The symmetry and exact solutions of the non-linear d'Alembert equation for complex fields

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The non-linear wave equations for the complex scalar field invariant under a conformal group are constructed and multiparametrical exact solutions of certain non-linear complex d'Alembert equations are found.

1. The non-linear wave equation

The non-linear wave equation

$$p_\mu p_\mu u + F(u) = 0$$

for the real function $u = u(x_0 \equiv t, x_1, \dots, x_n)$ is invariant under the extended Poincaré algebra $A_1P(1,n) \equiv \langle P_\mu, J_{\mu\nu}, D\rangle$

$$P_\mu = p_\mu = ig_{\mu\nu}\frac{\partial}{\partial x_\nu}, \quad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \tag{1}$$

where D is the dilation operator ($D = x_\mu p_\mu + \alpha u p_u$) iff $F(u) = \lambda u^k$ [4].

The classical and quantum scalar field, as is well known (see [1]), is described by the wave equation for the complex function u. Therefore it is interesting to construct the classes of non-linear wave equations invariant under wider groups than the Poincaré group. In the case of real fields, as was shown by Fushchych and Serov [4], there exist only two classes of such non-linear fields. In the complex case there are wide classes of fields invariant under groups which include the Poincaré group $P(1,n)$ as the subgroup.

In the present paper for the classical complex field u we construct the non-linear second-order wave equations

$$p_\mu p_\mu u + F(u, u^*, u_\alpha, u^*_\alpha) = 0, \quad u_\alpha \equiv \frac{\partial u}{\partial x_\alpha}, \quad u^*_\alpha \equiv \frac{\partial u^*}{\partial x_\alpha}, \quad \alpha, \mu = 0, 1, \dots, n \tag{2}$$

(the asterisk designates the complex conjugation and we indicate the sum by repeating indices: $p_\mu p_\mu = p_0^2 - p_1^2 - \dots - p_n^2$) invariant under the following Lie algebras (containing as subalgebra the Poincaré algebra $AP(1,n) = \langle P_\mu J_{\mu\nu}\rangle$ with the basic elements (1)):

$$A_1^{(1)} \equiv A_1^{(1)}P(1,n) \equiv \langle P_\mu, J_{\mu\nu}, D_1\rangle.$$

The dilation operator D_1 has the form

$$D_1 = x_\mu p_\mu - \lambda(up_u + u^* p_{u^*}), \quad p_u = -i\frac{\partial}{\partial u}, \quad p_{u^*} = -i\frac{\partial}{\partial u^*}.$$

$$A_1^{(2)} \equiv A_1^{(2)}P(1,n) \equiv \langle P_\mu, J_{\mu\nu}, D_2\rangle.$$

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The dilation operator D_2 has the form

$$D_2 = x_\mu p_\mu - \lambda(p_u + p_{u^*}).$$

$$A_2 \equiv A_2P(1,n) \equiv \langle P_\mu, J_{\mu\nu}, D_1, Q\rangle.$$

The operator of charge has the form

$$Q = u^* p_u - u p_{u^*}.$$

$$A_3^{(1)} \equiv A^{(1)}C(1,n) \equiv \langle P_\mu, J_{\mu\nu}, D_1, K_\mu^{(1)}\rangle.$$

The operators $K_\mu^{(1)}$ generating the conformal transformations have the form

$$K_\mu^{(1)} = 2x_\mu D_1 - x_\nu x_\nu p_\mu.$$

$$A_3^{(2)} \equiv A^{(2)}C(1,n) \equiv \langle P_\mu, J_{\mu\nu}, D_1, K_\mu^{(1)}, Q\rangle.$$

$$A_3^{(3)} \equiv A^{(3)}C(1,n) \equiv \langle P_\mu, J_{\mu\nu}, D_2, K_\mu^{(2)}\rangle.$$

$$K_\mu^{(2)} = 2x_\mu D_2 - x_\nu x_\nu p_\mu.$$

To describe the invariant equations of the form (2) we need the differential invariants of the zero and first order for the algebras $A_1^{(1)}, \dots, A_3^{(3)}$. As is well known (see, e.g. [7]) these invariants are solutions of the system

$$\overset{1}{X}_i \Phi(u, u^*, u_\alpha, u_\alpha^*) = 0, \tag{3}$$

where $\overset{1}{X}_i$ are the first prolongations of the basis operators of the corresponding algebras.

