On the new invariance algebras of relativistic equations for massless particles

W.I. FUSHCHYCH, A.G. NIKITIN

We show that the massless Dirac equation and Maxwell equations are invariant under a 23-dimensional Lie algebra, which is isomorphic to the Lie algebra of the group $C_4 \otimes U(2) \otimes U(2)$. It is also demonstrated that any Poincaré-invariant equation for a particle of zero mass and of discrete spin provide a unitary representation of the conformal group and that the conformal group generators may be expressed via the generators of the Poincaré group.

1. Introduction

Bateman [1] and Cunningham [3] discovered that Maxwell's equations for a free electromagnetic field were invariant under conformal transformations. Nearly fifty years ago the conformal invariance of an arbitrary relativistic equation, for a massless particle with discrete spin was established by Dirac [4] for a spin-$\frac{1}{2}$ particle and by McLennan [20] for a particle of any spin.

Until now the question of whether the conformal group is the maximally extensive symmetry group for the equations of motion for massless particles remained unsettled. A positive answer to this question has been obtained only in the frame of the classical Sofus Lie approach (Ovsjannicov [24]), but as has been found recently, Lie methods do not permit the possibility to obtain all possible symmetry groups of differential equations.

The restriction of the Lie method is that it applies only to those symmetry groups whose generators belong to the class of differential operators of first order. Using the non-Lie approach, in which the group generators may be differential operators of any order and even integro-differential operators, the new invariance groups of relativistic wave equations have been found (Fushchych [6–9]). It was demonstrated that any Poincaré-invariant equation for a free particle of spin $s \geq 12$ possessed additional invariance under the group $SU(2) \otimes SU(2)$ (Fushchych [6, 7]); that the Kemmer–Duffin–Petiau equation was invariant under the group $SU(3) \otimes SU(3)$, and that the Rarita–Schwinger equation was invariant under the group $O(6) \otimes O(6)$ was deminstrated by Nikitin et al [23] and by Fushchych and Nikitin [10]. The non-Lie approach was also used successfully to obtain the symmetry groups of the Dirac and Kemmer–Duffin–Petiau equations describing the particles in an external electromagnetic field (Fushchych and Nikitin [12]). Other examples of symmetries which cannot be obtained in the classical Lie approach are the symmetry groups of the non-relativistic oscillator (Levi–Leblond [16]) and of the hydrogen atom (Fock [5]).

In the present paper, we have found the new symmetry groups of the massless Dirac equation and of Maxwell's equations using a non-Lie approach. These groups are generated not by the transformations of coordinates, but by the transformations

J. Phys. A: Math. Gen, 1979, **12**, № 6, P. 747–757.

of the Dirac wave function Ψ and the vectors of the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$ of the type

$$\Psi \to \Psi' = f\left(\Psi, \frac{\partial \Psi}{\partial x_a}, \frac{\partial^2 \Psi}{\partial x_a \partial x_b}, \dots\right), \tag{1.1}$$

$$\begin{aligned} \boldsymbol{E} \to \boldsymbol{E}' = \boldsymbol{g}\left(\boldsymbol{E}, \boldsymbol{H}, \frac{\partial \boldsymbol{E}}{\partial x_a}, \frac{\partial \boldsymbol{H}}{\partial x_a}, \frac{\partial^2 \boldsymbol{E}}{\partial x_a \partial x_b}, \frac{\partial^2 \boldsymbol{H}}{\partial x_a \partial x_b}, \dots\right), \\ \boldsymbol{H} \to \boldsymbol{H}' = \boldsymbol{h}\left(\boldsymbol{E}, \boldsymbol{H}, \frac{\partial \boldsymbol{E}}{\partial x_a}, \frac{\partial \boldsymbol{H}}{\partial x_a}, \frac{\partial^2 \boldsymbol{E}}{\partial x_a \partial x_b}, \frac{\partial^2 \boldsymbol{H}}{\partial x_a \partial x_b}, \dots\right), \end{aligned} \tag{1.2}$$

where the functions f and $\boldsymbol{g}$, $\boldsymbol{h}$ may depend on any order derivatives of Ψ and $\boldsymbol{E}$, $\boldsymbol{H}$ respectively.

It is demonstrated that Maxwell's equations are invariant under the group $U(2) \otimes U(2)$; the explicit forms of the functions $\boldsymbol{g}$ and $\boldsymbol{h}$ in (1.2), which generate the transformations of such a group, are found. It is also shown that the Dirac equation (with $m = 0$) and Maxwell's equations are invariant under a 23-parametrical Lie group, which is isomorphic to the group $C_4 \otimes U(2) \otimes U(2)$. The results obtained admit immediate generalisation to the relativistic wave equations for massless particles of any spin. The conformal group generators which leave the Weyl equation and the massless Dirac equation invariant are expressed in a form which is transparently Hermitian. It is demonstrated that any (generally speaking, reducible) representation of a Poincaré group, which corresponds to zero mass and discrete spin, may be extended to the conformal group representation. The explicit expression for the generators of the conformal group C_4 via the generators of the Poincaré group $P(1,3)$ has been found. We therefore give a constructive proof of the statement that any relativistic equation for a discrete spin and zero-mass particle provides the unitary representation of the conformal group (for Maxwell and Bargman–Wigner equations this has been demonstrated by Gross [13]).

