On the non-relativistic motion equations in the Hamiltonian form

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The Galilean-invariant equations for particles with arbitrary spins have been obtained, which describe properly the spin-orbit and the Darwin interactions of a particle with an external field. The Hagen–Hurley non-relativistic equations have been reduced to the Hamiltonian form.

1. Introduction

It has been noted in many books and papers (see e.g. [2, 8, 10, 12, 13, 16]) that the Galilean invariant non-relativistic equations for particles with spins do not give the complete description of the particle movement in external electromagnetic fields, because such equations (of Pauli, or Levi–Leblond [16], of Hagen–Hurley [10, 12, 13]) do not take into account the spin-orbit and the Darwin interactions. It is generally accepted to think [16] that such interactions are truly relativistic effects, and, for instance, if the particle spin $s = 1/2$, only the Dirac relativistic equation describes them naturally. In our just published paper [7] this widespread opinion has been refuted, i.e. the Galilean invariant equations for the particles with the lowest spins $s = 1/2, 1, 3/2$ had been derived, which lead to the spin-orbit and to the Darwin interactions by the standard substitution $p_\mu \to \pi_\mu = p_\mu - eA_\mu$. In [6] the analogous equations have been obtained for a non-relativistic particle with any spin.

Peculiarity of such equations is that they have not redundant (unphysical) components unlike other known non-relativistic equations for arbitrary spin particles [10, 12, 13]. The wave function in the equations [7] has only $2(2s+1)$ components, and the energy operator has both positive and negative eigenvalues.

The present work has the two principal aims: first, to obtain the Galilean invariant equations for the particles with any spin in the Hamiltonian form without negative energy eigenvalues, which naturally describes not only the dipole, but also the spin-orbit and the Darwin interactions; and secondly, to establish the Hamiltonian form of the non-relativistic Levi–Leblond–Hagen–Hurley (LHH) equations.

2. The Hamiltonian form of the equations with redundant components

Galilean-invariant first-order wave equation for the particle with spin $s = \frac{1}{2}$ had been obtained by Levi–Leblond [16]. Then Hagen and Hurley [10, 12, 13] have obtained such equations for arbitrary spin particles.

It is convenient for our purposes to write the LHH equations [10, 12, 13] in the form

$$
\begin{gathered}
[\beta_\mu p_\mu + (1-\beta_0)2m]\,\Psi(t,\vec{x}) = 0, \\
\mu = 0,1,2,3, \qquad p_a = i\frac{\partial}{\partial x_a}, \qquad p_0 = i\frac{\partial}{\partial t},
\end{gathered}
\tag{2.1}
$$

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where $\Psi(t,\vec{x})$ is the $(6s+1)$ component wave function, and β_μ are the matrices having the following structure:

$$\beta_0 = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \beta_a = \frac{1}{s}\begin{pmatrix} 0 & S_a & K_a^\dagger \\ S_a & 0 & 0 \\ K-a & 0 & 0 \end{pmatrix}, \tag{2.2}$$

and I is the $(2s+1)$-dimensional unit matrix, S_a are the $(2s+1)$-dimensional matrices, which realize the irreducible representation of the algebra $O(3)$, K_a are the matrices with $(2s-1)$ rows and $(2s+1)$ columns, satisfying the condition

$$S_aS_b + K_a^\dagger K_b = is\varepsilon_{abc}S_c + s^2\delta_{ab}. \tag{2.3}$$

The peculiarity of equations (2.1) in comparison with the Dirac relativistic equation is that even for $s=1/2$ the matrix β_0 is singular. Therefore some difficulties arise in reducing the LHH equations to the Hamiltonian form. The analogous problems take place also in the relativistic Proca, Kemmer–Duffin, and Bhabha equations [1, 11, 15, 18, 19].