Not going into details we adduce the explicit form of the invariants for the algebras

$$\begin{aligned}
&AP(1,n): \quad u,\ u^*,\ r_1 = u_\alpha u_\alpha,\ r_2 = u_\alpha u_\alpha^*,\ r_3 = u_\alpha^* u_\alpha^*,\\
&A_1^{(1)}: \quad \frac{u}{u^*},\ \frac{r_1}{r_2},\ \frac{r_3}{r_2},\ \frac{r_1^2}{u^{2(\lambda-1)}},\\
&A_1^{(2)}: \quad u - u^*,\ \frac{r_1}{r_2},\ \frac{r_3}{r_2},\ r_1^{\lambda/2}\exp u,\\
&AP(1,n)\oplus Q: \quad u^2 + u^{*2},\ r_1 + r_3,\ r_2^2 - r_1 r_3,\ R = u^{*2} r_1 - 2uu^* r_2 + u^2 r_3,\\
&A_2: \quad \frac{(r_1 + r_3)^2}{(u^2 + u^{*2})^{\lambda-1}},\ \frac{r_2^2 - r_1 r_2}{(r_1 + r_3)^2},\ \frac{R}{(u^2 + u^{*2})(r_1 + r_3)},\\
&A_3^{(1)}\ (\lambda \neq 0): \quad \frac{u}{u^*},\ \frac{R}{u^{4-2/\lambda}},\\
&A_3^{(1)}\ (\lambda = 0): \quad u,\ u^*,\ \frac{r_1}{r_2},\ \frac{r_3}{r_2},\\
&A_3^{(2)}\ (\lambda \neq 0): \quad R(u^2 + u^{*2})^{1/\lambda - 2},\\
&A_3^{(2)}\ (\lambda = 0): \quad u^2 + u^{*2},\ \frac{r_2^2 - r_1 r_3}{(r_1 + r_3)^2},\ \frac{R}{r_1 + r_3},\\
&A_3^{(3)}\ (\lambda \neq 0): \quad u - u^*,\ (r_1 - 2r_2 + r_3)^{\lambda/2}\exp u.
\end{aligned} \tag{4}$$

These systems of invariants are complete when $n \geq 3$.

The classification of the non-linear equations for the complex scalar field invariant under the enumerated algebras gives the following theorem.

Theorem. *Equation (2) is invariant under the algebras*

$$\begin{array}{lll} AP(1,n) & \textit{when} & F=\varphi(\omega), \\ A_1^{(1)} & \textit{when} & F=u^{1-2/\lambda}\varphi(\omega), \\ A_1^{(2)} & \textit{when} & F=\exp(u)\varphi(\omega), \\ A_2 & \textit{when} & F=(u^2+u^{*2})^{-1/\lambda}(uf(\omega)+iu^*g(\omega)), \\ A_3^{(1)},\ \lambda\neq 0 & \textit{when} & F=\dfrac{2\lambda+n-1}{2\lambda u}r_1+u^{1-2/\lambda}\varphi(\omega), \end{array}$$

when $\lambda=0$ there are no invariant equations of the form (2);

$$A_3^{(2)},\ \lambda=\frac{1-n}{2} \quad \textit{when} \quad F=(u^2+u^{*2})^{2/(n-1)}(uf(\omega)+iu^*g(\omega))$$

(when $\lambda\neq(1-n)/2$ there are no invariant equations of the form (2));

$$A_3^{(3)},\ \lambda\neq 0 \quad \textit{when} \quad F=\frac{n-1}{2\lambda}r_1+\exp\left(-\frac{2}{\lambda}u\right)\varphi(\omega)$$

(here we designate as f and g arbitrary real and as φ arbitrary complex functions, ω are invariants of the corresponding algebras).

