On one- and two-particle Galilei-invariant wave eguations for any spin

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The problem of the motion of any spin charged particle in Coulomb field is solved by using the Galilei-invariant wave equations, which have been obtained by the authors recently. Galilei-invariant motion equations for a system of two interacting particles of any spin are deduced.

Решается проблема движения заряженной частицы с произвольным спином в Кулоновском поле, используя волновые уравнения, инвариантные относительно преобразований Галилея, которые были получены авторами ранее. Выводятся уравнения движения, инвариантные относительно преобразований Галилея, для системы двух взаимодействующих частиц с произвольным спином.

1. Introduction

The description of motion of the a charged spinning particle in a central field is one of the important problems of quantum mechanics. But the formulation of this problem for particles with spin $s > \frac{1}{2}$ is confronted with principle difficulties because of such fundamental relativistic equations as Kemmer–Duffin, Proca ones and others lead to contradictions when one tries to depict the interaction of the spin-one particle with the Coulomb field. Among them are the particle fall on centrum, the absence of stable solutions, nonrenormalizability and many others [1, 2]. The well-known paradoxes which arise by relativistic description of the interaction of highest-spin particles with an electromagnetic field are connected with the breakdown of causality (see, e.g., [3]).

In the present paper the problem of a the motion of charged particle with any spin in Coulomb field is solved by using Galilei-invariant wave equations (GIWE). The interest for such equations has been awaked by the paper of Levi-Leblond [4], who has obtained the GJWE for a particle of spin $\frac{1}{2}$. The Levi-Leblond equation as well as Dirac one gives the correct description of Pauli interaction of particle spin with a magnetic field. Unfortunately neither Levi-Leblond equation nor its generalization for any spin, obtained by Hagen and Hurley [5, 6], take into account such an important physical effect as spin-orbit coupling.

In papers [7–11] the GIWE for any spin particles are found which describe the spin-orbit interaction. These equations do not have pretensions to give a complete description of charged-particle interaction with an electromagnetic field, but they design adequately the physical situation in the cases in which the particle energy is too small to be enough for the pair creation, — i.e. when the one-particle Dirac equation is applicable. In spite of the absence of relativistic invariance, the equations found in [7–11] describe correctly the spin-orbit, Darwin and quadrupole couplings of any spin particle with an external field. It means specifically that the mentioned couplings are not to be interpreted as a relativistic corrections without fail, but may be described consistently in the frame of the Galilean-invariant approach.

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In this paper the explicit solutions of GIWE [9, 10] are found for the case of interaction of any spin particle with Coulomb field. The analog of Sommerfeld formula for any spin is obtained. It is demonstrated that by the solution of GIWE the difficulties do not arise, which characterize the relativistic equations, but at the same time the fine structure of Galilean particle spectrum contains the contribution from spin-orbit coupling.

Besides the problem of the description of particle interaction with an external field the two-body quantum-mechanical problem is of great interest for physics. In the last years such an interest is additionally stimulated by the successes in meson masses description in the frame of quark models.

In present paper, starting from one-particle equations [7, 9, 10] two-body GIWE are derived for particles of any spin. For the case in which the particle spins are equal to $\frac{1}{2}$ the equation is obtained, which leads to the same fine and hyperfine spectrum structure as the Breit one [12] and is explicitly invariant under the Galilei group, whereas the Breit equation is invariant neitlier under Galilean group nor under the Poincaré one.

2. GIWE of first order

Here we consider the systems of partial differential equations of a from

$$L\Psi \equiv (\beta_\mu p^\mu + \beta_5 m)\,\Psi = 0, \qquad \mu = 0,1,2,3, \tag{2.1}$$

where $p^0 = i(\partial/\partial t)$, $p^a = i(\partial/\partial x_a)$, β_μ and β_5 are $(n\times n)$-dimensional square matrices, Ψ is n-component function, m is c-number.

A great deal of papers are devoted to the description of relativistic equations of type (2.1), but Galilean-invariant equations of first order remain almost nonstudied ones.

A wide class of GIWE of type (2.1) is obtained in papers [9, 10], the main result of which are used here.

Equation (2.1) is invariant under Galilei transformations

$$\boldsymbol{x} \to \boldsymbol{x}' = R\boldsymbol{x} + \boldsymbol{v}t + \boldsymbol{a}, \qquad t \to t' = t + b \tag{2.2}$$

if a set of $(n\times n)$-dimensional matrices S_a and λ_a $(a = 1,2,3)$ exists, which satisfies the relations [9, 10]

$$[S_a, S_b] = i\varepsilon_{abc}S_c, \qquad [S_a, \lambda_b] = i\varepsilon_{abc}\lambda_c, \qquad [\lambda_a, \lambda_b] = 0; \tag{2.3}$$

$$\begin{gathered}\lambda_a^\dagger\beta_0 - \beta_0\lambda_a = 0, \qquad \lambda_a^\dagger\beta_5 - \beta_5\lambda_a = i\beta_a,\\ \lambda_a^\dagger\beta_b - \beta_b\lambda_a = -i\delta_{ab}\beta_0, \qquad [S_a, \beta_5] = [S_a, \beta_0] = 0.\end{gathered} \tag{2.4}$$

If eq.(2.1) admits the Lagrangian formulation, eqs.(2.3), (2.4) give necessary and sufficient conditions of its Galilean invariance [9]. The sufficiency of these conditions is rather obvious, as soon as the following relations may be obtained from (2.3), (2.4):

$$[L, P_\mu] = [L, J_a] = 0, \qquad [L, G_a] = \left(\lambda_a^\dagger - \lambda_a\right)L, \tag{2.5}$$

where P_μ, J_a, G_a are the Galilei group generators

$$\begin{gathered}P_0 = i\frac{\partial}{\partial t}, \qquad P_a = p_a = -i\frac{\partial}{\partial x_a},\\ J_a = \varepsilon_{abc}x_b p_c + S_a, \qquad G_a = tp_a - mx_a + \lambda_a.\end{gathered} \tag{2.6}$$

One concludes from (2.5) that the Lie algebra of Galilei group is realized on the set of eq.(2.1) solutions.

