On superalgebras of symmetry operators of relativistic wave equations

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It is well known that the classical Lie approach does not make it possible to describe completely the symmetry of systems of partial differential equations. Actually it gives the possibility of finding only such symmetry operators which are the first order differential operators.

Using the non-Lie approach, in which the invariance group generators may be differential operators of any order and even integro-differential operators, the new invariance groups of a number of relativistic wave equations have been found [1, 2]. It turns out that even such well studied equations as the Dirac and the Maxwell ones have more extensive symmetry then the relativistic and the conformal invariance [3]. A numerous examples of non-Lie symmetries had been collected in our book [4].

In this communication we give the description of any order symmetry operators for some class of relativistic wave equations (including the Dirac and the Kemmer–Duffin–Petiau equations) and determine superalgebraic structure of sets of symmetry operators of the Dirac and of the Maxwell equations.

Let us write an arbitrary linear system of partial differential equations in the following symbolic form

$$L\psi = 0, \tag{1}$$

where L is a linear differential operator defined on H, $\psi \in H$.

Let Q be a linear operator defined on H. We say that Q is the symmetry operator of the equation (1), if

$$L(Q\psi) = 0 \tag{2}$$

for any ψ satisfying (1).

Below we consider the symmetry operators of relativistic wave equations, the most famous of which is the Dirac one:

$$L\psi \equiv (\gamma_\mu p^\mu - m)\psi = 0, \qquad p_\mu = i\frac{\partial}{\partial x^\mu}, \quad \mu = 0,1,2,3. \tag{3}$$

Using the equation (3) as an example we shall give the definition of the first ($Q^{(1)}$), the second ($Q^{(2)}$), the third ($Q^{(3)}$), ..., order symmetry operator as a linear differential operator which satisfies (2) and has the form

$$\begin{aligned} &Q^{(1)} = a^\mu P_\mu + B, \qquad Q^{(2)} = a^{\mu\nu} p_\mu p_\nu + B^\mu p_\mu + B, \\ &Q^{(3)} = a^{\mu\nu\lambda} p_\mu p_\nu p_\lambda + B^{\mu\nu} p_\mu p_\nu + B^\mu p_\mu + B, \end{aligned} \tag{4}$$

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where $B, B^\mu, B^{\mu\nu}, \ldots$ are matrices depending on $x = (x_0, x_1, x_2, x_3)$, $a^\mu, a^{\mu\nu}, a^{\mu\nu\lambda}, \ldots$ are functions on x. For the Dirac equation all matrices, are 4×4 dimensional, in general the matrices dimension is determined by the number of components of wavefunction ψ.

It is well known that the complete set of first order symmetry operators of the Dirac equation is exhausted by the Poincaré group generators P_μ, $J_{\mu\nu}$

$$P_\mu = p_\mu, \qquad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + \frac{i}{4}[\gamma_\mu, \gamma_\nu], \tag{5}$$

which satisfy the commutation relations

$$\begin{aligned} &[P_\mu, P_\nu] = 0, \qquad [P_\mu, J_{\nu\lambda}] = i(g_{\mu\nu}P_\lambda - g_{\mu\lambda}P_\nu), \\ &[J_{\mu\nu}, J_{\lambda\sigma}] = i(g_{\mu\sigma}J_{\nu\lambda} + g_{\nu\lambda}J_{\mu\sigma} - g_{\mu\lambda}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\lambda}). \end{aligned} \tag{6}$$

It means that the Poincaré invariance is the most extensive symmetry of the Dirac equation in the Lie sense [5, 6].

Using higher order symmetry operators it is possible to extend the symmetry group of the Dirac equation to the 16-parametrical Lie group which includes the Poincaré group as a subgroup [4]. Higher-order symmetry operators are useful in construction of coordinate systems in which the solutions in separated variables exist [7 ,8]. These operators may be considered also as the generators of Lie–Bäcklund groups [9].

Below we present some our general results connecting with the symmetry operators of relativistic wave equations for any spin particles.

Definition. Equation (1) is Poincaré-invariant and describes a particle of mass m and spin s, if it has 10 symmetry operators $P_\mu, J_{\mu\nu}$ which satisfy the algebra (6), and any solution ψ satisfies the conditions

$$P_\mu P^\mu \psi = m^2\psi, \qquad W_\mu W^\mu \psi = -m^2 s(s+1)\psi, \tag{7}$$

where $W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}J^{\nu\rho}P^\sigma$ is the Lubanski–Pauli vector.

Besides the Dirac equation the well known examples of relativistic wave equations satisfying given definition are the Kemmer–Duffin–Petiau equations for particles of spin 0 and 1 and the Rarita–Schwinger equation for a particle of spin $\frac{3}{2}$.