In works [1, 11, 15, 18], the Kemmer–Duffin equation

$$\left(\tilde{\beta}_\mu p_\mu + m\right)\Psi = 0, \tag{2.4}$$

where $\tilde{\beta}_\mu$ are (10×10)-Kemmer–Duffin matrices, has been reduced to the form

$$i\frac{\partial}{\partial t}\Psi = H\Psi, \qquad H = \left(\tilde{\beta}_0\tilde{\beta}_a - \tilde{\beta}_a\tilde{\beta}_0\right)p_a + \tilde{\beta}_0 m, \tag{2.5}$$

$$\left[\left(1-\tilde{\beta}_0^2\right)m + \tilde{\beta}_a p_a\tilde{\beta}_0^2\right]\Psi = 0, \qquad a=1,2,3, \tag{2.6}$$

where H is the Kemmer–Duffin particle Hamiltonian, and (2.6) is the subsidiary condition, which removes the redundant components of the wave function Ψ.

The form (2.1) of the non-relativistic equations [10, 12, 13] shows that the methods of works [1, 11] may be used to reduce the LHH equations to the Schrödinger form

$$i\frac{\partial}{\partial t}\Psi = H\Psi. \tag{2.7}$$

Our task is to find the exact form of the Hamiltonian H.

The matrices β_0 and $(1-\beta_0)$ are the projectors on the subspaces of upper and lower components of the wave function Ψ. They satisfy the conditions

$$\beta_0^2 = \beta_0, \qquad (1-\beta_0)\beta_a = \beta_a\beta_0. \tag{2.8}$$

In order to reduce equation (2.1) to the form (2.7) we first multiply (2.1) by $(1-\beta_0)$. Using (2.8), one obtains

$$(1-\beta_0)\Psi = -\frac{\beta_a p_a}{m}\beta_0\Psi, \tag{2.9}$$

or after the multiplication by p_0,

$$(1-\beta_0)p_0\Psi = -\frac{\beta_a p_a}{2m}\beta_0 p_0\Psi. \tag{2.10}$$

On the other hand, multiplying (2.1) by β_0, one obtains

$$\beta_0 p_0 \Psi = -\beta_0 \beta_a p_a \Psi. \tag{2.11}$$

Substituting (2.11) into (2.10) and adding the result to (2.11), we come to the equation

$$i\frac{\partial}{\partial t}\Psi = \left[(1-\beta_0)\frac{(\beta_a p_a)^2}{2m} - \beta_a p_a - (1-\beta_0)2m\right]\Psi. \tag{2.12}$$

Equation (2.12) with the additional condition (2.10) is completely equivalent to (2.1).

Thus we have reduced the LHH equations to the Hamiltonian form.

3. Transition to the diagonal representation

Equations (2.12), (2.10) as well as equation (2.1) are invariant with respect to the Galilei group G. Indeed, on the set $\{\Psi\}$ of the solutions of these equations the following representation of the algebra G is realized:

$$\begin{gathered} P_0 = p_0 = i\frac{\partial}{\partial t}, \qquad P_a = p_a = -i\frac{\partial}{\partial x_a}, \qquad J_{ab} = x_a p_b - x_b p_a + S_{ab}, \\ G_a = t p_a - m x_a + \lambda_a, \qquad \lambda_a = -\frac{i}{2}\beta_a\beta_0, \end{gathered} \tag{3.1}$$

where the matrices S_{ab} realize the direct sum $D(s) \oplus D(s) \oplus D(s-1)$ of the algebra $O(3)$ representations. One can readily see that the generators (3.1) are non-Hermitian with respect to the usual scalar product

$$(\Psi_1, \Psi_2) = \int d^3x\, \Psi_1^\dagger \Psi_2. \tag{3.2}$$

The aim of this section is to transform equations (2.12), (2.10) and the operators (3.1) to such a form that the wave function $\Psi(t, \vec{x})$ has only $2s+1$ non-zero components and the generators of the Galilei group representation are Hermitian with respect to (3.2). It is achieved by the transformation to the new wave function

$$\Psi \to \Psi' = V\Psi, \qquad V = \exp\left(-i\frac{\vec{\lambda}\vec{p}}{m}\right). \tag{3.3}$$