To prove the theorem it is necessary to use the Lie invariance condition in the form

$$\overset{2}{X}_i L\Big|_{\substack{L=0\\ L^*=0}}=0$$

where $L=\Box u-F(u,u^*,u_\alpha,u^*_\alpha)$ ($\Box u=p_\mu p_\mu u$), $\overset{2}{X}_i$ are the second prolongations of the basis elements of the algebras being considered, which we resolve with respect to the unknown function for every algebra.

A similar theorem can be formulated and proved for the system of two wave equations for the pair of real functions.

The classification of the general quasilinear Poincaré-invariant equation for the complex scalar function is adduced by Fushchych and Yehorchenko [5].

2. The solutions of wave equations for the complex function

Let us consider the equation

$$\Box u=F(u,u^*) \tag{5}$$

which is invariant under the Poincaré algebra (1). Its solutions can be found with the help of the reduction with respect to subalgebras of $AP(1,n)$ as was done in the real case by Fushchych and Serov [4] or Winternitz et al [8] but such reduction leads mostly to systems of ordinary differential equations not solvable in quadratures; one of the ways to avoid this difficulty was suggested by Grundland and Tuszynski [6]. To find the exact solutions of (5) it is advisable to search especially for ansatze leading to systems of differential equations solvable in quadratures.

Using the ansatz (see, e.g., [3, 4])

$$u=\varphi(\omega), \quad u^*=\varphi^*(\omega), \quad \omega=\omega(x) \tag{6}$$

we come to the system

$$\begin{aligned} &\omega_\mu\omega_\mu\ddot{\varphi}+\Box\omega\dot{\varphi}=F(\varphi,\varphi^*),\\ &\omega_\mu\omega_\mu\ddot{\varphi}^*+\Box\omega\dot{\varphi}^*=F^*(\varphi,\varphi^*), \quad \dot{\varphi}=\frac{d\varphi}{d\omega}. \end{aligned} \tag{7}$$

The condition of separation of variables in the system (7) is that the new variable ω must satisfy the conditions

$$\Box\omega=\chi(\omega), \quad \omega_\mu\omega_\mu=T(\omega), \tag{8}$$

where χ, T are arbitrary functions (not equal simultaneously to zero).

Thus to find exact solutions of (5) in the form (6) it is sufficient to solve the system (8) and

$$\begin{aligned} &T(\omega)\ddot{\varphi}+\chi(\omega)\dot{\varphi}=F(\varphi,\varphi^*),\\ &T(\omega)\ddot{\varphi}^*+\chi(\omega)\dot{\varphi}^*=F^*(\varphi,\varphi^*). \end{aligned} \tag{9}$$

To solve the system (8) we use the results of Collins [2], where similar systems for the functions of three independent variables were investigated. The partial solutions of the system (8) when $\mu=0,1,\ldots,n$, $T(\omega)=1$, $\chi(\omega)=N(\omega-A)^{-1}$, $N=0,1,\ldots,n$, $A=\mathrm{const}$, are given in Table 1. Evidently when $n>2$ they are not general solutions.