Theorem 1. *Any Poincaré-invariant equation for a particle of mass m and spin $s = 0$ is invariant under the algebra $ASL(2, C)$ [10].*

Proof. Let $P_\mu, J_{\mu\nu}$ be the symmetry operators of the equation (1), satisfying the commutation relations (6). Then by the definition (2) the following combinations

$$Q^\pm_{\mu\nu} = \frac{1}{m^2}[\varepsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma \pm i(P_\mu W_\nu - P_\nu W_\mu)] \tag{8}$$

are also the symmetry operators of this equation.

Using (6), (7) and the relations $[W_\mu, W_\nu] = i\varepsilon_{\mu\nu\rho\sigma}P^\rho W^\sigma$, $[P_\mu, W_\nu] = 0$ can make sure that the operators (8) satisfy the conditions

$$\begin{aligned} &[Q^\pm_{\mu\nu}, Q^\pm_{\rho\sigma}] = i(g_{\mu\sigma}Q^\pm_{\nu\rho} + g_{\nu\rho}Q^\pm_{\mu\sigma} - g_{\mu\rho}Q^\pm_{\nu\sigma} - g_{\nu\sigma}Q^\pm_{\mu\rho}), \\ &Q^\pm_{\mu\nu}Q^{\pm\,\mu\nu}\psi = 2(l_0^2 - l_1^2 - 1)\psi, \qquad \frac{1}{4}\varepsilon_{\mu\nu\rho\sigma}Q^{\pm\,\mu\nu}Q^{\pm\,\rho\sigma}\psi = il_0 l_1\psi, \\ &\qquad l_0 = s, \quad l_1 = \pm(s+1), \end{aligned}$$

and so form the basis of the finite dimensional irreducible representation $D(s, \pm(s+1))$ of the algebra $ASL(2, C)$. Thus the theorem is proved.

We see that any relativistic wave equation for a particle of nonzero spin and mass is automatically invariant under the algebra $ASL(2, C)$ basis elements of which belong to the enveloping algebra of the Lie algebra of the Poincaré group. The operators (8) form the basis of the 16-dimensional Lie algebra together with P_μ and $J_{\mu\nu}$. For the Dirac equation they take the form [4]

$$Q^{\pm}_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu] + \frac{i}{m}(\gamma_\mu p_\nu - \gamma_\nu p_\mu)(1 \pm i\gamma_4). \tag{9}$$

The operators (5), (9) generate the 16-parametrical invariance group of the Dirac equation. The corresponding finite transformations mix ψ and $\partial\psi/\partial x_\mu$ and can be easily calculated using the relation $(Q^{\pm}_{\mu\nu})^2 = 1/4$ [4].

The following statement gives the basis of any order symmetry operators for a class of relativistic wave equations of a type

$$(\beta_\mu p^\mu - m)\psi = 0 \tag{10}$$

where β_μ are numerical matrices, β_0 is diagonalizable.

Theorem 2. *Any finite order symmetry operator of Poincaré-invariant equation for a particle of mass $m \neq 0$ and spin s (10) belongs to the enveloping algebra of the algebra $AP(1, 3)$.*

The proof can be carried out using the Theorem 1 and bearing in mind that the necessary conditions for the symmetry operators of the equation (10) is to be the symmetry operators of the equation (7).

Let us note that relativistic wave equations (10) also possess such additional invariance algebras which belong to the class of integro-differential operators [4] and generally speaking are not membered among the enveloping algebra of the algebra $AP(1, 3)$.

In contrast to the first order symmetry operators the higherorder ones in general do not form the basis of the Lie algebra. But as a rool the higher order symmetry operators have the structure of superalgebra. We shall demonstrated it for the Dirac and for the Maxwell equations.

Let us consider the complete set of the second order symmetry operators of the equation (3) commuting with P_μ. Using the Theorem 2 it is not difficult to find such a set in the form

$$I, \quad P_\mu, \quad \lambda_{\mu\nu} = p_\mu p_\nu, \quad W_\mu = \frac{i}{4}\gamma_4(\gamma_\mu m - p_\mu), \quad W^{\mu\nu} = \gamma_4(\gamma_\mu p_\nu - \gamma_\nu p_\mu), \tag{11}$$

where I is the unit matrix.