The transformed generators (3.1) take the form

$$\begin{gathered} P'_a = V P_a V^{-1}, \qquad J'_{ab} = V J_{ab} V^{-1} = J_{ab}, \\ P'_0 = V P_0 V^{-1} = p_0, \qquad G'_a = V G_a V^{-1} = t p_a - m x_a. \end{gathered} \tag{3.4}$$

These operators are apparently Hermitian in the scalar product (3.2). Equations (2.12), (2.10) after the transformation (3.4) have been reduced to the diagonal form

$$\frac{p_a^2}{2m}\Psi' = i\frac{\partial}{\partial t}\Psi', \tag{3.5}$$

$$(1-\beta_0)\Psi' = 0. \tag{3.6}$$

It follows from (3.6), (2.2) that the wave function Ψ' has only $2s+1$ non-zero components. Thus condition (2.10) (which is equivalent to (3.6) serves to remove $4s$ redundant components from the $(6s+1)$ component wave function $\Psi(t, \vec{x})$.

One can use the operator (3.3) to construct the positive definite scalar product on the set of the solutions of equations (2.12), (2.10). Indeed, it follows from the hermitici of the operators (3.4) with respect to (3.2) that the generators (3.4) are Hermitian with respect to

$$(\Psi_1, \Psi_2) = \int d^3x\ \Psi_1^\dagger M \Psi_2, \tag{3.7}$$

where

$$M = V^\dagger V = 1 - \frac{i}{m}(\vec{\lambda}\vec{p} - \vec{\lambda}^\dagger\vec{p}) + \frac{(\vec{\lambda}^\dagger\vec{p})\cdot(\vec{\lambda}\vec{p})}{m^2}. \tag{3.8}$$

For the case $s = 1/2$ the transformation operator (3.3) and the metric operators (3.8) have the form

$$V = \begin{pmatrix} I & 0 \\ -\dfrac{\vec{\sigma}\cdot\vec{p}}{m} & I \end{pmatrix}, \qquad M = \begin{pmatrix} I\left(1 + \dfrac{p_a^2}{m^2}\right) & -\dfrac{\vec{\sigma}\cdot\vec{p}}{m} \\ -\dfrac{\vec{\sigma}\cdot\vec{p}}{m} & I \end{pmatrix}, \tag{3.9}$$

where σ_a are the usual Pauli matrices.

It follows from the above that the transformation (3.3) may be considered as the non-relativistic analog of the Foldy–Wouthuysen transformation [3].

Equation (2.10) is not the only Gallilean invariant condition which can be added to (2.12) in order to remove the redundant components of the wave function Ψ. For instance, one can use for this purpose the subsidiary condition of the form

$$\left\{1 - \frac{1}{2}\left[\left(H - \frac{(\vec{\beta}\vec{p})^2}{4m} + m\right), \left[\left(H - \frac{(\vec{\beta}\vec{p})^2}{4m} + m\right)^2\right]^{-1/2}\right]_+\right\}\Psi = 0. \tag{3.10}$$

Equations (2.12), (3.10), as (2.12), (2.6), are Galilean invariant and can be reduced to the diagonal form (3.5), (3.6) by the unitary transformation

$$\Psi \to U\Psi, \qquad U = \frac{2m + (1 - 2\beta_0)\beta_a p_a}{\sqrt{4m^2 + (\beta_a p_a)^2}}. \tag{3.11}$$

On the set of the solutions of equations (2.12), (3.10) the Galilei group generators have the form

$$\begin{gathered} P_0 = i\frac{\partial}{\partial t}, \qquad P_a = p_a = -i\frac{\partial}{\partial x_a}, \qquad J_{ab} = x_a p_b - x_b p_a + S_{ab}, \\ G_a = tp_a - mX_a, \qquad X_a = x_a + [U^\dagger, x_a]_- U. \end{gathered} \tag{3.12}$$

The generators (3.12) are Hermitian with respect to the usual scalar product (3.2) but, in contrast to (3.1), are non-local (integral) operators.

4. The Hamiltonian equations without redundant components

In this section we obtain new (different from (2.1)) equations for arbitrary spin particles, which are invariant under the Galilei group G. The main property of these equations is that the wave function of a particle with spin s has $2(2s+1)$ components.