Table 1

N	Solutions	Conditions on parameters
0	$\omega+\alpha y+F(\beta y)$ $(\alpha y=\alpha_0y_0-$ $-\alpha_1y_1-\cdots-\alpha_ny_n)$	F is an arbitrary function of βy, $y_\nu=x_\nu+a_\nu$, $a_\nu=\mathrm{const}$, $\alpha^2=1$, $\beta^2=\alpha\beta=0$
$1,\ldots,n$	$\omega-A=[(\alpha^iy)(\alpha^iy)]^{1/2}$	$\alpha^i_\mu\alpha^j_\mu=\delta^{ij}$, $i,j=1,\ldots,N+1$, $y_\nu=x_\nu+a_\nu$, $a_\nu=\mathrm{const}$

Below we consider systems of the form (5)

$$\begin{aligned} &\Box u=\lambda u(uu^*)^k,\\ &\Box u^*=\lambda^*u^*(uu^*)^k, \end{aligned} \tag{10}$$

$$\begin{aligned} &\Box u=(\lambda_1u+i\lambda_2u^*)(u^2+u^{*2})^k,\\ &\Box u=(\lambda_1u^*-i\lambda_2u^*)(u^2+u^{*2})^k \end{aligned} \tag{11}$$

(λ_1, λ_2, k are arbitrary real numbers and λ is an arbitrary complex number), which are invariant corresponding with respect to the operators $Q_1=u\partial_u-u^*\partial_{u^*}$ and $Q_2=u^*\partial_u-u\partial_{u^*}$ (the operator of charge).

The system (9) with $T(\omega)=1$, $\chi(\omega)=N/(\omega-A)$ and ω from Table 1, for (10) takes the form

$$\begin{aligned} &\frac{N}{\omega-A}\dot{\varphi}+\ddot{\varphi}=\lambda\varphi(\varphi\varphi^*)^k,\\ &\frac{N}{\omega-A}\dot{\varphi}^*+\ddot{\varphi}^*=\lambda^*\varphi^*(\varphi\varphi^*)^k, \end{aligned} \tag{12}$$

where $N = 0, 1, \ldots, n$; for (11) the system (9) takes the form

$$\begin{aligned} &\frac{N}{\omega - A}\dot{\varphi} + \ddot{\varphi} = (\lambda_1\varphi + i\lambda_2\varphi^*)(\varphi^2 + \varphi^{*2})^k, \\ &\frac{N}{\omega - A}\dot{\varphi}^* + \ddot{\varphi}^* = (\lambda_1\varphi^* - i\lambda_2\varphi)(\varphi^2 + \varphi^{*2})^k. \end{aligned} \tag{13}$$

It is convenient to search for solutions of (12) in the form

$$\varphi = re^{i\theta}, \quad \varphi^* = re^{-i\theta} \tag{14}$$

for solutions of (13) in the form

$$\varphi = r\left(\frac{1+i}{2\sqrt{2}}e^{\theta} + \frac{1-i}{2\sqrt{2}}e^{-\theta}\right), \quad \varphi^* = r\left(\frac{1-i}{2\sqrt{2}}e^{\theta} + \frac{1+i}{2\sqrt{2}}e^{-\theta}\right). \tag{15}$$

For the real functions $r = r(\omega)$ and $\theta = \theta(\omega)$ we obtain the system

$$\begin{aligned} &\frac{N}{\omega - A}\dot{r} + \ddot{r} + \varkappa\dot{\theta}^2 r = \lambda_1 r^{2k+1}, \\ &\frac{N}{\omega - A}\dot{\theta}r + 2\dot{r}\dot{\theta} + r\ddot{\theta} = \lambda_2 r^{2k+1}. \end{aligned} \tag{16}$$

Here $\lambda = \lambda_1 + i\lambda_2$, $\varkappa = -1$ for (12) and $\varkappa = 1$ for (13).