So, to describe all GIWE in the form (2.1), it is necessary to solve the system of matrix relations (2.3), (2.4). The simplest (i.e. realized by the matrices of minimal dimensions) solutions of these relations are [10]

$$\begin{aligned}
&\beta_0 = \begin{pmatrix} a_{nn}\otimes I & 0 \\ 0 & 0_{n-1\,n-1}\otimes\hat{1} \end{pmatrix}, \qquad \beta_5 = 2\begin{pmatrix} b_{nn}\otimes I & 0 \\ 0 & c_{n-1\,n-1}\otimes\hat{1} \end{pmatrix},\\
&\beta_a = \frac{i}{s}\begin{pmatrix} d_{nn}\otimes\hat{S}_a & e_{n\,n-1}\otimes K_a^\dagger \\ (e_{n\,n-1})^\dagger\otimes K_a & 0_{n-1\,n-1}\otimes\hat{1} \end{pmatrix},\\
&S_a = \begin{pmatrix} I_{nn}\otimes\hat{S}_a & 0 \\ 0 & I_{n-1\,n-1}\otimes\hat{S}'_a \end{pmatrix},\\
&\lambda_a = \frac{1}{2s}\begin{pmatrix} f_n\otimes\hat{S}_a & g_{n\,n-1}\otimes K_a^\dagger \\ h_{n-1\,n}\otimes K_a & 0_{n-1\,n-1}\otimes\hat{1} \end{pmatrix},
\end{aligned} \tag{2.7}$$

where $\hat{S}_a$ and $\hat{S}'_a$ are the matrices, which realize irreducible representations $D(s)$ and $D(s-1)$ of O_3 algebra, K_a are the $(2s-1)\times(2s+1)$-dimensional matrices, determined by the relations

$$K_a\hat{S}_b - \hat{S}'_b K_a = i\varepsilon_{abc}K_c, \qquad S_aS_b + K_a^\dagger K_b = is\varepsilon_{abc}S_c + s^2\delta_{ab}, \tag{2.8}$$

I and $\hat{1}$ are unit matrices of dimension $(2s+1)\times(2s+1)$ and $(2s-1)\times(2s-1)$. The symbols A_{nl} signify $(n\times l)$-dimensional matrices $(n=2,3)$, whose nonzero matrix elements are

$$\begin{aligned}
&(a_{22})_{11} = (b_{22})_{22} = c^2c_{11} = (d_{22})_{12} = -(d_{22})_{21} = c(e_{21})_{11} = \\
&\quad = (f_{22})_{21} = c^{-1}(h_{12})_{11} = (I_{22})_{jj} = 1, \qquad j=1,2,
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
&(a_{33})_{12} = (a_{33})_{21} = (b_{33})_{23} = a(b_{33})_{13} = a(b_{33})_{22} = a(b_{33})_{31} = \\
&\quad = 2a^2(b_{33})_{12} = 2a^2(b_{33})_{21} = -a(c_{22})_{11} = -(d_{33})_{31} = (d_{33})_{13} = \\
&\quad = (f_{33})_{21} = (f_{33})_{32} = c^{-1}(h_{23})_{11} = c^{-1}(h_{23})_{22} = -c^{-1}(g_{32})_{31} = 1, \\
&(c_{22})_{12} = (c_{22})_{21} = c^{-1}(e_{32})_{12} = s-1, \qquad (e_{32})_{21} = cs, \\
&(I_{33})_{jj} = 1, \qquad j=1,2,3,
\end{aligned} \tag{2.10}$$

where $c=2s-1$, a is an arbitrary parameter.

The formulae (2.1), (2,7), (2.9) determine the GIWE, which are equivalent to Hagen–Hurley ones [5, 6]. These equations may be interpreted as Galilean-invariant motion equations of a free particle with spin s [6].

Equations (2.1) (2.7), (2.10) also describe a free Galilean particle of spin s and mass m, but in contrast to (2.1), (2.7), (2.9) these equations after minimal substitution

$$p_\mu \to \pi_\mu = p_\mu - eA_\mu \tag{2.11}$$

describe the spin-orbit coupling of a particle with a field. They are just the equations to be used to solve the problem of any spin particle motion in Coulomb field.

3. Equations in the Schrödinqer form

Consider together with (2.1) GIWE for spin-s particle in the Schrödinger form

$$i\frac{\partial}{\partial t}\Psi = H_s(\boldsymbol{p})\Psi, \tag{3.1}$$

where $H_s(\boldsymbol{p})$ is second-order differential operator, Ψ is $2(2s+1)$-component wave function. Such equations will be used as a basis for a construction of two-particle GIWE.

Equation (3.1) is invariant under Galilei transformations (2.2), if the Hamiltonian H_s, satisfy the following commutation relations:

$$[H_s, P_a] = [H_s, J_a] = 0, \qquad [H_s, G_a] = iP_a, \tag{3.2}$$

where P_a, G_a, J_a are the Galilei group generators (2.6). Without loss of generality the matrices S_a and λ_a from (2.6) may be taken in the form

$$S_a = \begin{pmatrix} \hat{S}_a & 0 \\ 0 & \hat{S}_a \end{pmatrix}, \qquad \lambda_a = \begin{pmatrix} 0 & 0 \\ \hat{S}_a & 0 \end{pmatrix}, \tag{3.3}$$

where $\hat{S}_a$ are the generators of irreducible representation $D(s)$ of O_3 group, 0 is the $(2s+1)$-row zero matrix.