It allows to establish the direct connection between our equations and the relativistic equations without redundant components.

We shall start from the following representation for the generators of the Galilei group

$$
\begin{aligned}
&P_0 = i\frac{\partial}{\partial t}, \qquad P_a = p_a = -i\frac{\partial}{\partial x_a}, \qquad J_{ab} = x_a p_b - x_b p_a + \hat{S}_{ab},\\
&G_a = tp_a - mx_a + \tilde{\lambda}_a, \qquad \hat{S}_{ab} = \begin{pmatrix} \hat{S}_c & 0 \\ 0 & \hat{S}_c \end{pmatrix}, \qquad (a,b,c) = (1,2,3),
\end{aligned}
\tag{4.1}
$$

where the matrices $\hat{S}_c$ realize the irreducible representation $D(s)$ of the group $O(3)$, and $\tilde{\lambda}_a$ are arbitrary numerical matrices which have to be such that the generators (4.1) satisfy the algebra G. It can be shown that the most general (up to equivalence) form of the matrices $\tilde{\lambda}_a$, satisfying such a requirement, is

$$
\tilde{\lambda}_a = k(\sigma_1 + i\sigma_2)S_a, \qquad S_a = \frac{1}{2}\varepsilon_{abc}\hat{S}_{bc},
\tag{4.2}
$$

where σ_1, σ_2 are the $2(2s+1)$-dimensional Pauli matrices which commute with $\hat{S}_{ab}$, k is an arbitrary constant.

To obtain the Galilean invariant equations in the form

$$
L_s\Phi(t,\vec{x}) = 0
\tag{4.3}
$$

we must find the operators L_s satisfying the conditions

$$
[P_\mu, L_s]_- = [J_{ab}, L_s]_- = [G_a, L_s]_- = 0.
\tag{4.4}
$$

Thus our problem has been reduced to the solution of the commutation relations (4.4).

In order to solve relations (4.4) we reduce the generators (4.4) to the diagonal representation

$$
\begin{aligned}
&P_0' = VP_0V^{-1} = i\frac{\partial}{\partial t}, \qquad P_a = Vp_aV^{-1} = p_a,\\
&J_{ab}' = VJ_{ab}V^{-1} = x_a p_b - x_b p_a + \hat{S}_{ab}, \qquad G_a' = VG_aV^{-1} = tp_a - mx_a.
\end{aligned}
\tag{4.5}
$$

The transition operator V has the form

$$
V = \exp\left(i\frac{\vec{\lambda}\vec{p}}{m}\right).
\tag{4.6}
$$

We require that the wave function of the spin-s particle has, in the diagonal representation (4.5), $2s+1$ non-zero components. This requirement may be written in the form of the Galilean invariant condition

$$
(1+\sigma_3)\Phi' = 0.
\tag{4.7}
$$

Another natural assumption is that each component of Φ' satisfies the non-relativistic Schrödinger equation

$$
i\frac{\partial}{\partial t}\Phi' = \frac{p_a^2}{2m}\Phi'.
\tag{4.8}
$$

One can write (4.7) and (4.8) in the form of the single equation

$$L'_s\Phi' = \left[\frac{1}{2}(\sigma_1 + i\sigma_2)\left(i\frac{\partial}{\partial t} - \frac{p_a^2}{2m}\right) + \frac{1}{2}(\sigma_1 - i\sigma_2)2m\right]\Phi' = 0. \tag{4.9}$$

Equation (4.9) is Galilean invariant inasmuch as the following relations are satisfied

$$[L'_s, P'_\mu]_- = [L'_s, J_{ab}]_- = [L'_s, G'_a]_- = 0. \tag{4.10}$$

To obtain equation (4.9) in the representation (4.1) it is sufficient to use the transition operator (4.6). Making the transformation

$$\Phi' \to \Phi = V^{-1}\Phi', \qquad L'_s \to L_s = V^{-1}L'_sV, \tag{4.11}$$

one obtains equation (4.3), where

$$L_s = \frac{1}{2}(\sigma_1 + i\sigma_2)\left(i\frac{\partial}{\partial t} - \frac{p_a^2}{2m} + k^2\frac{(\vec{S}\vec{p})^2}{2m}\right) + (\sigma_1 - i\sigma_2)m + \sigma_3 k(\vec{S}\vec{p}). \tag{4.12}$$

Thus we have found the Galilean invariant equation (4.3), (4.12) for the $2(2s+1)$-component wave function. For $s = 1/2$, $k = 1/s$ ((4.3), (4.12)) coincide with the Levi–Leblond equation [16].