Let $N = 0$. With an arbitrary k and $\lambda_2 = 0$ the system (16) has the general solution in the parametrical form ($\lambda = \lambda_1$):

$$\begin{aligned} &\omega = \int \left(\frac{\lambda}{k+1}r^{2k+2} + \frac{\varkappa c_1^2}{r^2} + c_2\right)^{-1/2} dr + c_3, \\ &\theta = c_1 \int r^{-2}\left(\frac{\lambda}{k+1}r^{2k+2} + \frac{\varkappa c_1^2}{r^2} + c_2\right)^{-1/2} dr + c_4, \end{aligned} \tag{17}$$

When $k = -2$ we obtain the general solution of (16) in explicit form in elementary functions

$$\begin{aligned} &r = \left(c_2(\omega + c_3)^2 + \frac{\lambda - \varkappa c_1^2}{c_2}\right)^{1/2}, \\ &\theta = \begin{cases} \dfrac{c_1}{(\lambda - \varkappa c_1^2)^{1/2}} \tan^{-1} \dfrac{c_2(\omega + c_3)}{(\lambda - \varkappa c_1^2)^{1/2}} + c_4, & \lambda - \varkappa c_1^2 > 0, \\ \dfrac{c_1}{2(\varkappa c_1^2 - \lambda)^{1/2}} \ln\left|\dfrac{\omega + c_3 + c_2^{-1}(\varkappa c_1^2 - \lambda)^{1/2}}{\omega + c_3 - c_2^{-1}(\varkappa c_1^2 - \lambda)^{1/2}}\right|, & \lambda - \varkappa c_1^2 < 0, \\ -\dfrac{c_1}{(c_2(\omega + c_3))}, & \lambda = \varkappa c_1^2, \end{cases} \end{aligned} \tag{18}$$

$c_1 \neq 0$, c_2, c_3, c_4 are arbitrary real numbers. (If $c_1 = 0$, $\theta = \text{const.}$)

Note 1. The solvability of systems of ordinary differential equations in quadratures is connected with their wide symmetry. Systems of the form (12) can be reduced to systems of four first-order equations and we may suppose that for their solvability in quadratures it is necessary for the range of basis of their invariance algebra to be not less than 4 [7]. However, this condition is not sufficient. The system (12) when $k = -2$, $N = 0$ has the maximal invariance algebra among systems of such form with the basis operators

$$\partial_\omega, \quad \omega\partial_\omega + \frac{1}{2}(\varphi\partial_\varphi + \varphi^*\partial_{\varphi^*}), \quad \omega^2\partial_\omega + \omega(\varphi\partial_\varphi + \varphi^*\partial_{\varphi^*}), \quad \varphi\partial_\varphi - \varphi^*\partial_{\varphi^*}$$

but when $\lambda_2 \neq 0$ it reduces to a Riccati equation not solvable in quadratures.

The system (16) when $N \neq 1$, $N \neq 2$, $k = (N-2)(N-1)^{-1}$ by the substitution $t = (\omega - A)^{N-1}$, $r = (\omega - A)^{1-N}\rho$ can be reduced to the form

$$\ddot\rho + \varkappa\dot\theta^2\rho = \lambda_1\rho^{2k+1}, \quad 2\dot\rho\dot\theta + \rho\ddot\theta = \lambda_2\rho^{2k+1}.$$

We obtain its solutions in parametrical form ($\lambda_2 = 0$) and from them we obtain the solutions of (16)

$$\begin{aligned}
r &= \rho\left[c_3 + (N-1)\int\left(\frac{\lambda}{k+1}\rho^{2k+2} + \frac{\varkappa c_1^2}{\rho^2} + c_2\right)^{-1/2} d\rho\right], \\
\theta &= c_1(N-1)^2\int\rho^{-2}\left(\frac{\lambda}{k+1}\rho^{2k+2} + \frac{\varkappa c_1^2}{\rho^2} + c_2\right)^{-1/2} d\rho + c_4, \\
\omega &= \left[c_3 + (N-1)\int\left(\frac{\lambda}{k+1}\rho^{2k+2} + \frac{\varkappa c_1^2}{\rho^2} + c_2\right)^{-1/2} d\rho\right]^{1/(N-1)} + A,
\end{aligned} \tag{19}$$

$c_1 \neq 0$, c_2, c_3, c_4 are arbitrary constants chosen for r, θ to be real.

