On the new symmetries of Maxwell equations

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The known symmetries of the Maxwell equations are summarized. Then new symmetries of these equations found by the authors are reviewed. These symmetries are generated by infinitesimal integrodifferential operators of the eight-dimensional Lie algebra. Their physical meaning is not clear.

Before considering the new symmetries of the Maxwell equations

$$\frac{\partial \boldsymbol{E}}{\partial t} = -\mathrm{rot}\,\boldsymbol{H}, \qquad \frac{\partial \boldsymbol{H}}{\partial t} = \mathrm{rot}\,\boldsymbol{E}, \\ \mathrm{div}\,\boldsymbol{E} = \mathrm{div}\,\boldsymbol{H} = 0 \tag{1}$$

it is natural to remind shortly the well-known data on symmetry properties of eqs. (1).

In 1890 Heaviside wrote the original Maxwell equations in the modern form (1) (independently it has been done by Hertz) and paid attention to their invariance under the transformation

$$\boldsymbol{E} \to \boldsymbol{H}, \qquad \boldsymbol{H} \to -\boldsymbol{E}. \tag{2}$$

Larmor [1] and Rainich [2] demonstrated that this symmetry may be extended to one-parametrical group of the following transformations

$$\boldsymbol{E} \to \boldsymbol{E}\cos\theta + \boldsymbol{H}\sin\theta, \\ \boldsymbol{H} \to \boldsymbol{H}\cos\theta - \boldsymbol{E}\sin\theta. \tag{3}$$

In 1904 Lorentz found the linear transformations of space and time variables and the corresponding transformations for $\boldsymbol{E}$ and $\boldsymbol{H}$, under which the free Maxwell equations remain invariant. In 1905 Poincaré and Einstein have demonstrated that the Maxwell equations are invariant under the Lorentz transformations also in the presence of charges and currents. Poincaré first established and studied in detail one of the most important properties of these transformations — their group structure and demonstrated that “Lorentz transformations are nothing but the rotations in the space of four dimensions, a point of which has coordinates $(x, y, z, \sqrt{-1}t)$”. So it was Poincaré who united the space and time into the four-dimensional space-time three years before Minkowski [3].

In 1909 Bateman [4] and Cunningham [5] proved eqs. (1) to be invariant under non-linear conformal transformations. Bateman demonstrated the conformal group $C(1,3)$ to be the maximal symmetry group of Maxwell equations with charges and currents.

One hundred years ago, S. Lie created the mathematical methods of group analysis of differential equations [6]. It is a surprising fact that these methods have only recently been used for the investigation of group properties of eqs. (1). It turned

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out that the maximal local invariance group of eqs. (1) is the direct product of the conformal group and of the Heaviside–Larmor–Rainich one (3), i.e. $C(1,3)\otimes H$ [7].

In connection with the facts mentioned above the impression may arise that the symmetry properties of eqs. (1) have been completely investigated and there is no hope to discover any new symmetry of these equations. But it is not true. It appears that all the relativistic equations of motion for spinning particles possess an additional non-geometrical symmetry under the group $U(2)\otimes U(2)$ (the only exception is the Weyl equation) [9–11]. The basis elements Q_A ($A=1,2,\ldots,8$) of Lie algebra of this group are not the first-order differential operators, but the integro-differential ones. That is why this non-geometrical symmetry could not be discovered by the infinitesimal Lie method.

In general case the non-geometrical symmetry of the relativistic equations of motion is more extensive than the symmetry $U(2)\otimes U(2)$ and increases with the rise of spin [9, 15].