The generators (2.6), (3.3) form (together with H_s) the Lie algebra of extended Galilei group, satisfying the following commutation relations:

$$\begin{aligned} &[P_\mu, P_\nu] = [G_a, G_b] = 0, \qquad [P_a, G_b] = i\delta_{ab}m, \\ &[m, J_a] = [m, P_\mu] = [m, G_a] = 0, \qquad [J_a, \{P_b, G_b, J_b\}] = i\varepsilon_{abc}\{P_c, G_c, J_c\}. \end{aligned} \tag{3.4}$$

Algebra $\{P_\mu, J_a, G_a, m\}$ has three invariant (Casimir) operators

$$C_1 = P_0 - \frac{P^2}{2m}, \qquad C_2 = m, \qquad C_3 = \sum_s (mJ_a - \varepsilon_{abc}p_bG_c)^2. \tag{3.5}$$

The eigenvalues of the operators C_1, C_2 and C_3 are associated with internal energy, mass and square of the spin of a particle, described by eq.(3.1).

So the problem of finding GIWE in the form (3.1) reduces to the determination of explicit expressions of operators H_s, satisfying the relations (3.2), (2.6), (3.3). In [7, 9, 10] the Hamiltonians H_s have been obtained in such a form:

$$H_s = \sigma_1 m + 2\sigma_3 S_a p_a + \frac{1}{2m}C_{ab}p_a p_b, \tag{3.6}$$

where

$$C_{ab} = \delta_{ab} - 2(\sigma_1 - i\sigma_2)(S_aS_b + S_bS_a), \tag{3.7}$$

σ_a are $2(2s+1)$-row Pauli matrices, commuting with S_a (3.3). One can make sure directly that the operators (3.6), (2.6), (3.3) satisfy conditions (3.2), and the operators (3.5) eigenvalues are equal to

$$c_1 = \pm m, \qquad c_2 = m, \qquad c_3 = m^2 s(s+1).$$

Therefore, one concludes that eqs.(3.1), (3.6) are Galilean invariant and describe a nonrelativistic partic1es of mass m and spin s.

The motion equation for a charged particle in an external electromagnetic field can be obtained from (3.1), (3.6) via standard substitution (2.11). As a result one obtains

$$i\frac{\partial}{\partial t}\Psi = \left(\sigma_1 m + 2\sigma_3 S_a\pi_a + \frac{1}{2m}C_{ab}\pi_a\pi_b + eA_0\right)\Psi. \tag{3.8}$$

Equation (3.8) (as well as the first-order equation (2.1) after substitution (2.11)) will be invariant under Galilei transformations [10] if the vector potential will be similtaneously transformed according to [4]

$$\boldsymbol{A} \to \boldsymbol{A}' = R\boldsymbol{A}, \qquad A_0 \to A_0' = A_0 + \boldsymbol{V}\cdot\boldsymbol{A}. \tag{3.9}$$

It is demonstrated in [7, 9] that eq.(3.8) describes dipole, quadrupole and spin-orbit coupling of a charged particle with an external field. In the case $s = \frac{1}{2}$ such a description is in good accordance with that given by Dirac equation.

4. Energy spectrum of any spin particle in Coulomb field

In this section GIWE are applied to solve the problem of the description of any spin particle movement in Coulombic field.

After minimal substitution (2.11) one comes from (2.1) to the equation

$$L(\pi)\Psi = 0, \qquad L(\pi) = \beta_\mu\pi^\mu + \beta_5 m. \tag{4.1}$$

Here we find the exact solutions of eqs.(4.1), (2.7), (2.10) for the case in which the external field is reduced to Coulomb potential

$$A = 0, \qquad A_0 = -\frac{ze}{x}. \tag{4.2}$$

Simultaneonsly we obtain an approximate solution of Schrödinger type equation (3.8), (4.2).

To simplify eqs.(4.1), (2.7), (2.10), it is convenient to use the transformation

$$\Psi \to \Psi' = U^{-1}\Psi, \qquad L(\pi) \to L'(\pi) = U^\dagger L(\pi) U, \tag{4.3}$$

where $U = \exp[i((\boldsymbol{\lambda}\cdot\boldsymbol{p})/m)]$. As a result, using Campbell–Hausdorf formula and taking into account relations (2.4), one obtains the following equivalent equation:

$$L'(\pi)\Psi' \equiv \left[\beta^0\left(\pi_0 - \frac{\boldsymbol{p}^2}{2m} + \frac{e}{m}\boldsymbol{\lambda}\cdot\boldsymbol{E}\right) + \beta_5 m\right]\Psi' = 0, \qquad \boldsymbol{E} = \frac{-ez\boldsymbol{x}}{x^3}. \tag{4.4}$$

Let us use the notation

$$\Psi' = \text{column}\,(\Psi_1, \Psi_2, \Psi_3, \chi_1, \chi_2), \tag{4.5}$$

where Ψ_a are $(2s+1)$-component functions, χ_α are $(2s+1)$-component ones. Substituting (4.5) into (4.4), one obtains using (2.7), (2.8), (2.10)

$$\left(p_0 + \frac{ze^2}{x} - \frac{\boldsymbol{p}^2}{2m} - \frac{ze^2 g}{2sm}\frac{\hat{S}\cdot\boldsymbol{x}}{x^3}\right)\Psi_1, \qquad g = \frac{a}{2}. \tag{4.6}$$

According to (4.4), (2.7), (2.10), $\chi_1 = \chi_2 = 0$ and the functions Ψ_2, Ψ_3 are expressed via Ψ_1:

$$\Psi_2 = -\frac{1}{a}\Psi_1, \qquad \Psi_3 = -\frac{1}{2m}\left(\pi_0 - \frac{\boldsymbol{p}^2}{2m} - \frac{m}{a^2}\right)\Psi_1. \tag{4.7}$$

So eqs.(4.1), (4.2), (2.7), (2.10) reduce to eq.(4.6) for the $(2s+1)$-component wave function Ψ_1. It is demonstrated in [7, 9] that Schrödinger-type equations (3.8), (4.2) also may be reduced to eq.(4.6) (with $g = \pm s$) by the consequent approximate transformations of Foldy–Wouthuysen [13] type.