From solutions (17)–(19) and substitutions (14) and (15) we obtain the solutions of (12) and (13) respectively. With ω from Table 1 we get solutions of the initial systems (10) and (11).

As (10) and (11) are invariant with respect to the scale transformations it is possible to find ansatze reducing them to the first-order differential equations which have more chances to be solved in quadratures. We search for such ansatze in the form

$$u = f(x)\Phi(\omega), \quad u^* = f(x)\Phi^*(\omega). \tag{20}$$

The corresponding conditions on f and ω are as follows:

$$\begin{aligned}
&\Box f(x) = F(\omega)f^{2k+1}(x), \\
&f\Box\omega + 2f_\mu\omega_\mu = G(\omega)f^{2k+1}(x), \\
&\omega_\mu\omega_\mu = 0,
\end{aligned} \tag{21}$$

where F, G are arbitrary functions.

It is interesting enough to investigate the system (21) itself but here we do not go into this matter and adduce only some solutions:

$$\begin{aligned}
&f(x) = [(\beta^i x)(\beta^i x)]^a, \quad \omega = \frac{\alpha x}{[(\beta^i x)(\beta^i x)]^b}, \\
&a = -\frac{1}{2k}, \quad \alpha^2 = \alpha\beta^i = 0, \quad \beta^i_\mu\beta^i_\mu = \delta^{ij}, \quad b = 0, 1,
\end{aligned} \tag{22}$$

the sum by i is implied, $i = 1, \ldots, m$, $m \leq n$, $1 - 2a \neq m$, when $b = 1$.

For the ansatz (20) with f, ω from (22) we obtain the reduced equations and their solutions.

For equations (10)

(i) $b = 0$, $m + 2a - 1 \neq 0$

$$\begin{gathered}\Phi'\omega + \Phi\frac{m+2a-1}{2} = \frac{\lambda}{4a}\Phi(\Phi\Phi^*)^k, \\ \Phi = c\left(\omega^{k(m+2a-1)} - c_1\right)^{-1/2k} \times \\ \times \exp\frac{i\lambda_2}{\lambda_1}\left((\ln\omega)\frac{m+2a-1}{2} - \frac{1}{2k}\ln\left(\omega^{k(m+2a-1)} - c_1\right)\right), \\ cc^* = \left(\frac{1}{\lambda_1}2a(m+2a-1)\right)^{1/k};\end{gathered} \tag{23}$$

(ii) $b = 0$, $m + 2a - 1 = 0$

$$\Phi = c(\lambda_1 k^2 \ln\omega + c_1)^{-(\lambda_1 + i\lambda_2)/2k\lambda_1}, \quad cc^* = 1; \tag{24}$$

(iii) $b = 1$

$$\begin{gathered}\Phi'\omega - a\Phi = \frac{\lambda}{2(m+2a-1)}\Phi(\Phi\Phi^*)^k, \\ \Phi = c\omega^{-i\lambda_2/2k\lambda_1}|1 - c_1\omega|^{-\lambda_2/2k\lambda_1}, \quad cc^* = \left(-\frac{2a(m+2a-1)}{\lambda_1}\right)^{1/k}.\end{gathered} \tag{25}$$

In a similar way solutions of (11) can be obtained; if Φ has the form (15) then

(i) $b = 0$, $m + 2a - 1 \neq 0$

$$\begin{gathered}r = \left(1 - c_1\omega^{k(m+2a-1)}\right)^{-1/2k}[(m+2a-1)4a]^{1/2k}(\lambda_1 + \lambda)^{-1/2k}, \\ \theta = \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}\left(\frac{m+2a-1}{2}\ln\omega - \frac{1}{2k}\ln\left|1 - c_1\omega^{k(m+2a-1)}\right|\right);\end{gathered} \tag{26}$$

