On the new invariance group of Maxwell equations

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It was Heaviside [1] who first called attention to the invariance of Maxwell equations under the transformations

$$\boldsymbol{E} \to \pm\boldsymbol{H}, \qquad \boldsymbol{H} \to \pm\boldsymbol{E}.$$

Then Larmor [2] and Rainich [3] generalized this fact and demonstrated that Maxwell equations well invariant under the one-parametrical group of transformations of a kind

$$\boldsymbol{E} \to \boldsymbol{E}\cos\theta + \boldsymbol{H}\sin\theta,$$
$$\boldsymbol{H} \to \boldsymbol{H}\cos\theta - \boldsymbol{E}\sin\theta.$$

It is well-known also, that Maxwell equations are invariant under 15-parametrical conformal group C_4, which includes 10-parametrical Poincaré group and 5-parametrical conformal transformations [4].

Within the framework of Lie approach [5] the 16-parametrical group of the abovementioned transformations is maximally extensive symmetry group of Maxwell equations.

It was demonstrated in works [6, 7] that all relativistic motion equations possess an additional (non obvious) invariance which in principle could not be found by the classical Lie method. Specifically in [7] there was formulated the theorem about the additional invariance of Maxwell equations

$$\begin{aligned} &\boldsymbol{p}\times\boldsymbol{H} = -i\frac{\partial \boldsymbol{E}}{\partial t}, \qquad \boldsymbol{p}\times\boldsymbol{E} = i\frac{\partial \boldsymbol{H}}{\partial t}, \\ &\boldsymbol{p}\cdot\boldsymbol{H} = 0, \qquad \boldsymbol{p}\cdot\boldsymbol{E} = 0, \qquad p_a = -i\frac{\partial}{\partial x_a}, \end{aligned} \tag{1}$$

under the group $U_2 \otimes U_2$. This group is generated not by the local co-ordinate transformations, but by the transformations of vectors of electric and magnetic fields of the kind

$$\begin{aligned} &\boldsymbol{E} \to \boldsymbol{E}' = \boldsymbol{f}\left(\boldsymbol{E}, \boldsymbol{H}, \frac{\partial \boldsymbol{E}}{\partial x_a}, \frac{\partial \boldsymbol{H}}{\partial x_a}, \frac{\partial^2 \boldsymbol{E}}{\partial x_a \partial x_b}, \frac{\partial^2 \boldsymbol{H}}{\partial x_a \partial x_b}, \dots\right), \\ &\boldsymbol{H} \to \boldsymbol{H}' = \boldsymbol{g}\left(\boldsymbol{E}, \boldsymbol{H}, \frac{\partial \boldsymbol{E}}{\partial x_a}, \frac{\partial \boldsymbol{H}}{\partial x_a}, \frac{\partial^2 \boldsymbol{E}}{\partial x_a \partial x_b}, \frac{\partial^2 \boldsymbol{H}}{\partial x_a \partial x_b}, \dots\right), \end{aligned} \tag{2}$$

where the functions $\boldsymbol{f}$ and $\boldsymbol{g}$ in general depend on any-order derivatives of $\boldsymbol{E}$ and $\boldsymbol{H}$ and on eight arbitrary parameters. However the explicit form of the functions $\boldsymbol{f}$ and $\boldsymbol{g}$ had not been found in [7].

The aim of the prevent work is to find the explicit form of the transformations (2), by which eqs. (1) remain invariant, and which form the representation of the group

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$U_2 \otimes U_2$. The result will be obtained with the help of non-Lie method for investigation of the group properties of differential equations, which has been proposed and developed in [6–10].

We shall prove the following assertion:

Theorem 1. *The Maxwell equations (1) are invariant with respect to the eight-parametrical transformations*

$$\begin{aligned} H_a &\to H_a \cos\theta_1 + iD_{ab}E_b \sin\theta_1, \\ E_a &\to E_a \cos\theta_1 + iD_{ab}H_b \sin\theta_1, \end{aligned} \tag{3a}$$

$$\begin{aligned} H_a &\to H_a \cos\theta_2 + i\varepsilon_{abc}\hat{p}_b D_{cd}E_d \sin\theta_2, \\ E_a &\to E_a \cos\theta_2 + i\varepsilon_{abc}\hat{p}_b D_{cd}H_d \sin\theta_2, \end{aligned} \tag{3b}$$

$$\begin{aligned} H_a &\to H_a \cos\theta_3 - \varepsilon_{abc}\hat{p}_b H_c \sin\theta_3, \\ E_a &\to E_a \cos\theta_3 - \varepsilon_{abc}\hat{p}_b E_c \sin\theta_3, \end{aligned} \tag{3c}$$

$$\begin{aligned} H_a &\to H_a \cos\theta_4 - i\varepsilon_{abc}\hat{p}_b D_{cd}H_d \sin\theta_4, \\ E_a &\to E_a \cos\theta_4 + i\varepsilon_{abc}\hat{p}_b D_{cd}E_d \sin\theta_4, \end{aligned} \tag{3d}$$