The solutions of eq.(4.6), which correspond to states with energy ε, can be written in a form $\Psi_1 = \exp[-i\varepsilon t]\Psi(\boldsymbol{x})$. Taking into account the symmetry of eq.(4.6) under group O_3, it is convenient to represent $\Psi(\boldsymbol{x})$ as a linear combination of spherical spinors

$$\Psi(\boldsymbol{x}) = \varphi_\lambda(\boldsymbol{x})\Omega^s_{j\,j-\lambda m}, \\ \lambda = -s, -s+1, \ldots, -s+2n_{sj}, \qquad n_{sj} = \min(s,j), \tag{4.8}$$

where $\Omega^s_{j\,j-\lambda m} = \Omega^s_{j\,j-\lambda m}(\boldsymbol{x}/x)$ are the eigenfunction of the operators $\boldsymbol{J}^2$, J^3 and $\boldsymbol{L}^2$ ($\boldsymbol{J} = \boldsymbol{x}\times\boldsymbol{p} + \hat{\boldsymbol{S}} \equiv \boldsymbol{L} + \boldsymbol{S}$) with eigenvalues $j(j+1)$, m, and $(j-\lambda)(j-\lambda+1)$. Substituting (4.8) into (4.6), one obtains the following equations for radial functions $\varphi_\lambda(\boldsymbol{x})$:

$$D\varphi_\lambda(\boldsymbol{x}) = x^{-2}b_{\lambda\lambda'}\varphi_{\lambda'}, \tag{4.9}$$

where

$$D = 2m\left(\varepsilon + \frac{\alpha}{x}\right) + \frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} - \frac{j(j+1)}{x^2}, \\ b_{\lambda\lambda'} = \left[\lambda^2 - \lambda(2j+1)\right]\delta_{\lambda\lambda'} + \frac{g\alpha}{s}a_{\lambda\lambda'}, \qquad \alpha = ze^2, \tag{4.10}$$

$a_{\lambda\lambda'}$ are the matrix elements of operator $\boldsymbol{S}\cdot\boldsymbol{x}/x$ in basis $\{\Omega^s_{j\,j-\lambda m}\}$, determined by the relation

$$\frac{\boldsymbol{S}\cdot\boldsymbol{x}}{x}\Omega^s_{i\,j-\lambda m} = a_{\lambda\lambda'}\Omega^s_{j\,j-\lambda' m}. \tag{4.11}$$

The values of $a_{\lambda\lambda'}$ for $s = \frac{1}{2}$ are well known (see, e.g., [14]). These values for spin are calculated in the appendix and are

$$a_{\lambda\lambda'} = -\frac{1}{2}(\delta_{\lambda\,\lambda'+1}a_{\lambda+s} + \delta_{\lambda\,\lambda'-1}a_{\lambda+s+1}), \\ a_\mu = \left[\frac{\mu(d_j-\mu)(d_s-\mu)(d_{js}-\mu)}{(d_{sj}-2\mu-1)(d_{sj}-2\mu+1)}\right]^{1/2}, \tag{4.12}$$

where

$$d_s = 2s+1, \qquad d_j = 2j+1, \qquad d_{sj} = d_s + d_j, \\ \mu = s+\lambda = 0, 1, 2, \ldots, 2n_{sj}, \qquad n_{sj} = \min(s,j). \tag{4.13}$$

The matrix $\|b_{\lambda\lambda'}\|$ commutes with the operator D (4.10) and is diagonalizable, so the system (4.9) can be reduced to the system of noncoupled equations

$$D\varphi = x^{-2}b^{sj}\varphi, \tag{4.14}$$

where D is operator (4.9), b^{sj} are the matrix $\|b_{\lambda\lambda'}\|$ eigenvalues. Any equation (4.14) in ones turn reduces to the well-known equation [15]

$$z\frac{d^2y}{dz^2} + \frac{dy}{dz} + \left(\beta - \frac{z}{4} - \frac{k^2}{4z}\right)y = 0, \tag{4.15}$$

where

$$y = \sqrt{z}\varphi, \qquad z = 2\sqrt{-2m\varepsilon}x, \qquad \beta = \sqrt{\frac{-m}{2\varepsilon}}\alpha^2, \qquad k^2 = d_j^2 + 4b^{sj}. \tag{4.16}$$

The eq.(4.15) solutions for coupled states $|\varepsilon < 0|$ are expressed via Laguerre polinomials, and parameter β takes the values [15]

$$\beta = \frac{k+1}{2} + n', \qquad n' = 0, 1, 2, \ldots. \tag{4.17}$$

From (4.16), (4.17) one obtains

$$\varepsilon = -\frac{m\alpha^2}{\left(\sqrt{\left(j+\frac{1}{2}\right)^2 + b^{sj}} + n' + \frac{1}{2}\right)^2}. \tag{4.18}$$

Formula (4.18) gives the energy levels of a nonrelativistic spinning particle in Coulomb field. Parameter b^{sj} in (4.18) takes the values determined as the roots of the matrix (4.10) characteristic equation

$$\det\|b_{\lambda\lambda'} - b^{sj}\delta_{\lambda\lambda'}\| \equiv \det\left\|\left(\lambda^2 - \lambda d_j - b^{sj}\right)\delta_{\lambda\lambda'} + \frac{\alpha g}{s}a_{\lambda\lambda'}\right\| = 0, \tag{4.19}$$

where $a_{\lambda\lambda'}$ are given in (4.12). The eq.(4.18) solutions and the analysis of spectrum (4.18) are given in the next section.

5. Discussion of formula (4.18)

Formula (4.19) determines an algebraic equation of order $2n_{sj} + 1$. This equation can be resolved in radicals only for $s \leq \frac{3}{2}$ or $j \leq \frac{3}{2}$. To analyse the spectrum (4.18) for arbitrary s and j it is convenient represent the eq.(4.19) solutions in such a form:

$$b^{sj} = \lambda^2 - \lambda d_j + (g\alpha)^2 b^{sj} + o(g\alpha)^4, \tag{5.1}$$

where we suppose that $\alpha \ll 1$. By using (4.12), (4.19), (5.1), it is not difficult to obtain the explicit expressions for

$$b_\lambda^{sj} = \frac{1}{8s^2}\left(\frac{a_{\lambda+s}^2}{j+1-\lambda} - \frac{a_{\lambda+s+1}^2}{j-\lambda}\right), \tag{5.2}$$

where a_μ are the coefficients (4.12).