Equations (4.3), (4.12), as well as equation (2.1), may be reduced to the Hamiltonian form. Indeed, multiplying (4.12) by $i\sigma_2$, one obtains from (4.3), (4.12) the following expression:

$$\left[\tilde{\beta}_0 B - \tilde{\beta}_a\tilde{p}_a + (1 - \tilde{\beta}_0)2m\right]\Phi = 0, \tag{4.13}$$

where

$$B = i\frac{\partial}{\partial t} - \frac{p_a^2}{2m} + \frac{k^2(\vec{S}\vec{p})^2}{2m}, \qquad \tilde{\beta}_0 = \frac{1}{2}(1 + \sigma_3), \qquad \tilde{\beta}_0 = -k\sigma_1 S_a. \tag{4.14}$$

The matrices $\tilde{\beta}_0$, $\tilde{\beta}_a$ satisfy thereby relations (2.8) as the β_0, β_a. Repeating the computations (2.9)–(2.12) one easily obtains from (4.13) the equations

$$i\frac{\partial}{\partial t}\Phi = H\Phi, \qquad H = \frac{p_a^2}{2m} - \tilde{\beta}_0\tilde{\beta}_a\tilde{\beta}_b\frac{(p_ap_b + p_bp_a)}{4m} + \tilde{\beta}_a p_a - (1 - \tilde{\beta}_0)2m, \tag{4.15}$$

$$(1 - \tilde{\beta}_0)\Phi = -\frac{\tilde{\beta}_a p_a}{2m}\tilde{\beta}_0\Phi. \tag{4.16}$$

The system of equations (4.15), (4.16) is completely equivalent to (4.3), (4.12).

Thus we have obtained Galilean invariant equations (4.15), (4.16) for a particle with arbitrary spin s, moreover, the wave function has $2(2s+1)$ components. As in Section 2, the subsidiary condition (4.16) is not the only one which can be added to (4.15) in order to remove the redundant components of the wave function Φ. For instance, it is possible to postulate that the wave function Φ satisfy instead of (4.16) the following equation

$$\hat{\varepsilon}\Phi = \Phi, \quad \hat{\varepsilon} = \frac{1}{2}\left[\left\{H - \frac{p_a^2}{2m} + \frac{(k\vec{S}\vec{p})^2}{4m}\right\}, \left\{\left[H - \frac{p_a^2}{2m} + \frac{(k\vec{S}\vec{p})^2}{4m}\right]^2\right\}^{-\frac{1}{2}}\right]_+. \tag{4.17}$$

Equations (4.15), (4.17) are Galilean invariant. On the set of the solutions of these equations the following representation of the algebra G is realized

$$
\begin{aligned}
&P_0 = p_0 = i\frac{\partial}{\partial t}, \qquad P_a = p_a = -i\frac{\partial}{\partial x_a}, \qquad J_{ab} = x_a p_b - x_b p_a + \hat{S}_{ab},\\
&G_a = t p_a - m X_a, \qquad X_a = x_a + [U^\dagger, x_a]U, \\
&U = \left(1 + \tilde{\beta}_4\hat{\varepsilon}\right)\left(2 + \tilde{\beta}_4\hat{\varepsilon} + \hat{\varepsilon}\tilde{\beta}_4\right)^{-1/2}, \qquad \tilde{\beta}_4 = 2\tilde{\beta}_0 - 1.
\end{aligned}
\tag{4.18}
$$

The generators (4.18), as (3.12), are Hermitian with respect to the usual scalar product (3.2).