$$\begin{aligned} H_a &\to H_a \cos\theta_5 + D_{ab}H_b \sin\theta_5, \\ E_a &\to E_a \cos\theta_5 - D_{ab}E_b \sin\theta_5, \end{aligned} \tag{3e}$$

$$\begin{aligned} H_a &\to H_a \cos\theta_6 + E_a \sin\theta_6, \\ E_a &\to E_a \cos\theta_6 - H_a \sin\theta_6, \end{aligned} \tag{3f}$$

$$\begin{aligned} H_a &\to H_a \cos\theta_7 + i\varepsilon_{abc}\hat{p}_b E_c \sin\theta_7, \\ E_a &\to E_a \cos\theta_7 - i\varepsilon_{abc}\hat{p}_b H_c \sin\theta_7, \end{aligned} \tag{3g}$$

$$H_a \to H_a \exp[i\theta_8], \qquad E_a \to E_a \exp[i\theta_8], \tag{3h}$$

where

$$\begin{aligned} D_{ad} &= \left[\left(p_a^2 p_b^2 + p_a^2 p_c^2 - p_b^2 p_c^2\right)\delta_{ad} + p_1 p_2 p_3 (p_b \delta_{cd} + p_c \delta_{bd} - p_a \hat{p}_d)\right] L^{-1}, \\ L &= \frac{1}{\sqrt{2}}\left[p_1^4\left(p_2^2 - p_3^2\right)^2 + p_2^4\left(p_3^2 - p_1^2\right)^2 + p_3^4\left(p_1^2 - p_2^2\right)^2\right], \end{aligned}$$

$\hat{p}_a = p_0/p$, (a, b, c) *is the cycle* $(1, 2, 3)$, θ_k *are arbitrary real parameters,*

$$p = \left(p_1^2 + p_2^2 + p_3^2\right)^{1/2}.$$

Proof. The invariance of eqs. (1) under the trivial phase transformations (3h) is obvious. It is not difficult to make sure by the direct verification, that eqs. (1) are invariant also under the rest of the transformations (3a)–(3g), and that the infinitesimal operators of transformations (3) satisfy Lie algebra of the group $U_2 \otimes U_2$. But such a way is too cumbersome. More constructive approach, which gives the method for finding the explicit form of transformations (3) consists in reduction of the eqs. (1) to such a form, for which the theorem statements become obvious.

Let us write eqs. (1) in the form [11]

$$\begin{aligned} &L_1\Psi = 0, \qquad L_1 = i\frac{\partial}{\partial t} - \sigma_2 \boldsymbol{S}\cdot\boldsymbol{p}, \\ &L_2\Psi \neq 0, \qquad L_2 = \sigma_2 \boldsymbol{S}\cdot\boldsymbol{p}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} &\Psi = \begin{pmatrix} \boldsymbol{H} \\ \boldsymbol{E} \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} \hat{0} & -\hat{I} \\ \hat{I} & \hat{0} \end{pmatrix}, \quad S_a = \begin{pmatrix} \hat{S}_a & \hat{0} \\ \hat{0} & \hat{S}_a \end{pmatrix}, \\ &\hat{S}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \hat{S}_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \qquad \hat{S}_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{5}$$

$\hat{I}$ and $\hat{0}$ are the three-row square unit and zero matrices.

Following the main algorithm of non-Lie approach of investigation of differential-equation group properties [6–10], let us transform the eqs. (4) to the canonical quasi-diagonal form. Using for this purpose the operator

$$W = \exp\left[(\sigma_2 - 1)D\boldsymbol{S}\cdot\hat{\boldsymbol{p}}\,\frac{\pi}{4}\right]\exp\left[-i\frac{S_a\hat{p}_a}{\tilde{p}}\operatorname{arctg}\frac{\tilde{p}}{p_1+p_2+p_3}\right], \tag{6}$$

where

$$\begin{aligned} &\tilde{p}_a = p_b - p_c, \qquad \tilde{p} = \left(\tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{p}_3^2\right)^{1/2}, \\ &D = \left\{ \sum_{a\neq b\neq c} \left[\left(p_a^2p_b^2 + p_a^2p_c^2 - p_b^2p_c^2\right)\left(1 - S_a^2\right) + \right.\right. \\ &\qquad \left.\left. + p_1p_2p_3S_aS_bp_c\right] - pp_1p_2p_3\left[1 - (\boldsymbol{S}\cdot\hat{\boldsymbol{p}})^2\right]\right\} L^{-1}, \end{aligned} \tag{7}$$

and taking into account the relations

$$D\boldsymbol{S}\cdot\boldsymbol{p} = -\boldsymbol{S}\cdot\boldsymbol{p}D, \qquad D\left(\boldsymbol{S}\cdot\hat{\boldsymbol{p}}\right)^2 = D, \qquad \boldsymbol{S}\cdot\boldsymbol{p}D^2 = \boldsymbol{S}\cdot\boldsymbol{p} \tag{8}$$

one obtains from (5) the following equations:

$$\begin{aligned} &L_1'\Psi' = 0, \qquad L_1' = WL_1W^{-1} = i\frac{\partial}{\partial t} - \Lambda p, \qquad \Psi' = W\Psi, \\ &L_2'\Psi' \neq 0, \qquad L_2' = WL_2W^{-1} = \Lambda p, \qquad \Lambda = \frac{1}{\sqrt{3}}(S_1 + S_2 + S_3), \end{aligned} \tag{9}$$