Using (5.2) and expanding the function (4.18) in powers of α^2, one obtains

$$\begin{gathered} \varepsilon = -\frac{m\alpha^2}{n^2} + \frac{mg^2\alpha^4 b_\lambda^{sj}}{n^2\left(l+\frac{1}{2}\right)} + o\left(\alpha^6\right), \\ n = n' + j - \lambda + 1 = 1, 2, \ldots, \qquad l = j - \lambda = 0, 1, \ldots, n-1. \end{gathered} \tag{5.3}$$

Formula (5.3) determines the fine structure of the energy spectrum of any spin particle in Coulomb field. The parameters b_λ^{sj} in (5.3) are easily calculated by formulae (5.2), (4.12).

The first member on the r.h.s. of (5.3) gives the well-known Schrödinger energy levels of a nonrelativistic particle in Coulombic field. The second term gives the correction of order α^4 which is connected with the existence of particle spin. As will be shown below, this correction corresponds to spin-orbit and Darwin coupling of a particle with a field.

According to (5.2), (5.3) any energy level, corresponding to possible value of main quantum number n, is splitted to $n-1$ sublevels, corresponding to possible values of l. Besides any level with fixed n and l is additionally splitted into sublevels, the number of which is $2n_{sj}+1$, $n_{sj}=\min(s,j)$. In contrary to the relativistic case the energy levels of a nonrelativistic particle of spin $\frac{1}{2}$ in Coulomb field are nondegenerated.

Consider the spectrum (5.2), (5.3) for $s\leq 1$ and $j\leq 1$. Using (4.12), one obtains from (5.2)

$$\begin{aligned} & b_\lambda^{0j}=0, \qquad \lambda=0; \qquad b_\lambda^{\frac{1}{2}j}=2\lambda d_j^{-1}, \qquad \lambda=\pm\frac{1}{2}; \\ & b_\lambda^{1j}=\lambda\frac{d_j+\lambda}{2d_j(d_j-\lambda)}-\frac{2(1-\lambda^2)}{d_j^2-1}, \qquad \lambda=\begin{cases} -1, & j=0, \\ -1,0,1, & j\neq 0; \end{cases} \\ & b_\lambda^{s0}=0, \qquad \lambda=-s; \qquad b_\lambda^{s\frac{1}{2}}=(-1)^{s+\lambda+1}(2sd_s)^{-1}, \qquad \lambda=-s,-s+\frac{1}{2}; \\ & b_\lambda^{s1}=\frac{(s+\lambda-1)(d_s+s+\lambda-1)}{2sd_s(d_s-s-\lambda+1)}+\frac{(s+\lambda-1)^2-1}{2s^2(s+1)}, \\ & \lambda=-s,-s+1,-s+2. \end{aligned} \tag{5.4}$$

For $s=0$ formulae (5.3), (5.4) give the well-known energy spectrum of a spinless nonrelativistic particle in Coulomb field. For $s=\frac{1}{2}$ one obtains from (5.3), (5.4)

$$\varepsilon=-\frac{m\alpha^2}{n^2}+\frac{\lambda m g^2\alpha^4}{n^3\left(l+\frac{1}{2}\right)\left(l+\frac{1}{2}+\lambda\right)}, \qquad \lambda=\pm\frac{1}{2}. \tag{5.5}$$

It is interesting to compare (5.5) with the fine-structure formula for a Dirac electron interacting with Coulomb field. One can make sure itself that for $g^2=-1$ the energy levels (5.5) may be represented as

$$\varepsilon=\varepsilon_D-\left\langle\frac{p^4}{8m^2}\right\rangle, \tag{5.6}$$

where ε_D gives the energy levels of a Dirac electron in Coulombic field, and average is taken in Schrödinger wave functions.

According to (5.6), formula (5.5) takes into account all "relativistic" corrections predicted by Dirac equation, except the relativistic kinetic-energy correction $\langle p^4/8m^2\rangle$. It means that formula (5.5) takes into account the contributions of spin-orbit and Darwin couplings, and so these couplings may be described in frame of Galilei-invariant theory.

Let us give for the completeness the exact solutions of eqs.(4.19) for $s\leq 1$ and $j\leq 1$

$$\begin{aligned} & b^{0j}=0, \qquad b^{\frac{1}{2}j}=\frac{1}{4}\pm\frac{1}{2}\sqrt{d_j^2+4(g\alpha)^2}, \\ & b^{1\frac{1}{2}}=\frac{c}{3}+2\sqrt{-c}\cos\left[\frac{1}{3}\left(\gamma+\lambda\frac{\pi}{2}\right)\right], \qquad \lambda=0,\pm 1, \quad j\neq 0, \\ & b^{s0}=0, \qquad b^{s\frac{1}{2}}=\frac{1}{4}(d_s^2-3)\pm\frac{1}{2}\sqrt{d_s^2+\left(\frac{g\alpha}{s}\right)^2}, \end{aligned} \tag{5.7}$$

$$b^{s1} = s(s+1) - 2 + \frac{d}{3} + 2\sqrt{-d}\cos\left[\frac{1}{3}\left(\xi + \mu\frac{\pi}{2}\right)\right], \quad \mu = 0, \pm 1, \ s \neq 0,$$

where

$$\cos\gamma = \frac{b}{\sqrt{-c^3}}, \qquad b = \frac{2}{3}(g\alpha)^2 + \frac{1}{3}d_j^2 - \frac{1}{27}, \qquad a = -(g\alpha)^2 - \frac{4}{27} - b,$$

$$\cos\xi = \frac{f}{\sqrt{-d_3}}, \qquad f = \frac{2}{3}\left(\frac{g\alpha}{s}\right)^2 + \frac{1}{3}d_s^2 - \frac{1}{27}, \qquad d = -\left(\frac{g\alpha}{s}\right)^2 - \frac{4}{27} - f.$$

Contrary to the approximate formulae (5.3), (5.4) relations (4.18), (5.7) give the exact values of the energy levels predicted by eqs.(4.1), (4.2). Using (4.18), (4.19) it is not difficult to obtain the exact spectrum also for $s = \frac{3}{2}$ and any j, and for $j = \frac{3}{2}$ and any s. We do not give the corresponding cumber-some formulae here.