5. The non-relativistic particles in an external electromagnetic field

It is known from the relativistic equation theory that the equation of motions which are mathematically equivalent in the case of a free particle, leads to different physical consequences after the introduction of an interaction. It means that various mathematically equivalent representations for the equations are physically non-equivalent. The classical example of such a situation is the equation for an electron in the Dirac and in the Foldy–Wouthuysen (FW) [3] representations. If one introduces the minimal interaction into the free equation in the Dirac representation, the result is obtained which is in a good accordance with experimental data. If, however, one introduces the interaction into the free equation in the FW representation, any sensible result will not be obtained. Another example is the Kemmer–Duffin equation which does not lead to the spin-orbit and to the Darwin couplings by introducing the minimal interaction into the original free equation, but describes these couplings if one introduces the interaction into the mathematically equivalent equation in the Hamiltonian form [9].

It turns out that the analogous situations takes place also for the non-relativistic equations. We shall see, that equations (2.12), (4.15) in contrast to (2.1) and (4.3), (4.13), lead to the spin-orbit and to the Darwin couplings.

First we consider equation (2.12). After the replacement $p_\mu \to \pi_\mu = p_\mu - eA_\mu$ one obtains

$$
i\frac{\partial}{\partial t}\Psi = H(\vec{\pi})\Psi = \left\{(1-\beta_0)\frac{(\beta_a\pi_a)^2}{2m} - \beta_a\pi_a - (1-\beta_0)2m + eA_0\right\}\Psi. \tag{5.1}
$$

In order to obtain from (5.1) the equation for the $2(2s+1)$ component wave function it is necessary to remove the "odd" terms $\beta_a\pi_a$ in (5.1), i.e. to diagonalize the operator $H(\vec{\pi})$. In the presence of the interaction such a problem may be solved only approximately as in the relativistic case [3]. We shall solve this problem up to terms of order $1/m^2$ with the help of a set of successive unitary transformations. After the first transformation

$$
U_1 = \exp\left(-\beta_4\frac{\beta_a\pi_a}{2m}\right), \qquad \beta_4 = 2\beta_0 - 1 \tag{5.2}
$$

one obtains

$$
\begin{aligned}
H(\vec{\pi}) \to H^{(1)}(\vec{\pi}) = U_1 H(\vec{\pi}) U_1^\dagger = (1+\beta_4)\frac{(\beta_a\pi_a)^2}{4m} - m(1-\beta_4) + eA_0 - \\
-\frac{ie}{2m}\beta_4(\beta_a E_a) - \frac{ie}{8m^2}[\beta_a\pi_a, \beta_b E_b]_- + \frac{1}{12m^2}(\beta_a\pi_a)^3 + O\left(\frac{1}{m^3}\right),
\end{aligned}
\tag{5.3}
$$

$$E_a = -\frac{\partial A_0}{\partial x_a} - \frac{\partial A_a}{\partial t},$$

where $O(1/m^3)$ possesses the terms of a power $1/m^3$.

After the second transformation

$$U_2 = \exp\left(-\frac{ie}{4m^2}(\beta_a E_a)\right] \tag{5.4}$$

the operator $H^{(2)}(\vec{\pi})$ becomes

$$\begin{gathered} H^{(2)}(\vec{\pi}) = U_2 H^{(1)}(\vec{\pi}) U_2^\dagger - (1+\beta_4)\frac{(\beta_a\pi_a)^2}{4m} - m(1-\beta_4) + eA_0 + \\ + \frac{1}{12m^2}(\beta_a E_a)^3 - \frac{ie}{8m^2}[\beta_a\pi_a, \beta_b E_b]_- + O\left(\frac{1}{m^3}\right). \end{gathered} \tag{5.5}$$

At least, with the help of the operator

$$U_3 = \exp\left[\frac{1}{24m^3}\beta_4(\beta_a\pi_a)^3\right], \tag{5.6}$$

one obtains the final form of $H^{(3)}(\vec{\pi})$:

$$\begin{gathered} H^{(3)}(\vec{\pi}) = U_3 H^{(2)}(\vec{\pi}) U_3^\dagger = (1+\beta_4)\frac{(\beta_a\pi_a)^2}{4m} - \\ -(1-\beta_4)m + eA_0 - \frac{ie}{8m^2}[\beta_a\pi_a, \beta_b E_b]_- + O\left(\frac{1}{m^3}\right). \end{gathered} \tag{5.7}$$

