Conformal symmetry and new exact solutions of SU_2 Yang–Mills theory

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Conformally invariant and some others exact solutions of the SU_2 Yang–Mills (YM) theory are found. The final conformal transformations for the YM potentials and the formulae of generating new solutions from known ones are presented.

The field equations of the SU_2 invariant YM theory are formidable system of partial differential equations yet possessing a wide symmetry group of local transformations. This group is known to be the 15-parametrical conformal group $C_{1,3}$ and the gauge group SU_2. Recently [1] it was shown that SU_2 YM equations do not allow any other Lie symmetries.

By now the interest in classical YM theory has become so widespread that many workers are involved in the seach for new solutions. A comprehensive review of the known exact solutions of SU_2 YM theory is presented in [2]. These solutions are obtained by introducing various ansätze for the YM potentials. But the very question of finding relevant ansätze is rather obscure, although there can be no doubt that successes achieved are connected with the symmetry properties of YM equations.

In this note we exploit the symmetry of SU_2 YM equations to obtain some new exact solutions. Besides that we present the final conformal transformations for the YM potentials and construct formulae allowing us to generate "new" solutions of the equations, starting from an "old" known one.

The SU_2-invariaut YM equations have the form

$$\partial^\nu G^a_{\mu\nu} = e\varepsilon_{abc} g^{\alpha\nu} G^b_{\mu\alpha} W^c_\nu, \qquad a,b=1,2,3, \quad \mu,\nu=0,1,2,3, \tag{1}$$

where

$$G^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\nu + e\varepsilon_{abc} W^b_\mu W^c_\nu, \qquad g_{\alpha\nu} = (1,-1,-1,-1)\delta_{\alpha\nu}$$

or at greater length

$$\begin{aligned} &\Box W^a_\nu - \partial_\nu(\partial^\mu W^a_\mu) + e\varepsilon_{abc}\left[(\partial^\mu W^b_\mu)W^c_\nu - g^{\alpha\mu} W^b_\alpha(\partial_\nu W^c_\mu) + 2W^b_\mu(\partial^\mu W^c_\nu)\right] + \\ &\quad + e^2 W^b_\mu g^{\mu\alpha}(W^b_\nu W^a_\alpha - W^b_\alpha W^a_\nu) = 0. \end{aligned} \tag{2}$$

As was previously mentioned these equations are invariant under the conformal group $C_{1,3}$ the special conformal transformations having the form

$$\begin{aligned} &x'_\mu = \frac{x_\mu - c_\mu x^2}{\sigma(x)}, \qquad \sigma(x) = 1 - 2cx + c^2x^2, \\ &cx \equiv c^\nu x_\nu, \qquad c^2 \equiv c^\nu c_\nu, \qquad x^2 \equiv x^\nu x_\nu, \\ &W^{a\prime}_\mu(x') = \left[\sigma(x)\delta^\nu_\mu + 2\left(x_\mu c^\nu - c_\mu c^\nu + 2cxc_\mu x^\nu - c^2 x_\mu x^\nu - x^2 c_\mu c^\nu\right)\right] W^2_\nu(x). \end{aligned} \tag{3}$$

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One can directly verify that (3) leaves eq. (2) invariant. Expressions (3) can be obtained by solving Lie equations for the infinitesimal generator of conformal transformations

$$Q_{\text{conf}} = 2cxx\partial - x^2c\partial + 2(cxI_4 + S_{\mu\nu}c^\mu x^\nu) \times I_3, \tag{4}$$

where I_3, I_4 are unit matrices 3×3 and 4×4, respectively, $\partial = \{\partial/\partial x_\nu\}$, c_ν are constants, $S_{\mu\nu} = -S_{\nu\mu}$ are (4×4)-matrices realizing the $D\left(\frac{1}{2},\frac{1}{2}\right)$ representation of the $SO_{1,2}$ algebra

$$\begin{gathered}
S_{01} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad
S_{02} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad
S_{03} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\
S_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad
S_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad
S_{31} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.
\end{gathered} \tag{5}$$

The general form of the operator from the $C_{1,3}$ invariance group of eq. (2) is

$$Q = \xi^\mu(x)\partial_\mu + \eta(x) \times I_3, \tag{6}$$

where $\xi^\mu(x)$ are scalar functions, $\eta(x)$ denotes a (4×4)-matrix. Following [3, 4] we set for the solutions to be found

$$W_\nu^a(x) = a_\nu^\alpha(x)\varphi_\alpha^a(\omega), \tag{7}$$

where the nonsigular matrix $A(x) = \{a_\nu^\alpha(x)\}$ and new variables $\omega = \omega(x)$ are determined from the conditions

$$QA(x) \equiv [\xi^\mu(x)\partial_\mu + \eta(x)]\, A(x) = 0, \qquad \xi^\mu(x)\partial_\mu\omega(x) = 0, \tag{8}$$

where Q is an infinitesimal operators of the form (6) admitted by eq. (2). The unknown functions $\varphi_\nu^a(\omega)$ are to be determined from eq. (2).

Let us begin from the conformally invariant solutions. One can make sure that the following functions satisfy eq. (8) with the operator Q (4)

$$a_{\nu\alpha} = \frac{g_{\nu\alpha}}{x^\sigma x_\sigma} - 2\frac{x_\nu x_\alpha}{(x^\sigma x_\sigma)^2}, \qquad \omega = \frac{\beta x}{x^\nu x_\nu}, \qquad \beta c = 0, \tag{9}$$

where β^ν are arbitrary real constants.