2. The Hermitian representation of the conformal group generators for any spin

The conformal invariance properties of any relativistic equation of motion for a particle of zero mass and of discrete spin may be formulated by the following statement.

Theorem 1. *Any Poincaré-invariant equation for a zero-mass and discrete spin particle is invariant under the conformal algebra C_4*, basis elements of which are given by the operators P_μ, $J_{\mu\nu}$ and*

$$\begin{aligned} &D = -\frac{1}{2}[P_0 P_a/P^2, J_{0a}]_+, \\ &K_\mu = \frac{1}{2}\left([P_0/P^2, [J_{0b}, J_{\mu b}]_+]_+ - [P_\mu/P^2, J_{0b}J_{0b}]_+\right) + g_{\mu\nu}\left(P_\nu/P^2\right)\left(\Lambda^2 - \frac{1}{2}\right), \end{aligned} \tag{2.1}$$

where P_μ and $J_{\mu\nu}$ are the basis elements of algebra $P(1,3)$,

$$[A,B]_+ = AB + BA, \qquad P^2 = P_1^2 + P_2^2 + P_3^2, \qquad \Lambda = \frac{1}{2}\varepsilon_{abc} J_{ab} P_c P_0^{-1}$$

*We use the same notation for the groups and for the corresponding Lie algebras.

and D, K_μ are the operators which extend the algebra $P(1,3)$ to the algebra C_4.

Proof. Inasmuch as the operators P_μ and $J_{\mu\nu}$ by definition satisfy the algebra

$$
\begin{aligned}
&[P_\mu, P_\nu]_- = 0, \qquad [J_{\mu\nu}, P_\lambda]_- = i(g_{\nu\lambda}P_\mu - g_{\mu\lambda}P_\nu),\\
&[J_{\mu\nu}, J_{\lambda\sigma}]_- = i(g_{\nu\lambda}J_{\mu\sigma} + g_{\mu\sigma}J_{\nu\lambda} - g_{\mu\lambda}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\lambda}),
\end{aligned}
\tag{2.2}
$$

the theorem proof is reduced to the verification of the correctness of the following commutation relations:

$$
\begin{aligned}
&[J_{\mu\nu}, K_\lambda]_- = i(g_{\nu\lambda}K_\mu - g_{\mu\lambda}K_\nu),\\
&[K_\mu, P_\nu]_- = 2i(g_{\mu\nu}D - J_{\mu\nu}),\\
&[D, P_\mu]_- = iP_\mu, \qquad [D, K_\mu]_- = -iK_\mu,\\
&[K_\mu, K_\nu]_- = 0, \qquad [J_{\mu\nu}, D]_- = 0,
\end{aligned}
\tag{2.3}
$$

which determine together with (2.2) the algebra C_4 (see, e.g., Mack and Salam [19]). It is not difficult to carry out such a verification, bearing in mind that for the set of solutions of any relativistic equation for a particle of zero mass and of discrete spin the following relations are satisfied:

$$
P_\mu P^\mu = 0, \qquad W_\mu W^\mu = 0, \qquad W_\mu = \Lambda P_\mu,
\tag{2.4}
$$

where W_μ is the Lubansky–Pauli vector

$$
W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}J_{\nu\rho}P_\sigma.
$$

So the formulae (2.1) have determined the explicit form of the conformal group generators via the given generators P_μ, $J_{\mu\nu}$ of the group $P(1,3)$. The theorem is proved.

We note that the generators K_μ and D are written in a transparently Hermitian form, and hence they generate the unitary representation of the conformal group. The constructive character of theorem 1 will be demonstrated in the next section.

3. Manifestly Hermitian representation of the conformal group generators for Dirac and Weyl equations

The results given above may be used to find the explicit form of the generators of the conformal group representation, which is realised on the set of solutions of any relativistic equation for a massless particle. In this section we shall demonstrate it by the examples of the massless Dirac equation and of the Weyl equation.

The Dirac equation for a massless particle of spin $\frac{1}{2}$ may be written in the form

$$
L\Psi = 0, \qquad L = i\frac{\partial}{\partial t} - \gamma_0\gamma_a p_a, \qquad p_a = -i\frac{\partial}{\partial x_a},
\tag{3.1}
$$

where γ_μ are the four-row Dirac matrices.

$\{Q_A\}$ denotes the set of the generators of some Lie group G. Equation (3.1) is by definition invariant under G if the operators Q_A satisfy the relations

$$
[L, Q_A]_- = F_A L,
\tag{3.2}
$$

where F_A are some operators which are defined on the set of the solutions of equation (3.1).