Let us denote by $\{Q_A\}$ the basis elements of a finite-dimensional Lie algebra. This algebra is by the definition the invariance algebra of Maxwell equations if Q_A are defined on the set of solutions of eqs. (1), i.e. satisfy the conditions

$$\begin{aligned} &L_1Q_A\psi=0, \qquad L_1=i\frac{\partial}{\partial t}-\sigma_2\boldsymbol{S}\cdot\boldsymbol{p}, \qquad \psi=\text{column}\,(\boldsymbol{E},\boldsymbol{H}),\\ &L_2Q_A\psi=0, \qquad L_2=p_1-\boldsymbol{S}\cdot\boldsymbol{p}S_1, \end{aligned} \tag{4}$$

where

$$S_a=\begin{pmatrix}\hat{S}_a & 0\\ 0 & \hat{S}_a\end{pmatrix}, \qquad \sigma_2=i\begin{pmatrix}\hat{0} & I\\ I & \hat{0}\end{pmatrix}, \tag{5}$$

I and $\hat{0}$ are unit and zero 3×3 matrices, $\hat{S}_a$ are the spin matrices, $(\hat{S}_a)_{bc}=i\varepsilon_{abc}$.

Theorem 1 [11, 14, 16]. *The Maxwell equations (1) are invariant under the 8-dimensional Lie algebra A_8, basis elements of which are the integro-differential operators of the form*

$$\begin{aligned} &Q_1=\sigma_3\frac{\boldsymbol{S}\cdot\boldsymbol{p}}{p}D, \qquad Q_2=i\sigma_2, \qquad Q_3=\sigma_1\frac{\boldsymbol{S}\cdot\boldsymbol{p}}{p}D,\\ &Q_{3+a}=-i\sigma_2\frac{\boldsymbol{S}\cdot\boldsymbol{p}}{p}Q_a, \qquad Q_7=I, \qquad Q_8=i\sigma_2\frac{\boldsymbol{S}\cdot\boldsymbol{p}}{p}, \end{aligned} \tag{6}$$

where

$$\begin{aligned} D=&\left\{\sum_{a\neq b\neq c}\left[\left(p_a^2p_b^2+p_a^2p_c^2-p_b^2p_c^2\right)\left(1-S_a^2\right)+p_1p_2p_3S_aS_bp_c\right]-\right.\\ &\left.\qquad -pp_1p_2p_3\left[1-\left(\frac{\boldsymbol{S}\cdot\boldsymbol{p}}{p}\right)^2\right]\right\}\varphi^{-1},\\ \varphi=&\frac{1}{\sqrt{2}}\left[p_1^4\left(p_2^2-p_3^2\right)^2+p_2^4\left(p_3^2-p_1^2\right)^2+p_3^4\left(p_1^2-p_2^2\right)^2\right]^{1/2}, \end{aligned} \tag{7}$$

σ_a are the Pauli matrices, commuting with S_a, $p = \left(\sum p_a^2\right)^{1/2}$. The operators (6) satisfy the algebra

$$\begin{aligned} &[Q_a, Q_b] = -[Q_{3+a}, Q_{3+b}] = -\varepsilon_{abc} Q_c, \\ &[Q_{3+a}, Q_b] = \varepsilon_{abc} Q_{3+c}, \qquad [Q_7, Q_A] = [Q_8, Q_A] = 0, \end{aligned} \tag{8}$$

which is isomorphic to the Lie algebra of the group $U(2) \otimes U(2)$.

Proof. The correctness of the Theorem, i.e. that the operators (6) satisfy the commutation relations (8) and the conditions of invariance (4), may be verified by the direct calculations. But it is necessary to note that such a verification is very complicated. There exists another, more constructive proof of Theorem 1, which explains the nature of the non-geometrical symmetry of the relativistic equations of motion. This proof consists in the diagonalization (decomposition) of the Maxwell equations into independent subsystems [11, 14, 15]. Such a diagonalization of the operator L_1 is carried out by the operator