(ii) $b = 0$, $m + 2a - 1 = 0$

$$\begin{gathered}r = (2k^2(\lambda_1 + \lambda_2)\ln\omega + c_1)^{-1/2k}, \\ \theta = -\frac{1}{2k}\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}\ln|k^2(\lambda_1 + \lambda_2)\ln\omega + c_1|;\end{gathered} \tag{27}$$

(iii) $b = 1$, $m + 2a - 1 \neq 0$

$$\begin{gathered}r = \left(-\frac{a(m+2a-1)}{\lambda_1 + \lambda_2}\right)^{1/2k}(1 - c_1\omega)^{-1/2k}, \\ \theta = -\frac{1}{2k}\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}(\ln\omega + \ln|1 - c_1\omega|).\end{gathered} \tag{28}$$

Substituting the obtained solutions (23)–(28) of the reduced equations into the ansatz (20) and (22) we get the multiparametrical families of exact solutions of (10) and (11) correspondingly.

The ansatz (20) and (22) when $b \neq 0$, $b \neq 1$ allows us to obtain the reduced equations of the second order

$$\begin{aligned}
&\Phi''\omega^2 4b(b-1) + \Phi'\omega[4b^2 - 2b(m+1) + 4a(1-2b)] + \\
&\qquad + 2a\Phi(2a+m-1) = \lambda\Phi F(\Phi, \Phi^*), \\
&F = (\Phi\Phi^*)^k \quad \text{for (10)}, \quad F = (\Phi^2 + \Phi^{*2})^k \quad \text{for (11)}.
\end{aligned} \tag{29}$$

We can adduce the parametrical solutions of (29) when

$$\begin{aligned}
&b = \frac{2a}{m+4a-1}, \quad \lambda = \lambda^* \quad (\lambda_2 = 0,\ \lambda = \lambda_1), \\
&\omega = \int \left(\frac{\lambda}{k+1} r^{2k+2} + \frac{\varkappa c_1^2}{r^2} + c_2 + Br^2 \right)^{-1/2} dr + c_3, \\
&\theta = c_1 \int r^{-2} \left(\frac{\lambda}{k+1} r^{2k+2} + \frac{\varkappa c_1^2}{r^2} + c_2 + Br^2 \right)^{-1/2} dr + c_4, \\
&B = \frac{1}{4}(r+4a-1)^2,
\end{aligned} \tag{30}$$

$\varkappa = -1$ and the representation (14) for Φ is taken for (10), and $\varkappa = 1$ and the representation (15) for Φ is taken for (11).

3. The conformally invariant families of solutions

Let us consider the conformally invariant system of the form (2) $(n = 2)$

$$\Box u = u^3 F\left(\frac{u}{u^*}, \frac{R}{(uu^*)^3}\right), \quad \Box u^* = u^{*3} F^*\left(\frac{u}{u^*}, \frac{R}{(uu^*)^3}\right), \tag{31}$$

where R is defined in (4).

We obtain here the conformally invariant families of solutions of (31) with certain F using the formulae of conformal reproduction of solutions.

We used the ansätze (20), where

$$\omega = \alpha x, \quad f = (x^2)^{-1/2}, \tag{32a}$$

$$\omega = \alpha x / x^2, \quad f = (x^2)^{-1/2}, \tag{32b}$$

$$\omega = \alpha x, \quad f = [x^2 - 2\varepsilon x \delta x + \delta^2(\varepsilon x)^2]^{-1/2}, \tag{33a}$$

$$\omega = \alpha x, \quad f = [2\alpha x \beta x - \beta^2(\alpha x)^2 + c(\alpha x)]^{-1/2}, \tag{33b}$$

where $\alpha^2 = \varepsilon^2 = \alpha\varepsilon = \alpha\delta = 0$, $\alpha\beta = \varepsilon\delta = 1$.

