Poincaré-invariant equations with a rising mass spectrum

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In recent years many papers have been devoted to the construction of infinite-component wave equations to describe properly the spectrum of strongly interacting particles [1, 2]. As a rule, the derived equations have a number of pathological properties: the unrealistic mass spectra, the appearence of spacelike solutions ($p_\mu^2 < 0$), the breakdown of causality etc. [2].

In this note we shall construct, in the framework of relativistic quantum mechanics, the Poincaŕe-invariant motion equations with realistic mass spectra. These equations describe a system with mass spectra of the form $m^2 = a^2 + b^2 s(s+1)$, where a and b are arbitrary parameters. Such equations are obtained by a reduction of the motion equation for two particles to a one-particle equation which describes the particle in various mass and spin states. It we impose a certain condition on the wave function of the derived equation, such an equation describes the free motion of a fixed-mass particle with arbitrary (but fixed) spin s.

Let us consider the motion equation for two free particles with masses $m_1 = m_2 = m$ and spins s_1 and s_2 in the Thomas–Bakamjian–Foldy form [3]

$$i\frac{\partial\Phi(t,\boldsymbol{x},\boldsymbol{\xi})}{\partial t} = (P_a^2 + M^2)^{1/2}\Phi(t,\boldsymbol{x},\boldsymbol{\xi}), \tag{1}$$

where

$$P_a = p_a^{(1)} + p_a^{(2)}, \qquad M = 2(m^2 + \boldsymbol{k}^2)^{1/2},$$

$p_a^{(1)}$, $p_a^{(2)}$ are components of the momenta of the two particle, $\boldsymbol{k}$ the relative momentum, $\boldsymbol{x}$ the co-ordinate of the centre of mass, $\boldsymbol{\xi}$ is the relative co-ordinate.

On the manifold of solutions $\{\Phi\}$ of eq. (1) the generators of the Poincaré group $P_{1,3}$ have the form

$$\begin{gathered}
P_0 = (P_a^2 + M^2)^{1/2}, \qquad P_a = p_a = -i\frac{\partial}{\partial x_a}, \quad a = 1,2,3, \\
J_{ab} = M_{ab} + L_{ab}, \qquad M_{ab} = x_a p_b - x_b p_a, \qquad L_{ab} = m_{ab} + S_{ab}, \\
m_{ab} = \xi_a k_b - \xi_b k_a, \qquad S_{ab} = s_{ab}^{(1)} + s_{ab}^{(2)}, \qquad [x_a, p_b]_- = i\delta_{ab}, \\
[\xi_a, k_b]_- = i\delta_{ab}, \qquad [\xi_a, p_b]_- = 0,
\end{gathered} \tag{2}$$

where $s_{ab}^{(1)}$ and $s_{ab}^{(2)}$ are the spin matrices satisfying the Lie algebra of the rotation group O_3.

Equations (1) is invariant with respect to algebra (2) since the condition

$$\left[i\frac{\partial}{\partial t} - (P_a^2 + M^2)^{1/2}, J_{\mu\nu}\right]\Phi = 0, \qquad \mu = 0,1,2,3, \tag{3}$$

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is satisfied. In spherical co-ordinates the operator $\boldsymbol{k}^2$ is

$$\boldsymbol{k}^2 = \frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial}{\partial\xi}\right) + \frac{1}{\xi^2}m_{ab}^2, \qquad \xi \equiv \boldsymbol{\xi}^2 = \xi_1^2+\xi_2^2+\xi_3^2, \tag{4}$$

where m_{ab} is the square of the angular momentum with respect to the centre of mass.

Let us impose on the function $\Phi(t,\boldsymbol{x},\xi,\theta,\varphi)$ the condition

$$\frac{\partial\Phi(t,\boldsymbol{x},\theta,\varphi)}{\partial\xi} = 0. \tag{5}$$

This condition means that the wave function Φ constant on the sphere of radius $r_0 = \xi \equiv \sqrt{\xi^2}$ with respect to internal variables ξ_1, ξ_2, ξ_3. If we take into account the condition (5), eq. (1) now becomes

$$i\frac{\partial\Phi(t,\boldsymbol{x},\theta,\varphi)}{\partial t} = \left(p_a^2 + 4m^2 + \frac{4}{r_0^2}m_{ab}^2\right)^{1/2}\Phi(t,\boldsymbol{x},\theta,\varphi). \tag{6}$$

Equation (6) may yield the mass spectrum only for the bosons so that m_{ab} should be replaced by L_{ab}. Having done this, we obtain the equation

$$i\frac{\partial\Phi(t,\boldsymbol{x},\theta,\varphi)}{\partial t} = \left(p_a^2 + 4m^2 + \frac{4}{r_0^2}L_{ab}^2\right)^{1/2}\Phi(t,\boldsymbol{x},\theta,\varphi). \tag{7}$$