So we get the conformally invariant ansatz

$$W_\nu^a(x) = \frac{\varphi_\nu^a(\omega)}{x^\sigma x_\sigma} - 2x_\nu\frac{x^\alpha\varphi_\alpha^a(\omega)}{(x^\sigma x_\sigma)^2}, \qquad \omega = \frac{\beta x}{x^\nu x_\nu}. \tag{10}$$

Using (10), the field equations (2) can ho rewritten as follows:

$$\begin{gathered}
\beta^2\ddot\varphi_\nu^a - \beta_\nu\beta^\mu\ddot\varphi_\mu^a + e\varepsilon_{abc}\left[\beta^\mu\dot\varphi_\mu^b\varphi_\nu^c - \beta_\nu g^{\alpha\mu}\varphi_\alpha^b\dot\varphi_\nu^c + 2\beta^\mu\varphi_\mu^b\dot\varphi_\nu^c\right] + \\
+ e^2\varphi_\mu^b g^{\mu\alpha}\left(\varphi_\nu^b\varphi_\alpha^a - \varphi_\alpha^b\varphi_\nu^a\right) = 0,
\end{gathered} \tag{11}$$

where the dot means $d/d\omega$.

The simplest solutions of eq. (11) are

$$\varphi^a_\nu(\omega) = g^a(\omega)\beta_\nu, \tag{12}$$

where $g^a(\omega)$ are arbitrary differentiable functions.

$$\begin{gathered}\varphi^a_\nu(\omega) = \left(\varkappa\varepsilon_{abc}c^b_1 c^c_2\omega^2 + c^a_2\omega + c^a_1\right)\alpha_\nu,\\ 2\varkappa(\beta_\mu\alpha\beta - \alpha_\mu\beta^2) = e(\alpha_\mu\alpha\beta - \beta_\mu\alpha^2),\end{gathered} \tag{13}$$

where c^a_1, c^a_2, α_ν, $\varkappa$ are arbitrary real constants.

Now it is easy to write down the conformally invariant solutions of eq. (2)

$$W^a_\nu(\omega) = g^a(\omega)\partial_\nu\omega, \qquad \omega = \beta x/x^\nu x_\nu, \tag{14}$$

$g^a(\omega)$ are arbitrary differentiable function.

$$\begin{gathered}W^a_\nu(x) = \left(\varkappa\varepsilon_{abc}c^b_1 c^c_2\omega^2 + c^a_2\omega + c^a_1\right)\left(\frac{\alpha_\nu}{x^\sigma x_\sigma} - 2x_\nu\frac{\alpha x}{(x^\sigma x_\sigma)^2}\right),\\ 2\varkappa\left(\beta_\mu\alpha\beta - \alpha_\mu\beta^2\right) = e\left(\alpha_\mu\alpha\beta - \beta_\mu\alpha^2\right).\end{gathered} \tag{15}$$

Let us note that

$$W^a_\nu(x) = g^a(f)\partial_\nu f \tag{16}$$

satisfy eq. (2) with the arbitrary differentiable function $f = f(x)$. A solution invariant under the displacements obtained in the same may has the form

$$\begin{gathered}W^a_\nu(x) = \left(\varkappa\varepsilon_{abc}c^b_1 c^c_2(kx)^2 + c^a_2(kx) + c^a_1\right)b_\nu,\\ 2\varkappa\left(k_\mu bk - b_\mu k^2\right) = e\left(b_\mu bk - k_\mu b^2\right),\end{gathered} \tag{17}$$

where $\varkappa$, c^a_1, c^a_2, b_ν, k_ν are arbitrary real constants.

In [4, 5] it is shown that if an equation is invariant under transformations of the form

$$x \to x' = f(x,\theta), \qquad \Psi(x) \to \Psi'(x') = R(x,\theta)\Psi(x), \tag{18}$$

where $R(x,\theta)$ is a nonsingular matrix, θ are parameters of transformations, then the formula

$$\psi_{\text{new}}(x) = R^{-1}(x,\theta)\psi_{\text{old}}(x) \tag{19}$$

gives a "new" solution $\psi_{\text{new}}(x)$ of the equation, starting from an "old" one $\psi_{\text{old}}(x)$.

In the case of special conformal transformations (3) the formula of generating new solutions of eq. (2) takes the form

$$\begin{gathered}W^a_{\mu(\text{new})}(x) = \left\{\frac{\delta^\nu_\mu}{\sigma(x)} + \frac{2}{\sigma^2(x)}\left(c_\mu x^\nu - x_\mu c^\nu + 2cx x_\mu c^\nu - \right.\right.\\ \left.\left. - c^2 x_\mu x^\nu - x^2 c_\mu c^\nu\right)\right\}W^a_{\nu(\text{old})}(x'),\\ x'_\mu = \frac{x_\mu - c_\mu x^2}{\sigma(x)}, \qquad \sigma(x) = 1 - 2cx + c^2x^2,\\ cx \equiv c^\nu x_\nu, \qquad c^2 \equiv c^\nu c_\nu, \qquad x^2 \equiv x^\nu x_\nu.\end{gathered} \tag{20}$$

Upon application of this formula to some known solution of eq. 2, one obtains new family of exact solutions of eq. (2).

Now everybody can easily write down analogous formulae of generating exact solution of eq. (2) for the rest transformations of the invariance group of eq. (2).

In conclusion let us note that we have obtained exact solutions of the nonlinear Dirac equation [4, 6], the relativistic eikonal equation [5], the quantum electrodynamics nonlinear equations [7] by the same method.

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