Equations (9) by definition are invariant with respect to the transformations $\Psi' \to Q_k'\Psi'$ if an operator Q_k' satisfies the conditions

$$[L_1', Q_k']\Psi' = [L_2', Q_k']\Psi' = 0. \tag{10}$$

Conditions (10) are obviously satisfied by the matrices

$$Q_a' = \sigma_a\Lambda, \qquad Q_{a+3}' = \sigma_a\Lambda^2, \qquad Q_7' = \Lambda, \qquad Q_8' = I, \qquad a = 1, 2, 3. \tag{11}$$

The operators (11) satisfy the commutation relations

$$\begin{aligned} &[Q'_a, Q'_b] = [Q'_{a+3}, Q'_{3+b}] = i\varepsilon_{abc}Q'_c, \\ &[Q'_b, Q'_{3+b}] = i\varepsilon_{abc}Q'_{3+c}, \qquad [Q'_7, Q'_k] = [Q'_8, Q'_k] = 0, \end{aligned} \tag{12}$$

i.e. form the Lie algebra of the group $U_2 \otimes U_2$. Since $(Q'_k)'\Psi' = \Psi'$, the operators Q'_a, Q'_{3+a} the representation $D\left(\frac{1}{2}, 0\right) \oplus D\left(0, \frac{1}{2}\right)$ of the group $SU_2 \otimes SU_2$. It is not difficult make certain to show, that formulae (11) give the complete set of numerical matrices, satisfying the conditions (10).

It follows from the above, that eqs. (9) are invariant under an arbitrary transformation from the group $U_2 \otimes U_2$

$$\Psi' \to \exp\left[iQ'_k\theta_k\right]\Psi' = \left(\cos\theta_k + iQ'_k\sin\theta_k\right)\Psi', \tag{13}$$

where θ_k is an arbitrary parameter, and Q'_k is any generator, given by formula (11) (no sum over k!). Returning with the help of the operator (6) to the initial Ψ-representation one obtains from (13) the transformations, under which eqs. (4) remain invariant

$$\Psi \to (\cos\theta_k + iQ_k\sin\theta_k)\Psi, \tag{14}$$

where

$$\begin{aligned} &Q_k = W^{-1}Q'_kW, \qquad Q_1 = \sigma_1 D, \qquad Q_2 = i\sigma_2\boldsymbol{S}\cdot\hat{\boldsymbol{p}}D, \\ &Q_3 = \boldsymbol{S}\cdot\hat{\boldsymbol{p}}, \qquad Q_{3+a} = \sigma_2\boldsymbol{S}\cdot\hat{\boldsymbol{p}}, \qquad Q_7 = \sigma_2\boldsymbol{S}\cdot\hat{\boldsymbol{p}}, \qquad Q_8 = I. \end{aligned} \tag{15}$$

Substituting (15), (8), (6) into (14), one comes to formulae (3). The theorem is proved.

So we have found new eight-parametrical symmetry group of Maxwell equations, given by the transformations (3). These transformations are unitary with respect to the scalar product

$$(\Psi_1, \Psi_2) = \int d^3x\, \Psi_1^\dagger\Psi_2. \tag{16}$$

Substituting expression (6) for the function Ψ into (16) and claiming for the $\boldsymbol{E}$ and $\boldsymbol{H}$ to be Hermitian, one comes to the conclusion, that the transformations (3) preserve the value

$$\mathcal{E} = \int d^3x\,(\boldsymbol{E}^2 + \boldsymbol{H}^2),$$

which determines the energy of an electromagnetic field.

The main property of the transformations (3) is that they are carried out by nonlocal (integro-differential) operators (the only exceptions are the phase transformation (3h) and the transformation (3f), which coincides with Heaviside–Larmor–Rainich one). It is to be emphasized, that tin transformations (3) have nothing to do with Lorentz transformations, since they realize unitary finite-dimensional representation of compact group $U_2 \otimes U_2$.

In analogy with theorem 1 it can be demonstrated, that the first couple pair of Maxwell equations

$$\boldsymbol{p}\times\boldsymbol{E} = i\frac{\partial\boldsymbol{H}}{\partial t}, \qquad \boldsymbol{p}\times\boldsymbol{H} = -i\frac{\partial\boldsymbol{E}}{\partial t}$$

is invariant under the group $U_2 \otimes U_2 \otimes U_2$.

Let us note in conclusion that transformations (3) do not form a closed group jointly with ordinary local Lorentz and conformal transformations. But the representation of the comformal group C_4 on the set of solutions of eqs. (1) may be realized also in the class of nonlocal integro-differential operators [12]. In such a way the following statement may be proved [10]:

Theorem 2. *Maxwell equations (1) are invariant under 23-dimensional Lie algebra of group* $C_4 \otimes U_2 \otimes U_2$.

We do not give here the explicit form of this algebra basis elements.

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