A well known example of such operators is the set of Poincaré group generators

$$P_0 = H = \gamma_0\gamma_a p_a, \qquad P_a = p_a,$$
$$J_{ab} = x_a p_b - x_b p_a + S_{ab}, \qquad J_{0a} = x_0 p_a - \frac{1}{2}[x_a, H]_+, \tag{3.3}$$

where

$$x_0 = t, \qquad S_{ab} = \frac{1}{4} i(\gamma_a\gamma_b - \gamma_b\gamma_a).$$

According to theorem 1, the representation (3.3) may be extended to the representation of Lie algebra of the conformal group. Substituting (3.3) into (2.4), one obtains the operators

$$D = \frac{1}{2}[x_\mu, P_\mu],$$
$$K_\mu = [J_{\mu\nu}, x^\nu]_+ + \frac{1}{2}[P_\mu, x_\nu x^\nu]_+ \tag{3.4}$$

which satisfy the invariance condition (3.2) (where $F_A \equiv 0$) and the commutation relations (2.5). The operators (3.3) and (3.4) are transparently Hermitian under the usual scalar product

$$(\Psi_1, \Psi_2) = \int d^3x\, \Psi_1^\dagger \Psi_2 \tag{3.5}$$

and therefore generate the unitary representation of the conformal group.

Let us note that on the set of solutions of equation (3.1) the generators (3.3) and (3.4) may also be written in the usual form (see e.g. Mack and Salam [19])

$$P_\mu = p_\mu = i g_{\mu\nu}\frac{\partial}{\partial x_\nu}, \qquad D = x_\mu p^\mu + \frac{3}{2} i,$$
$$J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + \frac{1}{4} i[\gamma_\mu, \gamma_\nu]_-, \tag{3.6}$$
$$K_\nu = 2x_\nu D - x_\mu x^\mu p_\nu - \frac{1}{2} x^\mu [\gamma_\nu, \gamma_\mu]_-,$$

which is not, however, manifestly Hermitian.

The Weyl equation for the neutrino,

$$i\frac{\partial\phi}{\partial t} = \sigma_a p_a \phi, \tag{3.7}$$

where σ_a are Pauli matrices, is equivalent to the equation (3.1) with the Poincaré-invariant subsidiary condition

$$(1 + i\gamma_4)\Psi = 0, \qquad \gamma_4 = \gamma_0\gamma_1\gamma_2\gamma_3. \tag{3.8}$$

The exact form of the Hermitian generators of the conformal group which are provided by equation (3.7) may be obtained from (3.3) and (3.4) by the substitution

$$p_0 \to \sigma_a p_a, \qquad S_{ab} \to \frac{1}{4} i(\sigma_a\sigma_b - \sigma_b\sigma_a). \tag{3.9}$$

Finally, if P_μ and $J_{\mu\nu}$ are the generators of the irreducible representation of the Poincaré group in Lomont–Moses [18] form, then the formulae (2.1) give the conformal group generators in the form of Bose and Parker [2].

4. The additional symmetry of the Dirac equation with mass $m = 0$

Some years ago the new invariance algebra of equation (3.1) was found (Fushchych [6, 7]); this is different from the algebra of the conformal group generators. The basis elements of this algebra have the form

$$\begin{aligned}
&\Sigma_{ab} = S_{ab} - \frac{1}{2}\left(\gamma_a\hat{p}_b - \gamma_b\hat{p}_a\right)\left(1 + \gamma_a\hat{p}_a\right), \\
&\Sigma_{4a} = \frac{1}{2}\gamma_4\gamma_a + \frac{1}{2}\gamma_4\hat{p}_a\left(1 + \gamma_b\hat{p}_b\right),
\end{aligned} \tag{4.1}$$

where

$$\hat{p}_a = p_a p^{-1}, \qquad p = \left(p_1^2 + p_2^2 + p_3^2\right)^{1/2}, \qquad a, b = 1, 2, 3.$$

The operators (4.1) realise the representation $D\left(\frac{1}{2}, 0\right) \otimes D\left(0, \frac{1}{2}\right)$ of the Lie algebra of the group $O(4) \sim SU(2) \otimes SU(2)$, but do not form the closed algebra together with (3.3), (3.4) or (3.8). Below we will obtain the 23-dimensional invariance algebra of equation (3.1), which includes the Lie algebras of the groups C_4 and $U(2) \otimes U(2)$.

Theorem 2. *The Dirac equation (3.1) is invariant under the 23-dimensional Lie algebra, which is isomorphic to the algebra of generators of the group $C_4 \otimes U(2) \otimes U(2)$. The basis elements of this algebra have the form*

$$\begin{aligned}
&P_0 = p_0 = i\frac{\partial}{\partial t}, \qquad P_a = p_a = -i\frac{\partial}{\partial x_a}, \qquad J_{ab} = x_a p_b - x_b p_a + S_{ab}, \\
&J_{0a} = x_0 p_a - x_a p_0 - \frac{iH}{2p}(1 - i\gamma_4)\gamma_a\gamma_b\hat{p}_b + \hat{\Sigma}_{0a}, \qquad D = x_\mu p^\mu + i, \\
&K_\mu = \left(-x_\nu x^\nu + J_{ab}S_{ab}p^{-2} + p^{-2}\right)p_\mu + 2\left[x_\mu + (1 - \delta_{\mu 0})(1 - \gamma_0)S_{\mu b}\hat{p}_b\right]D, \\
&\hat{\Sigma}_{0c} = \frac{1}{2}\gamma_4\left(\hat{p}_a + \gamma_0 S_{ab}\hat{p}_b\right), \qquad \hat{\Sigma}_5 = \frac{H}{p}, \\
&\hat{\Sigma}_{ab} = \frac{1}{2}\varepsilon_{abc}\frac{H}{p}\hat{\Sigma}_{0c}, \qquad \hat{\Sigma}_6 = 1, \qquad a, b, c = 1, 2, 3,
\end{aligned} \tag{4.2}$$