6. Two-particle equations

The Breit equation [12] is an important and often used one in the quantum-mechanical two-body problem. Besides a lot of incontrovertible merits, this equation has the shortcoming of principle — it is not invariant either under Poincaré or under Galilei group. So the Breit equation does not satisfy any relativity principle accepted in physics.

In this section we find two-particle wave equation, which describes the system of electrically charged spin-$\frac{1}{2}$ particles with the same accuracy as the Breit equation, but is Galilei invariant.

Starting from one-particle Schrödinger-like GIWE (3.1), one may write the equation for a system of two noninteracting particles in such a form:

$$i\frac{\partial}{\partial t}\Psi\left(t, \boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}\right) = \left(H_{s_{(1)}} + H_{s_{(2)}}\right)\Psi\left(t, \boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}\right), \tag{6.1}$$

where $\Psi\left(t, \boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}\right)$ is the $2(2s_{(1)}+1) \times 2(2s_{(2)}+1)$-component wave function, $H_{s_{(1)}}$ and $H_{s_{(2)}}$ are the Hamiltonians of the first and second particle — i.e. differential operators of form (3.6). Here and below we use the indices (1) and (2) to distinquish the quantitics related to the first and second particle.

Equation (6.1) is manifestly invariant under Galilei group. The Galilei group generators on the set of eq.(6.1) solutions are represented as a direct sum of single-particle generators (2.6), (3.3).

It is convenient to arrive at (6.1) from individual variables $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ to c.m. ones. Previously we transform eq.(6.1) to such a representation, in which the internal energy operator

$$C_1 = H_{s_{(1)}} + H_{s_{(2)}} - \frac{\boldsymbol{P}^2}{2(m_{(1)} + m_{(2)})}$$

does not depend on total momentum $\boldsymbol{P} = \boldsymbol{p}^{(1)} + \boldsymbol{p}^{(2)}$. Using for this purpose the transformation operator (compare (4.3))

$$U = \exp\left[\frac{i\boldsymbol{\lambda}\cdot\boldsymbol{P}}{M}\right], \qquad \boldsymbol{\lambda} = \boldsymbol{\lambda}_{(1)} + \boldsymbol{\lambda}_{(2)}, \qquad M = m_{(1)} + m_{(2)}, \tag{6.2}$$

where $\boldsymbol{\lambda}_{(\alpha)} = \left(\sigma_1^{(\alpha)} - i\sigma_2^{(\alpha)}\right)\boldsymbol{S}^{(\alpha)}$, $\alpha = 1, 2$, one obtains from (6.1) the following equivalent equation:

$$i\frac{\partial}{\partial t}\Phi = \hat{H}\Phi, \qquad \Phi = U\Phi, \qquad \hat{H} = UHU^{-1} = \frac{\boldsymbol{P}^2}{2M} + E, \tag{6.3}$$

where E is the internal energy operator

$$\begin{aligned} E = {} & \sigma_1^{(1)} m_{(1)} + \sigma_1^{(2)} m_{(2)} + 2\sigma_3^{(1)}\boldsymbol{S}^{(1)}\cdot\boldsymbol{p} - 2\sigma_3^{(2)}\boldsymbol{S}^{(2)}\cdot\boldsymbol{p} + \\ & + \left[\frac{1}{\mu} - \left(\sigma_1^{(1)} - i\sigma_2^{(1)}\right)\frac{1}{m_{(1)}} - \left(\sigma_1^{(2)} - i\sigma_2^{(2)}\right)\frac{1}{m_{(2)}}\right]\frac{\boldsymbol{p}^2}{2}, \\ \mu = {} & \frac{m_{(1)}m_{(2)}}{M}, \qquad \boldsymbol{p} = \frac{m_{(1)}\boldsymbol{p}^{(1)} - m_{(2)}\boldsymbol{p}^{(2)}}{M}. \end{aligned} \tag{6.4}$$

So to describe the system of free nonrelativistic particles of spins $s_{(1)}$ and $s_{(2)}$ one may use the motion equation (6.1) in single particle variables, or eq.(6.3) in c.m. ones. The Galilei group generators on the sets of this equation solutions have the form

$$\begin{aligned} & P_0 = H_{s_{(1)}} + H_{s_{(2)}}, \qquad \boldsymbol{P} = \boldsymbol{p}^{(1)} + \boldsymbol{p}^{(2)}, \\ & \boldsymbol{J} = \boldsymbol{x}^{(1)} \times \boldsymbol{p}^{(1)} + \boldsymbol{x}^{(2)} \times \boldsymbol{p}^{(2)} + \boldsymbol{S}^{(1)} + \boldsymbol{S}^{(2)}, \\ & \boldsymbol{G} = t\boldsymbol{P} - m_{(1)}\boldsymbol{x}^{(1)} - m_{(2)}\boldsymbol{x}^{(2)} + \boldsymbol{\lambda} \end{aligned} \tag{6.5}$$

for eq.(6.1), and

$$\begin{aligned} & P_0^1 = \hat{H} + E + \frac{\boldsymbol{P}^2}{2m}, \qquad \boldsymbol{P}' = -i\frac{\partial}{\partial \boldsymbol{X}}, \\ & \boldsymbol{J}' = \boldsymbol{X} + \boldsymbol{P} + \boldsymbol{S}, \qquad \boldsymbol{S} = \boldsymbol{x} \times \boldsymbol{p} + \boldsymbol{S}^{(1)} + \boldsymbol{S}^{(2)}, \\ & \boldsymbol{G}' = t\boldsymbol{P} - M\boldsymbol{X}, \qquad \boldsymbol{X} = \left(m_{(1)}\boldsymbol{x}^{(1)} + m_{(2)}\boldsymbol{x}^{(2)}\right)M^1, \qquad \boldsymbol{x} = \boldsymbol{x}^{(1)} - \boldsymbol{x}^{(2)}, \end{aligned} \tag{6.6}$$

for eq.(6.3).