Direct verification can make sure that the operators (11) do not form the basis of the Lie algebra. But these operators together with $J_{\mu\nu}$ (5) form the Lie superalgebra with the basis elements (12)

$$\{W_\mu, W_{\mu\nu};\ J_{\mu\nu}, P_\lambda, \lambda_{\mu\nu}, I\}. \tag{12}$$

The operators W_μ, $W_{\mu\nu}$ satisfy the anticommutation relations

$$[W_\mu, W_\nu]_+ = W_\mu W_\nu + W_\nu W_\mu = \frac{1}{2}(\lambda_{\mu\nu} - g_{\mu\nu} I),$$
$$[W_\mu, W_{\lambda\sigma}]_+ = i(g_{\mu\nu} P_\lambda - g_{\mu\lambda} P_\nu),$$
$$[W_{\mu\nu}, W_{\rho\lambda}]_+ = 2(g_{\mu\lambda}\lambda_{\nu\sigma} + g_{\nu\sigma}\lambda_{\mu\lambda} - g_{\mu\sigma}\lambda_{\nu\lambda} - g_{\nu\lambda}\lambda_{\mu\sigma}),$$

the commutation relations W_μ, $W_{\mu\nu}$ with P_μ, $J_{\mu\nu}$, $\lambda_{\mu\nu}$, I and between P_μ, $J_{\mu\nu}$, $\lambda_{\mu\nu}$, I are obvious.

So the Dirac equation is invariant under the 27-dimensional Lie superalgebra which contains the subalgebra $AP(1,3)$. Basis elements of this superalgebra are second order symmetry operators.

Consider the Maxwell equations with currents and charges

$$\frac{\partial\vec{E}}{\partial t} = \vec{\nabla}\times\vec{H}+\vec{j}, \quad \frac{\partial\vec{H}}{\partial t} = -\vec{\nabla}\times\vec{E}, \quad \vec{\nabla}\cdot\vec{E} = j_0, \quad \vec{\nabla}\cdot\vec{H} = 0. \tag{13}$$

The symmetry superalgebra of the equations (13) is formed by the set of the operators

$$\{Q^{ab};\ P_\mu, J_{\mu\nu}, \eta_{ab} = \nabla_a\nabla_b D, \eta_{abcd} = \nabla_a\nabla_b\nabla_c\nabla_d\}$$

where P_μ, $J_{\mu\nu}$ are the Poincaré group generators, $a, b, c,\ d = 1, 2, 3$, and Q^{ab}, D are the additional symmetry operators of the Maxwell equations [11] which act on E_a, H_a j_a and j_0 as follows

$$\begin{aligned}
Q^{ab}: \quad & E_c \to q_{cd}^{ab}E_d, \quad H_c \to -q_{cd}^{ab}H_d,\\
& j_c \to q_{cd}^{ab}j_d, \quad j_0 \to (\delta_{ab}\Delta - \nabla_a\nabla_b)j_0;\\
D: \quad & E_c \to \nabla_c\nabla_d E_d, \quad H_c \to \nabla_c\nabla_d H_d,\\
& j_c \to \nabla_c\nabla_d j_d, \quad j_0 \to \Delta j_0,
\end{aligned}$$

where

$$\begin{aligned}
& q_{cd}^{ab} = f_{cd}^{ab} + f_{cd}^{ba} + f_{dc}^{ab} + f_{dc}^{ba};\\
& f_{cd}^{ab} = \delta_{ad}\nabla_b\nabla_c + \frac{1}{4}\delta_{cd}(\delta_{ab}\Delta - \nabla_a\nabla_b) - \frac{1}{2}\delta_{ac}\delta_{bd} - \frac{1}{2}\delta_{ab}\nabla_c\nabla_d.
\end{aligned}$$

The operators Q^{ab} satisfy the anticommutation relations

$$[Q^{ab}, Q^{a'b'}]_+ = f_{klnm}^{aba'b'}\eta_{klnm} + g_{kl}^{aba'b'}\eta_{kl},$$

where

$$\begin{aligned}
f_{klnm}^{aba'b'} &= 2(\delta_{aa'}\delta_{kl} - \delta_{ak}\delta_{a'l})(\delta_{bb'}\delta_{nm} - \delta_{bn}\delta_{b'm}) -\\
&\quad -(\delta_{ab}\delta_{kl} - \delta_{ak}\delta_{bl})(\delta_{a'b'}\delta_{nm} - \delta_{a'n}\delta_{b'm}) + (a \leftrightarrow b),\\
g_{kl}^{aba'b'} &= 2(\delta_{aa'}\delta_{bk}\delta_{b'l} - \delta_{a'b'}\delta_{ak}\delta_{bl})\\
&\quad +(\delta_{ab}\delta_{a'b'} - \delta_{ab'}\delta_{a'b})\delta_{kl} + (a \leftrightarrow b) + (a' \leftrightarrow b') + (a \leftrightarrow b,\ a' \leftrightarrow b').
\end{aligned}$$

The remaining commutation relations for the operators (14) can be easily calculated.

It is interesting to note that the symmetry operators Q^{ab} do not belong to the enveloping algebra of the Lie algebra of the conformal group. These and other problems connecting with the symmetry of relativistic and nonrelativistic wave equations, the description of classes of equations with given symmetry, the exact solutions of linear and nonlinear wave equations are discussed in our book [11] which will be published this year.

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