relativistic Barut–Komy equation [8]. Apparently it is possible to continue the list of the equations for which operator (5) is the motion constant (for instance it is the case for equation (2) with arbitrary $O(3)$ and P-invariant potential V).

One can demonstrate that the spectrum of the operator (4) is discrete and is given by the formula

$$Q\psi = \varepsilon j(j+1)\psi, \quad \varepsilon = \pm 1, \quad j = 0, 1, 2, \ldots, \quad \psi \in L_2(R_4).$$

In conclusion we give the new constants of motion for the equation describing two interacting particles with spins $\frac{1}{2}$ and 1 [9] and for the three-particle equation of Krolikowski [10]. They have the form

$$Q = r[q^3 - q^2 - (7J_aJ_a + S_aS_a)q + (4S_aS_a - 6)J_bJ_b + 9/4]$$

where $q = 2\hat{S}_a J_a - \frac{3}{2}$, $J_a = \varepsilon_{abc} x_b p_c + \hat{S}_a$. For the two-particle equation [9]

$$r = \gamma_0(1 - 2\beta_0), \quad \hat{S}_a = i\varepsilon_{abc}\left(\frac{1}{4}\gamma_b\gamma_c + \beta_b\beta_c\right),$$

$\{\gamma_\mu\}$ and $\{\beta_\mu\}$ are the commuting sets of the Dirac and of the Kemmer–Duffin–Petiau matrices. For the three-particle equation [10]

$$r = \gamma_0^{(1)}\gamma_0^{(2)}\gamma_0^{(3)}, \quad \hat{S}_a = \frac{1}{4} i\varepsilon_{abc}\big(\gamma_b^{(1)}\gamma_c^{(1)} + \gamma_b^{(2)}\gamma_c^{(2)} + \gamma_b^{(3)}\gamma_c^{(3)}\big),$$

$\{\gamma_\mu^{(1)}\}$, $\{\gamma_\mu^{(2)}\}$ and $\{\gamma_\mu^{(3)}\}$ are the commuting sets of the Dirac matrices.

Constants of motion for arbitrary spin particles are discussed in [11].

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