It follows from (2.2) that the Hamiltonian $H^{(3)}(\vec{\pi})$ is a completely even (commuting with β_4) operator. On the set of Ψ^+, satisfying the condition

$$\frac{1}{2}(1+\beta)\Psi^+ = \Psi^+, \tag{5.8}$$

the Hamiltonian $H^{(2)}(\vec{\pi})$ has the form

$$\begin{gathered} H^{(3)}(\vec{\pi})\Psi^+ = \left\{\frac{\pi_a^2}{2m} - e\frac{\vec{S}\vec{H}}{2ms} + eA_0 - \right. \\ \left. -\frac{ie}{8m^2 s}\vec{S}\cdot(\vec{E}\times\vec{\pi} - \vec{\pi}\times\vec{E}) - \frac{e}{8m^2}\operatorname{div}\vec{E}\right\}\Psi^+, \\ H_a = -\frac{i}{2}\varepsilon_{abc}[\pi_b, \pi_c]_-. \end{gathered} \tag{5.9}$$

Thus, starting from the non-relativistic equation (5.1), we have obtained the approximate Hamiltonian (5.9) which describes not only the dipole $\left(-e\frac{\vec{S}\vec{H}}{2ms}\right)$, but also the spin-orbit $\left(-\frac{ie}{8m^2 s}\vec{S}(\vec{\pi}\times\vec{E} - \vec{E}\times\vec{\pi})\right)$ and the Darwin $\left(-\frac{e}{8m^2}\operatorname{div}\vec{E}\right)$ interaction of a charged particle with an external electromagnetic field. For the spin

$s=1/2$ particle the Hamiltonian (5.9) coincides with the one, obtained by Foldy and Wouthuysen [3] from the relativistic Dirac equation.

Now we appeal to equation (4.15) and introduce to it the minimal interaction $p_\mu \to \pi_\mu$. It leads to the Hamiltonian

$$\tilde{H}(\vec{\pi}) = \frac{\pi_a^2}{2m} - \tilde{\beta}_0 \frac{(\tilde{\beta}_a \pi_a)^2}{2m} - \tilde{\beta}_0 k^2 \frac{\vec{S}\vec{H}}{4m} + \tilde{\beta}_a \pi_a + (1-\tilde{\beta}_0)2m + eA_0. \tag{5.10}$$

This Hamiltonian, as well as (5.1), cannot be diagonalized exactly. By the analogy with (5.2)–(5.7), one can diagonalize (5.10) approximately with the help of the operator

$$\tilde{U} = \exp(iB_3)\cdot\exp(iB_2)\cdot\exp(iB_1), \tag{5.11}$$

where

$$B_1 = -\tilde{\beta}_4 \frac{\tilde{\beta}_a \pi_a}{2m}, \qquad \tilde{\beta}_4 = 2\tilde{\beta}_0 - 1, \tag{5.12}$$

$$B_2 = -e\frac{\tilde{\beta}_a E_a}{4m^2}, \tag{5.13}$$

$$\begin{aligned} B_3 = \frac{1}{8m^3}\Bigg\{ &-\frac{i}{k}\tilde{\beta}_4 (\tilde{\beta}_a \pi_a)^3 - [\tilde{\beta}_a \pi_a, \pi_a^2]_- - \\ &- \frac{ek^2}{4}[\tilde{\beta}_a \pi_a, S_b H_b]_- - \frac{ek^2}{4}[(\tilde{\beta}_a \pi_a), \tilde{\beta}_4 (S_b H_b)]_- \Bigg\}. \end{aligned} \tag{5.14}$$