Proof. Let us transform equation (3.1) and the generators (4.2) to a representation in which the theorem statements may easily be verified immediately. Using for this purpose the operator

$$V = V^{-1} = \frac{1}{2}\left[1 + \gamma_0 + (1 - \gamma_0)\varepsilon_{abc}S_{ab}\hat{p}_c\right] \tag{4.3}$$

one obtains

$$L'\Psi' = 0, \qquad \Psi' = V\Psi, \qquad L' = VLV^{-1} = i\frac{\partial}{\partial t} - i\gamma_4 p, \tag{4.4}$$

$$
\begin{aligned}
&P'_\mu = VP_\mu V^{-1} = P_\mu, \qquad J'_{ab} = VJ_{ab}V^{-1} = J_{ab},\\
&J'_{0a} = VJ_{0a}V^{-1} = x_0p_a - x_ap_0 + \frac{1}{2}i\gamma_0\gamma_a,\\
&D' = VDV^{-1} = D = x_\mu p^\mu + i,\\
&K'_\mu = VK_\mu V^{-1} = -x_\nu x^\nu p_\mu + 2x_\mu D',\\
&\hat\Sigma'_{ab} = V\hat\Sigma_{ab}V^{-1} = S_{ab}, \qquad \hat\Sigma'_{0a} = V\hat\Sigma_{ab}V^{-1} = \frac{1}{2}i\gamma_0\gamma_a,\\
&\hat\Sigma'_5 = V\hat\Sigma_5V^{-1} = i\gamma_4, \qquad \hat\Sigma'_6 = V\hat\Sigma_6V^{-1} = \hat\Sigma_6,
\end{aligned}
\tag{4.5}
$$

It is not difficult to be convinced that the operators (4.4) and (4.5) satisfy the invariance condition (3.2):

$$
\begin{aligned}
&[L', P'_\mu]_- = [L', J'_{ab}]_- = [L', \hat\Sigma_{\mu\nu}]_- = [L', \hat\Sigma'_\alpha]_- = 0,\\
&[L', K'_0]_- = 2i\left[x_0 + (x_ap_a - i)i\gamma_4p^{-1}\right]L', \qquad [L', K'_a]_- = 2i(x_a + i\hat p_ax_0\gamma_4)L',\\
&[L', D]_- = iL', \qquad [L', J'_{0a}]_- = \gamma_4\hat p_aL'
\end{aligned}
$$

and the commutation relations for $Q'_A \subset \left\{P'_\mu, J'_{\mu\nu}, K'_\mu, D', \hat\Sigma'_{\mu\nu}, \Sigma'_\alpha\right\}$

$$
\begin{aligned}
&[P'_\mu, P'_\nu]_- = 0, \qquad [P'_\mu, J'_{\nu\lambda}]_- = i(g_{\mu\lambda}P'_\nu - g_{\nu\lambda}P'_\mu),\\
&[J'_{\mu\nu}, J'_{\lambda\sigma}]_- = i(g_{\mu\sigma}J'_{\nu\lambda} + g_{\nu\lambda}J'_{\mu\sigma} - g_{\mu\lambda}J'_{\nu\sigma} - g_{\nu\sigma}J'_{\mu\lambda}),\\
&[P'_\mu, D']_- = -iP'_\mu, \qquad [K'_\mu, D']_- = iK'_\mu, \qquad [J'_{\mu\nu}, D']_- = 0,\\
&[P'_\mu, K'_\nu]_- = 2i(J'_{\mu\nu} - \hat\Sigma'_{\mu\nu} - g_{\mu\nu}D'),\\
&[J'_{\mu\nu}, \hat\Sigma'_{\lambda\sigma}]_- = [\hat\Sigma'_{\mu\nu}, \hat\Sigma'_{\lambda\sigma}]_- = i(g_{\mu\sigma}\hat\Sigma'_{\nu\lambda} + g_{\nu\lambda}\hat\Sigma'_{\mu\sigma} - g_{\mu\lambda}\hat\Sigma'_{\nu\sigma} - g_{\nu\sigma}\hat\Sigma'_{\mu\lambda}),\\
&[\hat\Sigma'_{\mu\nu}, P'_\lambda]_- = [\hat\Sigma'_{\mu\nu}, D']_- = [\hat\Sigma'_{\mu\nu}, K'_\lambda]_- = [\hat\Sigma'_\alpha, Q'_A]_- = 0.
\end{aligned}
$$

The algebra (4.6) is isomorphic to the algebra of generators of the group $C_4 \otimes U(2) \otimes U(2)$. The theorem is therefore proved.

We note that the subsidiary condition (3.8) is not invariant under the transformations which are generated by the operators $\hat\Sigma_{\mu\nu}$. Therefore the Weyl equation (3.7) is not invariant relative to the whole algebra (4.2), but is invariant with respect to its subalgebra C_4.

It should be emphasised that the generators (4.2) belong to the class of nonlocal integro-differential operators, and therefore one cannot obtain them in the classical Lie approach.

