On approximate symmetry and approximate solutions of the non-linear wave equation with a small parameter

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The concept of approximate symmetry is introduced. We describe all nonlinearities $F(u)$ with which the non-linear wave equation $\Box u + \lambda u^3 + \varepsilon F(u) = 0$ with a small parameter ε is approximately scale and conformally invariant. Some approximate solutions of wave equations in question are obtained using the approximate symmetry.

Let us consider the non-linear wave equation

$$\Box u + \lambda u^3 + \varepsilon F(u) = 0, \tag{1}$$

where $\Box = \partial_\mu \partial^\mu$ is the d'Alembertian, $\mu = \overline{0,3}$; λ is an arbitrary constant; $\varepsilon \ll 1$ is a small parameter; $u = u(x)$, $x \in R(1,3)$; $F(u)$ is an arbitrary smooth function. By means of Lie's method (see [5, 4]) one can make sure that when $F(u) \neq 0$ and $F(u) \neq u^3$, equation (1) is invariant under the Poincaré group $P(1,3)$ only, because the term $\varepsilon F(u)$ breaks down the scale and conformal symmetry of the equation $\Box u + \lambda u^3 = 0$.

Below we describe all functions $F(u)$ with which equation (1) is approximately invariant under the scale and conformal transformations.

Let us represent an arbitrary solution, analytic in ε, of equation (1) in the form

$$u = w + \varepsilon v, \tag{2}$$

where w and v are some smooth functions of x. After substitution of (2) into (1) and equating to zero the coefficients of zero and first power of ε we get the following system of partial differential equations (PDE):

$$\begin{aligned} &\Box u + \lambda w^3 = 0, \\ &\Box v + 3\lambda w^2 v + F(w) = 0. \end{aligned} \tag{3}$$

Definition. *We shall call the approximate symmetry of equation (1) the (exact) symmetry of the system (3).*

Theorem 1. *Equation (1) is approximately scale invariant (in the sense of the above definition) if and only if*

$$F(u) = \begin{cases} \dfrac{2\lambda b}{k+1} u^3 + \dfrac{3\lambda c}{k} u^2 + a u^{2-k}, & k \neq 0, -1, \\ 2\lambda b u^3 + 3\lambda c u^2 \ln u + a u^2, & k = 0, \\ 2\lambda b u^3 \ln u - 3\lambda c u^2 + a u^3, & k = -1 \end{cases} \tag{4}$$

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(k, a, b, c are arbitrary constants), with the generator of scale transformations having the form

$$D = x\partial - w\partial_w + (kv + bw + c)\partial_v. \tag{5}$$

Proof. Using Lie's algorithm [5, 4] we find from the condition of invariance that the generator of scale transformations should have the form

$$D = x\partial - w\partial_w + \eta^2(v, w)\partial_v$$

provided (following from the invariance of the second equation of system (3)) that

$$\begin{aligned} &\eta^2_{vv} = \eta^2_{ww} = \eta^2_{wv} = 0 \quad \Rightarrow \quad \eta^2 = kv + bw + c, \\ &2\lambda bw^3 + 3\lambda cw^2 + (2-k)F - w\frac{dF}{dw} = 0. \end{aligned} \tag{6}$$

The general solution of equations (6) is given in (4). Thus the theorem is proved.

In particular, as follows from Theorem 1, the equation

$$\Box u + \lambda u^3 + \varepsilon u = 0 \tag{7}$$

is approximately scale invariant and the corresponding generator has the form $D = x\partial - w\partial_w + v\partial_v$. This statement holds true even if $\lambda = 0$.

Theorem 2. *Equation (1) is approximately conformally invariant if and only if*

$$F(u) = -3\lambda\beta u^2 + au^3 \tag{8}$$

with the generator of conformal transformations having the form

$$K = 2cx[x\partial - w\partial_w - (v - \beta)\partial_v] - x^2 c\partial, \tag{9}$$

where β, a, c_μ are arbitrary constants.

The proof of Theorem 2 is performed in the same spirit as that of Theorem 1.

Suppose that in (2)

$$v = f(w), \tag{10}$$

where f is an arbitrary differentiable function. In this case the system (3) takes the form

$$\Box w + \lambda w^3 = 0, \tag{11}$$

$$w_\mu w^\mu \ddot{f} + \Box w \dot{f} + 3\lambda w^2 f + F(w) = 0, \quad w_\mu \equiv \partial w/\partial x^\mu. \tag{12}$$

From the condition of splitting of equation (12) one has to put

$$w_\mu w^\mu = A(w), \tag{13}$$

where A is some function of w. Equation (13) is compatible with (11) if $A(w) = \lambda w^4$, i.e.