As a result one obtains

$$\begin{aligned} H^{(3)}(\vec{\pi}) = {}& \frac{\pi_a^2}{2m} - \frac{1}{2}(1-\tilde{\beta}_4)\frac{(k\vec{S}\vec{\pi})^2}{2m} + \frac{k^2}{2}(1+\tilde{\beta}_4)\frac{e\vec{S}\vec{H}}{4m} - \\ &- (1-\tilde{\beta}_4)m + eA_0 - \frac{ie}{8m^2}k^2[\vec{S}\vec{\pi}, \vec{S}\vec{E}]_- + O\left(\frac{1}{m^3}\right). \end{aligned} \tag{5.15}$$

On the set of $\Phi^+ = \tilde{\beta}_4 \Phi^+$ this Hamiltonian takes the form

$$H^{(3)}(\vec{\pi})\Phi^+ = \left\{ \frac{\pi_a^2}{2m} - k^2 e\frac{\vec{S}\vec{H}}{4m} + eA_0 - i\frac{ek^2}{8m^2}[\vec{S}\vec{\pi}, \vec{S}\vec{E}]_- \right\}\Phi^+. \tag{5.16}$$

Using the identity

$$\begin{aligned} [\vec{S}\vec{\pi}, \vec{S}\vec{E}]_- \equiv {}& -\frac{i}{6}(3[S_a, S_b]_+ - 2\delta_{ab}s(s+1))\frac{\partial E_a}{\partial x_b} - \\ &- \frac{ie}{3}s(s+1)\,\mathrm{div}\,\vec{E} - \frac{i}{2}\vec{S}(\vec{E}\times\vec{\pi} - \vec{\pi}\times\vec{E}), \end{aligned} \tag{5.17}$$

one can rewrite equation (5.16) in the form

$$\begin{aligned} H^{(3)}(\vec{\pi})\Phi^+ = \Bigg\{ &\frac{\pi_a^2}{2m} - ek^2\frac{\vec{S}\vec{H}}{4m} + eA_0 - \frac{k^2}{24}Q_{ab}\frac{\partial E_a}{\partial x_b} - \\ &- \frac{e}{24m^2}k^2 s(s+1)\,\mathrm{div}\,\vec{E} - \frac{ek^2}{16m^2}\vec{S}(\vec{E}\times\vec{\pi} - \vec{\pi}\times\vec{E}) \Bigg\}\Phi^+, \end{aligned} \tag{5.18}$$

where

$$Q_{ab} = \frac{e}{2m^2}\{3[S_a, S_b]_+ - 2\delta_{ab}s(s+1)\} \tag{5.19}$$

is the tensor of the quadrupole interaction.

Thus the non-relativistic equation without redundant components (4.15) allows us to obtain the description of a motion of the spin s particle in an external electromagnetic field. Such a description is in a good qualitative accordance with experimental data. For $s = 1/2$ (5.18) coincides with the FW Hamiltonian if one puts an arbitrary constant $k = 1/s$.

6. Conclusion

So we have demonstrated that the non-relativistic Hamiltonian equations (2.12), (4.15) give the consistent description of a charged particle of any spin in external fields. Thus we have shown that the spin-orbit, the Darwin and the electric quadrupole interactions are not the truly relativistic effects but may be considered within the framework of the non-relativistic mechanics.

It is interesting to compare the obtained results with the ones predicted by the relativistic theory. One can make sure that there is not only the qualitative but also the quantitative accordance between them. We have demonstrated this fact for the case $s = 1/2$. If one puts into (5.18) $k = \pm 2$, the resulting equation completely coincides with the one obtained in [5] from the relativistic equations for arbitrary spin particles without redundant components [4, 17, 20]. In the particular case $k = \pm 2$, $s = 1$, equation (5.18) possesses all terms, predicted by the Kemmer–Duffin equation [9], but additionally takes into account the quadrupole electric interaction of a particle with a field. At least, if one puts into (5.18) $k = 1/s$, the coefficients in the terms representing the spin-orbit, the Darwin and the quadrupole interactions are the same as calculated in [14] starting from the Feynman–Gell–Mann relativistic equations.

Note, that equations (4.15) and equation (2.12) with the redundant components lead to different physical consequences (see (5.9) and (5.18)). The analogous situation takes place in the relativistic case [9, 14].

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