Starting from (6.1) or (6.3) one may look for motion equation for interacting particles in such a form:

$$i\frac{\partial}{\partial t}\Psi = [H_{s_{(1)}} + H_{s_{(2)}} + V]\Psi, \tag{6.7}$$

or

$$i\frac{\partial}{\partial t}\Phi = [H + \hat{V}]\Phi, \tag{6.8}$$

where V and $\hat{V}$ are interaction Hamiltonians. The requirement of Galilei invariance reduces to commutation of operators V and $\hat{V}$ with generators (6.5) and (6.6). It means that the operators can depend upon internal variables $\boldsymbol{x}$ and $\boldsymbol{p}$ only and be scalars under spatial rotations. Besides that V must satisfy the condition

$$[V, \boldsymbol{\lambda}] = 0. \tag{6.9}$$

It follows from (6.3), (6.9) that eq.(6.7) may be reduced to the from (6.8) using the transformation operator (6.2). So formula (6.8) gives a wider class of GIWE, as soon as eq.(6.8) in general is not reducible to the form (6.7).

So the condition of Galilei invariance give a wide choice of interaction Hamiltonians. Consider some examples which are interesting from the physical point of view.

1) Central potential $\hat{V} = If(x)$, where I is the unit matrix. The corresponding equation (6.3) in the c.m. frame takes the form

$$i\frac{\partial}{\partial t}\Phi(t,\boldsymbol{x}) = H(\boldsymbol{x},\boldsymbol{p})\Phi(t,\boldsymbol{x}), \tag{6.10}$$

where $H(\boldsymbol{x},\boldsymbol{p}) = E + If(x)$, E is operator (6.4).

To analyse eq.(6.10) we suppose the momentum $\boldsymbol{p}$ to be small enough: $\boldsymbol{p}^2 \ll m^2$. Applying to (6.10) the standard approximate diagonalization procedure of Barker–Glover–Chraplivy (BGC) [16], one comes to Hamiltonian

$$\begin{gathered}H' = \sigma_1^{(1)}m_{(1)} + \sigma_1^{(2)}m_{(2)} + \frac{p^2}{2\mu} + If(x) + \\ + \frac{1}{x}\left[\sigma_1^{(1)}\frac{\boldsymbol{S}^{(1)}\cdot\boldsymbol{x}\times\boldsymbol{p}}{2m_{(1)}} + \sigma_1^{(2)}\frac{\boldsymbol{S}^{(2)}\cdot\boldsymbol{x}\times\boldsymbol{p}}{2m_{(2)}}\right]\frac{\partial f}{\partial x}.\end{gathered} \tag{6.11}$$

In spite of the spin independence of $\hat{V}$, the approximate Hamiltonian (6.11) contains terms, which correspond to-spin-orbit coupling.

2) Breit potential

$$V_B = -\frac{e^2}{x} + 2\sigma_3^{(1)}\sigma_3^{(2)}\frac{e^2}{x}\left[\boldsymbol{S}^{(1)}\cdot\boldsymbol{S}^{(2)} + \frac{\left(\boldsymbol{S}^{(1)}\cdot\boldsymbol{x}\right)\left(\boldsymbol{S}^{(2)}\cdot\boldsymbol{x}\right)}{x^2}\right]. \tag{6.12}$$

The BGC reduction for Hamiltonian $\hat{H}_B = E + V_B$, where E and $\hat{V}$ are given in (6.4), (6.12), leads to the following result:

$$H \to \frac{\boldsymbol{P}^2}{2M} + H^{\rm int}, \tag{6.13}$$

where

$$\begin{gathered}H^{\rm int} = \sigma_1^{(1)}m_{(1)} + \sigma_1^{(2)}m_{(2)} + \frac{\boldsymbol{p}^2}{2\mu} - \frac{e^2}{x} - \frac{e^2\sigma_1^{(1)}\sigma_1^{(2)}}{2\mu M}\left(\boldsymbol{p}\frac{1}{x}\cdot\boldsymbol{p} + \boldsymbol{p}\cdot\boldsymbol{x}\frac{1}{x^2}\boldsymbol{x}\cdot\boldsymbol{p}\right) + \\ + \frac{e^2}{x^3}\left[\frac{\sigma_1^{(1)}\sigma_1^{(2)}}{2m_{(1)}m_{(2)}}\left(\boldsymbol{S}^{(1)} + \boldsymbol{S}^{(2)}\right) + \frac{1}{2m_{(1)}^2}\boldsymbol{S}^{(1)} + \frac{1}{2m_{(2)}^2}\boldsymbol{S}^{(2)}\right]\cdot\boldsymbol{x}\times\boldsymbol{p} - \\ - \frac{e^2\sigma_1^{(1)}\sigma_1^{(2)}}{2m_{(1)}m_{(2)}x^3}\left(\boldsymbol{S}^{(1)}\cdot\boldsymbol{S}^{(2)} - \frac{3\boldsymbol{S}^{(1)}\cdot\boldsymbol{x}\boldsymbol{S}^{(2)}\cdot\boldsymbol{x}}{x^2}\right) + \\ + 4\pi e^2\left(\frac{2}{3}\frac{\sigma_1^{(1)}\sigma_1^{(2)}}{m_{(1)}m_{(2)}}\boldsymbol{S}^{(1)}\cdot\boldsymbol{S}^{(2)} + \frac{1}{8m_{(1)}^2} + \frac{1}{8m_{(2)}^2}\right)\delta(\boldsymbol{x}).\end{gathered}$$

On the set of functions, satisfying $\sigma_1^{(1)}\Psi = \sigma_1^{(2)}\Psi = \Psi$, Hamiltonian (6.14) may be represented as

$$H^{\rm int} = H_B + \frac{\boldsymbol{p}^4}{8}\left(\frac{1}{m_{(1)}^3} + \frac{1}{m_{(2)}^3}\right),$$

where H_B is the approximate Breit Hamiltonian in the c.m. frame [16]. So GIWE (6.8) with potential (6.12) leads in approximation $1/m^2$ to the results, which are analogous to the ones predicted by Breit equation, but do not take into account the relativistic correction to kinetic energy.