5. The symmetry of Maxwell's equations

The Maxwell equations for a free electromagnetic field have the form

$$
\begin{aligned}
&\boldsymbol{p}\times\boldsymbol{E} = i\frac{\partial\boldsymbol{H}}{\partial t}, \qquad \boldsymbol{p}\times\boldsymbol{H} = -i\frac{\partial\boldsymbol{E}}{\partial t},\\
&\boldsymbol{p}\cdot\boldsymbol{E} = 0, \qquad \boldsymbol{p}\cdot\boldsymbol{H} = 0,
\end{aligned}
\tag{5.1}
$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are the vectors of the electric and magnetic field strengths.

Equations (5.1) are invariant under the conformal group. It is well known that these equations are also invariant under the transformations (Heaviside [14], Larmor [15])

$$E_a \to H_a, \qquad H_a \to -E_a \tag{5.2}$$

and under the more general ones (Rainich [25])

$$\begin{aligned} &E_a \to E_a \cos\theta + H_a \sin\theta, \\ &H_a \to H_a \cos\theta - E_a \sin\theta, \end{aligned} \tag{5.3}$$

We now demonstrate that the summetry of the Maxwell equations is more extensive, namely that the equations (5.1) are invariant under the set of transformations which realise the representation of the group $U(2)\otimes U(2)$ and include (5.3) as a one-parameter subgroup. The theorem about such an invariance of the Maxwell equations in the class of transformations of kind (1.1) and (1.2) had been formulated by one of us (Fushchych [9]) without showing the exact form of the functions $\boldsymbol{g}$ and $\boldsymbol{h}$. Below we give the explicit transformation laws for E_a and H_a.

Theorem 3. *The Maxwell equations (5.1) are invariant under the transformations*

$$\begin{aligned} &H_a \to H'_a = H_a\cos\theta + \left[iD_{ab}E_b\theta_1 - \varepsilon_{abc}\hat{p}_b(H_c\theta_3 + iD_{cd}E_d\theta_2)\right]\frac{\sin\theta}{\theta}, \\ &E_a \to E'_a = E_a\cos\theta + \left[iD_{ab}H_b\theta_1 - \varepsilon_{abc}\hat{p}_b(E_c\theta_3 + iD_{cd}H_d\theta_2)\right]\frac{\sin\theta}{\theta}; \end{aligned} \tag{5.4a}$$

$$\begin{aligned} &H_a \to H''_a = H_a\cos\lambda - \left[i\varepsilon_{abc}\hat{p}_b D_{cd}H_d\lambda_1 + D_{ad}H_d\lambda_2 - E_a\lambda_3\right]\frac{\sin\lambda}{\lambda}, \\ &E_a \to E''_a = E_a\cos\lambda + \left[i\varepsilon_{abc}\hat{p}_b D_{cd}E_d\lambda_1 + D_{ad}E_d\lambda_2 - H_a\lambda_3\right]\frac{\sin\lambda}{\lambda}; \end{aligned} \tag{5.4b}$$

$$\begin{aligned} &H_a \to H'''_a = H_a\cos\eta - \varepsilon_{abc}\hat{p}_b E_c \sin\eta, \\ &E_a \to E'''_a = E_a\cos\eta + \varepsilon_{abc}\hat{p}_b H_c \sin\eta; \end{aligned} \tag{5.4c}$$

$$\begin{aligned} &H_a \to H''''_a = \exp(i\phi)H_a, \\ &E_a \to E''''_a = \exp(i\phi)E_a, \end{aligned} \tag{5.4d}$$

where

$$D_{ad} = \left[\left(p_a^2p_c^2 + p_a^2p_b^2 - p_b^2p_c^2\right)\delta_{ad} + p_1p_2p_3\left(p_b\delta_{cd} + p_c\delta_{bd} - p_a\hat{p}_d\right)\right]L^{-1},$$

$$L = \frac{1}{2}\sqrt{2}\left[\left(p_1^2 - p_2^2\right)p_3^4 + \left(p_1^2 - p_3^2\right)p_2^4 + \left(p_2^2 - p_3^2\right)p_1^4\right]^{1/2},$$

and where (a,b,c) *is a cyclic permutation of* $(1,2,3)$*;*

$$\lambda = \left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right)^{1/2}, \qquad \theta = \left(\theta_1^2 + \theta_2^2 + \theta_3^2\right)^{1/2}.$$

θ_a*,* λ_a*,* η *and* ϕ *are real parameters. The transformations (5.4) realise the representation of the group* $U(2)\otimes U(2)$.

Proof. One can be convinced by the direct verification that E'_a, H'_a, E''_a, H''_a, E'''_a, H'''_a, E''''_a, H''''_a satisfy equation (5.1) as well as the non-transformed vectors $\boldsymbol{E}$ and

$\boldsymbol{H}$ but a more elegant and constructive way, which shows the method of obtaining the group (5.4) is to transform the equations to a form for which the theorem statements become obvious.