$$W = U_4 U_3 U_2 U_1, \qquad W^{-1} = W^{\dagger}, \tag{9}$$

where

$$\begin{aligned} &U_1 = \frac{1}{2}\left[1 + \sigma_2 - (1 - \sigma_2) D \frac{\boldsymbol{S}\cdot\boldsymbol{p}}{p}\right], \qquad U_2 = \exp(-iS_a\tilde{\theta}_a), \\ &U_3 = \left[1 - i\left(S_1 S_2 + S_2 S_1 + 1 - S_3^2\right)\right] \Big/ \sqrt{2}, \qquad U_4 = \exp\left[(S_2 - S_1) i\pi/4\sqrt{2}\right], \\ &\tilde{\theta}_a = \varepsilon_{abc}(p_b - p_c) \operatorname{arctg}\left[\tilde{p}/(p_1 + p_2 + p_3)\right]/2\tilde{p}, \\ &\tilde{p} = \left[(p_1 - p_2)^2 + (p_3 - p_1)^2 + (p_2 - p_3)^2\right]^{1/2}. \end{aligned}$$

As a result one obtains

$$L_1' = W L_1 W^{-1} = i\frac{\partial}{\partial t} - \Gamma_0 p, \tag{10}$$

where Γ_0 is the diagonal matrix

$$\Gamma_0 = -i(S_1 S_2 + S_2 S_1) S_3 = \operatorname{diag}(1, -1, 0, 1, -1, 0). \tag{11}$$

Now it is not difficult to find eight linearly independent matrices Q_A', which commute with the operator L_1'. These matrices may be chosen in the form

$$Q_a' = i\sigma_a, \qquad Q_{3+a}' = -i\Gamma_0 Q_a', \qquad Q_7' = I, \qquad Q_8' = i\Gamma_0. \tag{12}$$

The matrices (12) satisfy the algebra (8) and are connected with Q_A (6) by the transformation $Q_A = W^{-1} Q_A W$. It is obvious that the matrices (12) operating on the vector-function $\psi' = W\psi$ change their components but do not alter the variables (t, x). The theorem is proved.

Since Q_A (6) are integro-differential operators, we give the finite transformations, generated by Q_A, for the Fourier-components of E_a and H_a

$$\begin{aligned} &\tilde{E}_a \to \tilde{E}_a' = \tilde{E}_a \cos\theta_1 + i\varepsilon_{abc}\hat{p}_b D_{cd} \tilde{E}_d \sin\theta_1, \\ &\tilde{H}_a \to \tilde{H}_a' = \tilde{H}_a \cos\theta_1 - i\varepsilon_{abc}\hat{p}_b D_{cd} \tilde{H}_d \sin\theta_1, \end{aligned} \tag{13a}$$

$$\tilde{E}_a \to \tilde{E}'_a = \tilde{E}_a \cos\theta_2 + \tilde{H}_a \sin\theta_2, \quad \tilde{H}_a \to \tilde{H}'_a = \tilde{H}_a \cos\theta_2 - \tilde{E}_a \sin\theta_2, \tag{13b}$$

$$\tilde{E}_a \to \tilde{E}'_a = \tilde{E}_a \cos\theta_3 - i\varepsilon_{abc}\hat{p}_b D_{cd}\tilde{H}_d \sin\theta_3, \quad \tilde{H}_a \to \tilde{H}'_a = \tilde{H}_a \cos\theta_3 - i\varepsilon_{abc}\hat{p}_b D_{cd}\tilde{E}_d \sin\theta_3, \tag{13c}$$

$$\tilde{E}_a \to \tilde{E}'_a = \tilde{E}_a \,\mathrm{ch}\,\theta_4 - D_{ab}\tilde{H}_b \,\mathrm{sh}\,\theta_4, \quad \tilde{H}_a \to \tilde{H}'_a = \tilde{H}_a \,\mathrm{ch}\,\theta_4 - D_{ab}\tilde{E}_b \,\mathrm{sh}\,\theta_4, \tag{13d}$$

$$\tilde{E}_a \to \tilde{E}'_a = \tilde{E}_a \,\mathrm{ch}\,\theta_5 + i\varepsilon_{abc}\hat{p}_b\tilde{E}_c \,\mathrm{sh}\,\theta_5, \quad \tilde{H}_a \to \tilde{H}'_a = \tilde{H}_a \,\mathrm{ch}\,\theta_5 + i\varepsilon_{abc}\hat{p}_b\tilde{H}_c \,\mathrm{sh}\,\theta_5, \tag{13e}$$