$$w_\mu w^\mu = \lambda w^4. \tag{14}$$

(For more details see [1, 2].) Taking account of (11) and (14) we rewrite (12) as

$$\lambda(w^2\ddot{f} - w\dot{f} + 3f) + w^{-2}F(w) = 0. \qquad (15)$$

So, if we find function $f(w)$ as a solution of equation (15), we thereby obtain by means of expressions (2) and (10) approximate solutions of equation (1). It will be noted that a subset of such solutions of equation (1) is approximately conformally invariant since the corresponding approximate system (11) and (14) is conformally invariant [1, 2]. Solutions of equation (15) for functions $F(w)$ given in (4) have the form

$$f(w) = \begin{cases} -\dfrac{a}{\lambda[k(k+2)+3]}w^{-k} - \dfrac{b}{k+1}w - \dfrac{c}{k}, & k \neq 0, -1, \\ -c\ln w - bw - \dfrac{1}{3}(2c + a/\lambda), & k = 0, \\ -w(a/2\lambda + b\ln w) + c, & k = -1. \end{cases} \qquad (16)$$

The solution of the system (11) and (14) is the function

$$w = \pm[\lambda(x_\nu + a_\nu)(x^\nu + a^\nu)]^{-1/2}, \qquad (17)$$

where a_ν are arbitrary constants.

When $\lambda = 0$, the non-trivial condition of splitting of equation (12) compatible with the equation $\Box w = 0$ is

$$w_\mu w^\mu = 1. \qquad (18)$$

So, in this case we find approximate solutions of equation (1) by means of expressions (2) and (10), where function $f(w)$ is determined from the equation

$$\ddot{f} + F(w) = 0 \qquad (19)$$

and w, in turn, is determined from the system

$$\Box w = 0, \quad w_\mu w^\mu = 1. \qquad (20)$$

The system (20) is invariant under the extended Poincaré group $\tilde{P}(1,4)$ and has solution [1]

$$w = \alpha x + a, \quad \alpha_\nu \alpha^\nu = 1, \qquad (21)$$

where a, α_ν are arbitrary constants.

In particular, equation

$$\Box u + \varepsilon u = 0 \qquad (22)$$

is approximately invariant under the group $\tilde{P}(1,4)$ on the subset of solutions

$$u = w - \varepsilon\left(\frac{1}{6}w^3 + a_1 w + a_2\right), \qquad (23)$$

where w is given in (21) and a_1, a_2 are arbitrary constants.

In conclusion, let us note some generalisations of the concept of approximate symmetry studied in this paper. First of all, obviously, one can consider higher orders of approximation of u in ε, i.e. $u = w + \varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + \cdots$, and can study the symmetry of the corresponding approximate system of PDE for functions w, $v^{(1)}$, $v^{(2)}$, and so on. Secondly, one can expand in ε-series not only dependent variables, but also independent ones, e.g. $x_0 \equiv t = x + \varepsilon z^{(1)} + \varepsilon^2 z^{(2)} + \cdots$, and can construct in this way the corresponding approximate system and then study its symmetry. Another approach to the study of approximate symmetry it to use some special approximations, say the two-point Padé approximants

$$u = \sum_{k=0}^{m} \varepsilon^k f_k \left(\sum_{j=0}^{n} \varepsilon^j g_j \right)^{-1}, \quad m, n < \infty, \tag{24}$$

where functions f_k, g_j are determined from the condition: when $\varepsilon \to 0$ expression (24) coincides with the expansion

$$u = v^{(0)} + \varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + \cdots, \quad \varepsilon \ll 1$$

and when $\varepsilon \to \infty$ (24) coinsides with the expansion

$$u = w^{(0)} + \varepsilon^{-1} w^{(1)} + \varepsilon^{-2} w^{(2)} + \cdots, \quad \varepsilon \gg 1.$$

We also note that the symmetry of a system of PDE which approximates the non-linear wave equation was studied by Shulga [6]. Using symmetry properties, Mitropolsky and Shulga [3] obtained some asymptotic solutions of the non-linear wave equation.

Note added. Readers who are less well acquinted with work in this might refer to the related work of Winternitz et al [7] which is also concerned with this type of non-linear wave equation from a symmetry point of view.

1. Fushchych W.I., Shtelen W.M., Serov N.I., Symmetry analysis and exact solutions of nonlinear equations of mathematical physics, Kyiv, Naukova Dumka, 1989.
2. Fushchych W.I., Zhdanov R.Z., Preprint N 468, University of Minnesota, 1988.
3. Mitropolsky Yu.A., Shulga M.W., *Dokl. Akad. Nauk*, 1987, **295**, № 1, 30–33.
4. Olver P., Applications of Lie groups to differential equations, Berlin, Springer, 1986.
5. Ovsyannikov L.V., Group analysis of differential equations, Moscow, Nauka, 1978.
6. Shulga M.W., in Symmetry and Solutions of Nonlinear Equations of Mathematical Physics, Kyiv, Institute of Mathematics, 1987, 96–99.
7. Winternutz P., Grundland A.M., Tuszynski J.A., *J. Math. Phys.*, 1987, **28**, 2194–2212.