Let us write equations (5.1) in the matrix form (Fushchych and Nikitin [10, 11], Nikitin and Fushchych [22])

$$i\frac{\partial}{\partial t}\Psi = \alpha_a p_a\Psi, \qquad \sigma_3 S_{4a} p_a\Psi = 0, \tag{5.5}$$

where Ψ is an eight-component wavefunction

$$\Psi = \text{column}(H_1, H_2, H_3, \phi_1, E_1, E_2, E_3, \phi_2) \tag{5.6}$$

and α_a, S_{4a} are matrices of the form

$$\alpha_a = 2\sigma_2\tau_a,$$
$$\sigma_2 = i\begin{pmatrix} \hat{0} & -\hat{I} \\ \hat{I} & \hat{0} \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} \hat{I} & \hat{0} \\ \hat{0} & -\hat{I} \end{pmatrix}, \qquad \tau_a = \begin{pmatrix} \hat{\tau}_a & 0 \\ 0 & \hat{\tau}_a \end{pmatrix}, \tag{5.7}$$

$$\hat{\tau}_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad \hat{\tau}_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},$$

$$\hat{\tau}_3 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \qquad S_{4a} = \begin{pmatrix} \hat{S}_{4a} & \hat{0} \\ \hat{0} & -\hat{S}_{4a} \end{pmatrix},$$

$$\hat{S}_{41} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad \hat{S}_{42} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},$$

$$\hat{S}_{43} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}.$$

$\hat{0}$ and $\hat{I}$ are four-row square zero and unit matrices. The matrices $\hat{S}_{4a}$ and

$$\hat{S}_{ab} = \frac{1}{2}\left(\hat{S}_{4c} + 2\hat{\tau}_c\right)\varepsilon_{abc}$$

realize the representation $D\left(\frac{1}{2}, \frac{1}{2}\right)$ of the algebra $O(4)$. Writing equations (5.5) by components, one obtains the usual form for the Maxwell equation (5.1) and the conditions for ϕ_1 and ϕ_2:

$$\phi_1 = C_1, \qquad \phi_2 = C_2,$$

where C_1 and C_2 are constants which may be equated to zero without loss of generality*.

Using the unitary operator

$$U = \exp\left(-i\frac{S_a\tilde{p}_a}{\tilde{p}}\tan^{-1}\frac{\tilde{p}}{p_1+p_2+p_3}\right), \tag{5.8}$$

where

$$\tilde{p}_a = p_b - p_c, \qquad \tilde{p} = \left(\tilde{p}_1^2+\tilde{p}_2^2+\tilde{p}_3^2\right)^{1/2}, \qquad S_a = \begin{pmatrix} \hat{S}_{bc} & \hat{0} \\ \hat{0} & \hat{S}_{bc} \end{pmatrix},$$

one reduces the equations (5.5) to the symmetrical form

$$\begin{aligned} &L_1'\Phi = 0, \qquad L_1' = UL_1U^\dagger = i\frac{\partial}{\partial t} - \frac{1}{\sqrt{3}}(\alpha_1+\alpha_2+\alpha_3)p; \\ &L_2'\Phi = 0, \qquad L_2' = UL_2U^\dagger = \frac{1}{\sqrt{3}}(S_{41}+S_{42}+S_{43}), \qquad \Phi = U\Psi. \end{aligned} \tag{5.9}$$

The operator (5.8) also transforms the helicity operator $S_p = S_a p_a p^{-1}$ to the symmetrical matrix form:

$$US_pU^\dagger = (S_1+S_2+S_3)/\sqrt{3}.$$

The invariance condition (3.2) for the equations (5.9) takes the form

$$[L_1', Q_A']_- = f_A^1 L_1' + f_A^2 L_2', \qquad [L_2', Q_A']_- = \tilde{f}_A^1 L_1' + \tilde{f}_A^2 L_2'. \tag{5.10}$$

The conditions (5.10) are obviously satisfied by any operator which commutes with the matrices

$$A = (\alpha_1+\alpha_2+\alpha_3)/\sqrt{3} \qquad \text{and} \qquad B = (S_{41}+S_{42}+S_{43})/\sqrt{3}. \tag{5.11}$$

We choose the complete set of such operators in the form

$$\begin{aligned} &Q_{12}' = (S_1+S_2+S_3)/\sqrt{3}, \qquad Q_{23}' = iQ_{12}'Q_{31}', \\ &Q_{31}' = \sum_a (S_b - S_c)p_a^2\left(p_b^2 - p_c^2\right)L^{-1}/\sqrt{3}, \\ &Q_{4a}' = AQ_{bc}', \qquad Q_5' = A, \qquad Q_6' = \sigma_0 = \begin{pmatrix} \hat{I} & \hat{0} \\ \hat{0} & \hat{I} \end{pmatrix}. \end{aligned} \tag{5.12}$$

Of course this is not the only possible basis set of the operators commuting with (5.11). However, we prefer the operators (5.12) because they are invariant under the permutation

$$S_a \to S_b, \qquad p_a \to p_b, \qquad a, b = 1, 2, 3.$$

*The analogous "Dirac-like" formulation of the Maxwell equations (but using a four-component wave function and subsidiary condition different from (5.5b) has been proposed previously by Lomont [17] and Moses [21].