3) Let us give the example of potential $\hat{V}'$, which, being substituted into (6.8), leads to the equation, which is Galilei invariant and is equivalent in approximation $1/m^2$ to the Breit one,

$$\hat{V}' = \frac{e_{(1)}e_{(2)}}{x}\left[1 - 2\sigma_3^{(1)}\sigma_3^{(2)}\left(a\boldsymbol{S}^{(1)}\cdot\boldsymbol{S}^{(2)} + b\frac{\boldsymbol{S}^{(1)}\cdot\boldsymbol{x}\boldsymbol{S}^{(2)}\cdot\boldsymbol{x}}{x^2}\right)\right] +$$
$$+ i\nu e_{(1)}e_{(2)}\left\{\left(1 - \tilde{\lambda}_{(1)} - \tilde{\lambda}_{(2)}\right)\left[\left(\frac{\tilde{\lambda}_{(1)}}{2m_{(1)}} + \frac{\tilde{\lambda}_{(2)}}{m_{(2)}}\right)\boldsymbol{S}^{(1)} - \left(\frac{\tilde{\lambda}_{(2)}}{2m_{(2)}} +\right.\right.\right.$$
$$\left.\left.\left. + \frac{\tilde{\lambda}_{(1)}}{m_{(1)}}\right)\boldsymbol{S}^{(2)}\right]\frac{\boldsymbol{x}}{x^3} + 2\pi i\tilde{\lambda}_{(1)}\tilde{\lambda}_{(2)}\left(1 + 2\boldsymbol{S}^{(1)}\cdot\boldsymbol{S}^{(2)}\right)\frac{\delta(\boldsymbol{x})}{m_{(1)}m_{(2)}}\right\},$$

where

$$\tilde{\lambda}_\alpha = \sigma_1^{(\alpha)} - i\sigma_2^{(\alpha)}, \qquad \alpha = 1 + \frac{1}{4}c, \qquad b = 1 + \frac{1}{2}c,$$
$$2\nu = a + b - 2, \qquad c = \left(m_{(1)}^2 + m_{(2)}^2\right)\left[m_{(1)}m_{(2)}\left(m_{(1)} + m_{(2)}\right)\right]^{-1}.$$

Applying the BGC reduction for the Hamiltonian $H = E + \hat{V}'$, one obtains the operator, which coincides for $\Psi' = \frac{1}{4}(1+\sigma_1^{(1)})(1+\sigma_1^{(2)})\Psi$ with the approximate Breit Hamiltonian in the c.m. frame.

One may conclude from the above that two-particle GIWE can be successfully applied for the description of interacting particles. The application of such equations to concrete physical problems will be considered in future publications.

Appendix

Explicit expression for the operator $(\boldsymbol{S}\cdot\boldsymbol{x})/x$ in spherical spinor basis

Spherical spinors $\Omega^s_{j\,j-\lambda m}$ are $(2s+1)$-component functions with the components

$$(\Omega^s_{j\,j-\lambda m})^\mu = C^{jm}_{j-\lambda\,m-\mu s\mu}Y_{j-\lambda\,m-\mu}, \tag{A.1}$$

where $C^{jm}_{j-\lambda\,m-\mu s\mu}$ are Clebah–Gordan coefficients, $Y_{j-\lambda\,m-\mu}$ are spherical harmonics. Substituting (A.1) into (4.11), choosing $\hat{\boldsymbol{x}} = \hat{x}_0 = (0,0,1)$ and taking into account that [17]

$$Y_{j-\lambda\,0}(\hat{x}_0) = \sqrt{\frac{2(j-\lambda)+1}{4\pi}}, \qquad (\boldsymbol{S}\cdot\hat{x}_0)_{\mu\mu'} = (S_2)_{\mu\mu'} = \mu\delta_{\mu\mu'}, \tag{A.2}$$

one comes to the following system of linear algebraic equations for $a_{\lambda\lambda'}$ [18]:

$$\sum_{\lambda'}(a_{\lambda\lambda'}-\mu\delta_{\lambda\lambda'})\sqrt{2(j-\lambda)+1}C^{j\mu}_{j-\lambda'\,0s\mu}=0, \tag{A.3}$$

where

$$\lambda,\lambda'=-s,-s+1,\ldots,-s+2n_{sj},\qquad n_{sj}=\min(s,j),$$
$$\mu=-n_{sj},-n_{sj}+1,\ldots,n_{sj}.$$

The solution of system (A.3) is given by formulae (4.12). For $s\le\frac{3}{2}$ one has specifically

$$s=0,\qquad a_{\lambda\lambda'}=0,\qquad s=\frac{1}{2},\qquad a_{\frac12\,-\frac12}=a_{-\frac12\,\frac12}=-1;$$

$$s=1,\qquad a_{10}=a_{01}=-\sqrt{\frac{j}{2j+1}},\qquad a_{0\,-1}=a_{-1\,0}=-\sqrt{\frac{j+1}{2j+1}},\qquad j\ne0;$$

$$s=\frac{3}{2},\qquad a_{\frac32\,\frac12}=a_{\frac12\,\frac32}=-\sqrt{\frac{j+1}{3j}},\qquad a_{-\frac32\,-\frac12}=a_{-\frac12\,-\frac32}=-\sqrt{\frac{j}{3(j+1)}};$$

$$a_{\frac12\,-\frac12}=a_{-\frac12\,\frac12}=-\frac{1}{3}\sqrt{\frac{(2j+3)(2j-1)}{j(j+1)}},\qquad j\ne\frac{1}{2};$$

$$a_{-\frac12\,-\frac32}=a_{-\frac32\,-\frac12}=-\frac{1}{3},\qquad j=\frac{1}{2}.$$

The remaining coefficients $a_{\lambda\lambda'}$ for $s\le\frac{3}{2}$ are equal to zero.

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