$$\tilde{E}_a \to \tilde{E}'_a = \tilde{E}_a \,\mathrm{ch}\,\theta_6 - D_{ab}\tilde{E}_b \,\mathrm{sh}\,\theta_6, \quad \tilde{H}_a \to \tilde{H}'_a = \tilde{H}_a \,\mathrm{ch}\,\theta_6 + D_{ab}\tilde{H}_b \,\mathrm{sh}\,\theta_6, \tag{13f}$$

$$\tilde{E}_a \to \tilde{E}'_a = \tilde{E}_a \exp\theta_7, \qquad \tilde{H}_a \to \tilde{H}'_a = \tilde{H}_a \exp\theta_7, \tag{13g}$$

$$\tilde{E}_a \to \tilde{E}'_a = \tilde{E}_a \cos\theta_8 + i\varepsilon_{abc}\hat{p}_b\tilde{H}_c \sin\theta_8, \quad \tilde{H}_a \to \tilde{H}'_a = \tilde{H}_a \cos\theta_8 - i\varepsilon_{abc}\hat{p}_b\tilde{E}_c \sin\theta_8, \tag{13h}$$

So the formulae (13) give the explicit form of the transformations from the group $U(2)\otimes U(2)$ which remain (1) invariant. The transformation (13b) coincides with the Heaviside–Larmor–Rainich one (3). The transformations for $E(t,x)$ and $H(t,x)$ may be obtained from (13) by the Fourier integral

$$E'(t,x) = \frac{1}{(2\pi)^{3/2}}\int d^3p\,\tilde{E}'\exp(ipx), \quad H'(t,x) = \frac{1}{(2\pi)^{3/2}}\int d^3p\,\tilde{H}'\exp(ipx). \tag{14}$$

One can make sure that from the invariance of eqs. (1) under the algebra (6) it follows the existence of the following new constants of motion for the electromagnetic field

$$I_1 = \int d^3x \sum_{a,b,c,b\neq c}\left(\frac{\partial^2 E_a}{\partial x_b^2}\frac{\partial^2 H_a}{\partial x_c^2} - \frac{\partial \dot{E}_a}{\partial x_a}\frac{\partial \dot{H}_a}{\partial x_a}\right),$$

$$I_2 = \int d^3x \sum_{a,b,c,b\neq c}\left(\frac{\partial^2 E_a}{\partial x_b^2}\frac{\partial^2 E_a}{\partial x_c^2} - \frac{\partial^2 H_a}{\partial x_b^2}\frac{\partial^2 H_a}{\partial x_c^2} - \frac{\partial \dot{E}_a}{\partial x_a}\frac{\partial \dot{E}_a}{\partial x_a} + \frac{\partial \dot{H}_a}{\partial x_a}\frac{\partial \dot{H}_a}{\partial x_a}\right), \tag{15}$$

$$\dot{A} = \frac{\partial A}{\partial t}.$$

Theorem 2. *The Maxwell equations (1) are invariant under the 23-dimensional Lie algebra, basis elements of which have the form (6) and (16):*

$$P_\mu = p_\mu, \quad J_{\mu\nu} = x'_\mu p_\nu - x'_\nu p_\mu, \quad D = x'_\mu p^\mu + i, \quad K_\mu = 2x'_\mu D - x'_\nu x'^\nu p_\mu, \tag{16}$$

where

$$x'_a = W^{-1}x_a W, \qquad x'_0 = x_0 = t.$$

This theorem may be considered as unification of the results of Bateman and of the ones, established in Theorem 1. Proof is adduced in [14, 16].

In conclusion we note that the equations for the electromagnetic potential A_μ possess the non-geometrical symmetry $U(3)$ [17].

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