The operators (5.12) satisfy the invariance condition (5.10) (with $f_A^1 = f_A^2 = \tilde{f}_A^1 = \tilde{f}_A^2 = 0$) and the commutation relations

$$[Q'_{kl}, Q'_{mn}]_- = 2i(\delta_{km}Q'_{ln} + \delta_{ln}Q'_{km} - \delta_{kn}Q'_{lm} - \delta_{lm}Q'_{kn}),$$
$$[Q'_5, Q'_{kl}]_- = [Q'_6, Q'_{kl}]_- = [Q'_5, Q'_6]_- = 0. \tag{5.13}$$

These operators also satisfy the conditions

$$(Q'_{kl})^2\Phi = (Q'_5)^2\Phi = (Q'_6)^2\Phi = \Phi,$$

i.e. they realise the representation of the Lie algebra of the group $U(2) \otimes U(2)$ and Q'_{kl} form the representation $D\left(0, \frac{1}{2}\right) \otimes D\left(\frac{1}{2}, 0\right)$ of the group $SU(2) \otimes SU(2)$.

It follows from the above that equations (5.9) are invariant under the arbitrary transformation from the group $U(2) \otimes U(2)$:

$$\Phi \to \Phi' = \exp\left(\frac{1}{2}i\varepsilon_{abc}Q'_{ab}\theta_c\right)\Phi = \left(\cos\theta + \frac{1}{2}i\theta^{-1}\varepsilon_{abc}Q'_{ab}\theta_c\right)\Phi,$$
$$\Phi \to \Phi'' = \exp(iQ'_{4a}\lambda_a)\Phi = \left(\cos\lambda + iS_{4a}\lambda_a\frac{\sin\lambda}{\lambda}\right)\Phi,$$
$$\Phi \to \Phi''' = \exp(iQ'_5\phi)\Phi = (\cos\phi + iQ'_5\sin\phi)\Phi,$$
$$\Phi \to \Phi'''' = \exp(iQ'_6\eta)\Phi = \exp(i\eta)\Phi. \tag{5.14}$$

Returning with the help of the operator (5.8) to the starting Ψ function one obtains from (5.14) the following transformation laws:

$$\Psi \to \Psi' = \left(\cos\theta + \frac{1}{2\theta}\varepsilon_{abc}Q_{ab}\sin\theta\right)\Psi,$$
$$\Psi \to \Psi'' = \left(\cos\lambda + \frac{i}{\lambda}Q_{4a}\lambda_a\sin\lambda\right)\Psi,$$
$$\Psi \to \Psi''' = (\cos\phi + iQ_5\sin\phi)\Psi,$$
$$\Psi \to \Psi'''' = \exp(i\eta)\Psi. \tag{5.15}$$

where

$$Q_{kl} = W^{-1}Q_{kl}W, \qquad Q_\lambda = W^{-1}Q_\lambda W, \quad \lambda = 5, 6,$$
$$Q_{12} = S_a\hat{p}_a, \qquad Q_{23} = \sigma_1 F, \qquad Q_{31} = i\sigma_1 S_a\hat{p}_a F,$$
$$Q_{4a} = \frac{1}{2}\sigma_2 S_b\hat{p}_b\varepsilon_{abc}Q_{bc}, \qquad Q_5 = \sigma_2 S_b\hat{p}_b, \qquad Q_6 = 1,$$
$$F = L^{-1}\left(\sum_{a\neq b\neq c}\left[\left(p_a^2p_c^2 + p_a^2p_b^2 - p_b^2p_c^2\right)\left(1 - S_a^2\right) + p_1p_2p_3p_aS_bS_c\right] - \right.$$
$$\left. -pp_1p_2p_3\left[1 - (S_a\hat{p}_a)^2\right]\right). \tag{5.16}$$

Substituting (5.6) and (5.16) into (5.15), we obtain the formulae (5.4). The theorem is proved.

So we have found a new eight-parameter symmetry group of the Maxwell equations which is given by the transformations (5.4). The main property of such transformations is that they are carried out by the nonlocal (integro-differential) operators.

It is necessary to emphasise that the transformations (5.4) have nothing to do with the Lorentz ones, inasmuch as they realise the unitary finite-dimensional representation of the compact group $U(2)\otimes U(2)$. If $\lambda_1=\lambda_2=0$, the formulae (5.4b) give the Heaviside–Larmor–Rainich transformation (5.3).

The transformations (5.4) are unitary under the usual scalar product (3.5). Substituing (5.6) into (3.5), we discover that the transformations (5.4) do not change the quantity

$$\mathcal{E}=\int d^3x\,\left(\boldsymbol{E}^2+\boldsymbol{H}^2\right),$$

which is associated with the full energy of an electromagnetic field.

If the parameters θ_a, λ_a, η and ϕ in (5.4) are the complex ones, the transformations (5.4) realise the representation of the group $GL(2)\otimes GL(2)$. Such transformations also leave the equations (5.1) invariant, but are, of course non-unitary.