When u, u^* are defined from the ansätze (20), (32) and (33), $R \equiv 0$. Then the reduced equations have the form

$$\varkappa\Phi - 2\dot{\Phi}\omega = \Phi^3 F\left(\frac{\Phi}{\Phi^*}\right), \quad \varkappa\Phi^* - 2\dot{\Phi}^*\omega = \Phi^{*3} F^*\left(\frac{\Phi}{\Phi^*}\right), \tag{34}$$

where $\varkappa = -1$ for (32), $\varkappa = 1$ for (33).

The solution of (34) in parametrical form can be obtained for arbitrary F.

The multiparametrical conformally invariant families of solutions we adduce for the equations are

$$\begin{aligned}\Box u &= (u^2 + u^{*2})(g_1 u + i g_2 u^*),\\ \Box u^* &= (u^2 + u^{*2})(g_1 u^* - i g_2 u),\end{aligned} \tag{35}$$

$$\begin{aligned}\Box u &= (g_1 + i g_2) u (u u^*),\\ \Box u^* &= (g_1 - i g_2) u^* (u u^*),\end{aligned} \tag{36}$$

where g_1, g_2 are real functions of $R(u^2 + u^{*2})^{-3}$.

Their families of solutions are non-reproducible by conformal transformations and given by the following formulae. The solutions of (35) are

$$\begin{aligned}u = f(x)\omega^{\varkappa/2}\Bigg(&c_2\frac{1+i}{4}|c_1 + \varkappa\omega^{\varkappa}A_1|^{-(A_1+A_2)/2A_1} +\\ &+ c_2^{-1}\frac{1-i}{4}|c_1 + \varkappa\omega^{\varkappa}A_1|^{-(A_1-A_2)/2A_1}\Bigg),\end{aligned}$$

$$A^j = g^j(o), \quad A_2 \neq 0, \quad c^j \in R, \quad c_1 = 0 \quad (j = 1, 2)$$

and the solutions of (36) are

$$u = f(x)\omega^{\varkappa/2}|A_1\varkappa\omega^{\varkappa} + c_1|^{-1/2}\exp\left[i\left(c_2 - \frac{A_2}{2A_1}\ln|c_1 + A_1\varkappa\omega^{\varkappa}|\right)\right],$$

$f(x)$ and ω being substituted from Table 2 and $\varkappa$ being defined from the corresponding ansätze (32) or (33).

Table 2

Ansatz	$\varkappa$	ω	$\{f(x)\}^{-2}$
(32b)	-1	$\dfrac{\alpha x + \alpha\tau x^2 + ab\sigma(\tau,x)}{x^2 + 2bx + 2b\tau x^2 + b^2\sigma(\tau,x)}$	$\sigma(\tau,x)[x^2 + 2bx + 2b\tau x^2 + b^2\sigma(\tau,x)]$
(32a)	-1		
(33b)	1	$(\sigma(\tau,x))^{-1}[\alpha x + \alpha\tau x^2 + \alpha b\sigma(\tau,x)]$	$\omega\sigma(\tau,x)[2(\beta x + \beta\tau x^2) - \beta^2(\alpha x + \alpha\tau x^2) + (c + 2b\beta - \beta^2\alpha b)\sigma(\tau,x)]$
(33a)	1		$-(\varepsilon x + \varepsilon\tau x^2)(\delta x + \delta\tau x^2) + \delta^2(\alpha x + \varepsilon\tau x^2)^2 + \sigma(\tau,x)[x^2 + 2bx + 2b\tau x^2 - 2b\delta(\varepsilon x + \varepsilon\tau x^2) - 2b\varepsilon(\delta x + \delta\tau x^2) + \delta^2 2\varepsilon b(\varepsilon x + 2\varepsilon\tau x^2) + \sigma(\tau,x)(b^2 - 2b\varepsilon b\delta + \delta^2(\varepsilon b)^2]$

$\sigma(\tau,x) = 1 + 2\tau x + \tau^2 x^2$, b_μ, τ_μ are arbitrary parameters.

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