Using theorem 1, we can show that equations (5.5) provide the Hermitian representation of the Lie algebra of the conformal group. The basis elements of this algebra have the form

$$\begin{aligned}
&P_0=\boldsymbol{\alpha}\cdot\boldsymbol{p},\qquad P_a=p_a,\\
&J_{ab}=x_ap_b-x_bp_a+S_{ab}=X_ap_b-X_bp_a+\hat{p}_c\Lambda,\\
&J_{0a}=tp_a-\frac{1}{2}[X_a,P_0]_+,\qquad D=\frac{1}{2}[x_a,p_a]_+-tP_0\equiv-\frac{1}{2}[X_\mu,P^\mu]_+,\\
&K_\mu=-[J_{\mu\nu},X^\nu]_++\frac{1}{2}[P_\mu,X_\nu X^\nu]_+-P_\mu\left(\Lambda^2+\frac{1}{4}\right)/p^2,
\end{aligned}\tag{5.17}$$

where

$$X_0=x_0=t,\qquad \Lambda=\frac{1}{2}\varepsilon_{abc}S_{ab}\hat{p}_cp^{-1},\qquad X_a=x_a+S_{ab}p_bp^{-2}.$$

But the generators (5.17) together with (5.16) do not form the closed algebra. The symmetry of equations (5.5) under the 23-dimensional Lie algebra, which includes the subalgebras C_4 and $U(2)\otimes U(2)$, is established in the following theorem.

Theorem 4. *Equations (5.5) are invariant under the 23-dimensional Lie algebra, basis elements of which are the operators (5.16) and the generators*

$$\begin{aligned}
&\hat{p}_\mu=p_\mu,\qquad \hat{J}_{\mu\nu}=x'_\mu p_\nu-x'_\nu p_\mu,\\
&\hat{D}=x'_\mu p^\mu+i,\qquad \hat{K}'_\mu=-x'_\nu x'^\mu p_\mu+2x'_\mu\hat{D},
\end{aligned}\tag{5.18}$$

where

$$\begin{aligned}
&x'_0=x_0,\\
&x'_a=x_a+(S_b-S_c)(\sqrt{3}p-p_1-p_2-p_3)+S_d\tilde{p}_d(\sqrt{3}\hat{p}_a+1)+\\
&\qquad+(p_b-p_c)(S_1+S_2+S_3)\{p[3p+\sqrt{3}(p_1+p_2+p_3)]\}^{-1}.
\end{aligned}$$

The proof may be carried out in full analogy with the proof of theorem 2 (but using the operator (5.8) instead of (3.3)). The operators (5.18) satisfy the algebra (2.2) and (2.3) and commute with (5.16).

It is not difficult to generalise the statements of theorem 4 to the case of "Dirac-like" equations for massless particles of any spin (Fushchych and Nikitin [11], Nikitin and Fushchych [22]).

We note that the generators (5.16) and (5.17) are nonlocal (integro-differential) ones. This means that the invariance algebra of the Maxwell equations which we have obtained in principle cannot be obtained in the classical Lie approach, where, as is well known, the group generators always belong to the class of differential first-order operators.

1. Bateman H., *Proc. London Math. Soc.*, 1909, **8**, 223–264.
2. Bose S.K., Parker R., *J. Math. Phys.*, 1969, **10**, 812–813.
3. Cunningham E., *Proc. Lond. Math. Soc.*, 1909, **8**, 77–97.
4. Dirac P.A.M., *Ann. Math.*, 1936, **37**, 429–435.
5. Fock V.A., *Z. Phys.*, 1935, **98**, 145–149.
6. Fushchych W.I., Preprint E-70-32, Institute for Theoretical Phisycs, Kiev, 1970.
7. Fushchych W.I., *Teor. Mat. Fiz.* 1971, **7**, 3–12 (transl. *Theor. Math. Phys.*, 1971, **7**, 3–11).
8. Fushchych W.I., *Nuovo Cim. Lett.*, 1973, **6**, 133–138.
9. Fushchych W.I., *Nuovo Cim. Lett.*, 1974, **11**, 508–512.
10. Fushchych W.I., Nikitin A.G., *Nuovo Cim. Lett.*, 1977, **19**, 347–352.
11. Fushchych W.I., Nikitin A.G., Preprint 77-3, Mathematical Institute, Kiev, 1977.
12. Fushchych W.I., Nikitin A.G., *Nuovo Cim. Lett.*, 1978, **21**, 541–546.
13. Gross L., *J. Math. Phys.*, 1964, **5**, 687–695.
14. Heaviside O., Electromagnetic Theory, London, 1893.
15. Larmor, Collected papers London, 1928.
16. Levi-Leblond, *Am. J. Phys.*, 1971, **39**, 502–506.
17. Lomont I.S., *Phys. Rev.*, 1958, **111**, 1700–1709.
18. Lomont I.S., Moses H.E., *J. Math. Phys.*, 1962, **3**, 405–408.
19. Mack G., Salam A., *Ann. Phys., NY*, 1969, **53**, 174–202.
20. McLennan A., *Nuovo Cim.*, 1956, **3**, 1360–1380.
21. Moses H.E., *Nuovo Cim. Suppl.*, 1958, **7**, 1–18.
22. Nikitin A.G., Fushchych W.I., *Teor. Mat. Fiz.*, 1978, **34**, 319–333.
23. Nitikin A.G., Segeda Yu.N., Fushchych W.I., *Teor. Mat. Fiz.*, 1976, **29**, 82–94 (transl. *Theor. Math. Phys.*, 1976, **29**, 943–954).
24. Ovsjannikov L.V., The Group Analyses of Differential Equations, Moscow, Nauka, 1978.
25. Rainich G.Y., *Trans. Am. Math. Soc.*, 1925, **27**, 106–125.