Equation (7) shows that the mass operator $M^2 = P_0^2 - P_a^2$ has on the set $\{\Phi(t,\boldsymbol{x},\theta,\varphi)\}$ the discrete mass spectrum of the form

$$M^2\Phi = \left(4m^2 + \frac{4}{r_0^2}L_{ab}^2\right)\Phi = \left\{4m^2 + \frac{4}{r_0^2}s(s+1)\right\}\Phi, \tag{8}$$

where

$$s = 0,1,2,\ldots \qquad \text{if} \qquad L_{ab} = m_{ab} = \xi_a k_b - \xi_b k_a, \tag{9}$$

$$s = \frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots \qquad \text{if} \qquad L_{ab} = \xi_a k_b - \xi_b k_a + S_{ab}, \tag{10}$$

$S_{ab} = \sigma_c/2$, σ_c are the 2×2 Pauli matrices.

In the case (9) the operator M^2 has a simple spectrum. In the case (10) the spectrum of M^2 is twofold degenerated. In the general case the measure of the degeneracy depends on the dimension of the matrices S_{ab} realizing representations of the group O_3.

If we suppose that the energy operator P_0 can have both the positive and negative spectrum, then for fermions (the spectum (10)) we find the equation

$$\begin{gathered} p_0\Phi(t,\boldsymbol{x},\theta,\varphi) = \gamma_0\left(p_a^2 + 4m^2 + \frac{4}{r_0^2}L_{ab}^2\right)^{1/2}\Phi(t,\boldsymbol{x},\theta,\varphi), \\ p_0 = i\frac{\partial}{\partial t}, \qquad \gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{gathered} \tag{11}$$

where Φ is the four-component wave function. The integro-differential equation (11) may be written in the symmetrical form with respect to the operators p_0, p_a if the transformation [4] is carried out on it

$$\mathcal{U}=\frac{1}{\sqrt{2}}\left(1+\frac{\gamma_0\mathcal{H}}{\sqrt{\mathcal{H}^2}}\right),\quad \mathcal{H}=\gamma_0\gamma_cp_c+\gamma_0\gamma_4\left(a^2+b^2L_{cd}^2\right)^{1/2},\quad c,d=1,2,3,\qquad(12)$$

where γ_0, γ_c, γ_4 are the 4×4 Dirac matrices, $a^2=4m^2$, $b^2=4/r_0^2$. After the transformation (12), eq. (11) takes the form

$$\begin{gathered}p_0\Psi(t,\boldsymbol{x},\theta,\varphi)=\left\{\gamma_0\gamma_cp_c+\gamma_0\gamma_4\left(a^2+b^2L_{cd}^2\right)^{1/2}\right\}\Psi(t,\boldsymbol{x},\theta,\varphi),\\ \Psi=\mathcal{U}\Phi.\end{gathered}\qquad(13)$$

We now summarize that eq. (7) describes a boson system with increasiftg mass spectrum if the operator L_{ab} has the form (9). Equation (13) (or eq. (7)) describes a fermion system with increasing mass spectrum if the operator L_{ab} has the form (10).

The four-component eq. (13) (or (7)) may be used for describing the free motion of a particle of nonzero mass with arbitrary half-integer spin s. Indeed, to do this it is sufficient to impose the Poincaré-invariant condition on the wave function Ψ, picking up a fixed spin from the whole discrete spectrum (10).

This condition has the form

$$\frac{1}{M^2}W_\mu W^\mu\Psi(t,\boldsymbol{x},\theta,\varphi)=L_{ab}^2\Psi(t,\boldsymbol{x},\theta,\varphi)=s(s+1)\Psi,\qquad(14)$$

where

$$W_\mu=\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}P^\nu J^{\alpha\beta},\qquad(15)$$

s is an arbitrary but fixed number from the set (10).

Equations (7), (13) may be obtained in another way. Let us consider the equation

$$i\frac{\partial\Phi(t,x_1,x_2,\ldots,x_6)}{\partial t}=\left(p_1^2+p_2^2+\cdots+p_6^2+\varkappa^2\right)^{1/2}\Phi(t,x_1,x_2,\ldots,x_6),\qquad(16)$$

where $p_k=-i(\partial/\partial x_k)$, $k=1,2,\ldots,6$, $\varkappa$ is a constant. The equation is invariant under the generalized Poincaré group $P_{1,6}$ [5].

$P_{1,6}$ is the group of rotations and translations in $(1+6)$-dimensional Minkowski space. Equation (16) is invariant with respect to the algebra [5]

$$\begin{gathered}P_0=p_0=i\frac{\partial}{\partial t},\qquad P_k=p_k=-i\frac{\partial}{\partial x_k},\quad k=1,2,\ldots,6,\\ J_{\mu\nu}=x_\mu p_\nu-x_\nu p_\mu+S_{\mu\nu},\qquad \mu,\nu=0,1,2,\ldots,6.\end{gathered}\qquad(17)$$

Equation (16), together with the supplementary condition of the type (5), is equivalent to eq. (7). This may be shown by passing from the variables x_4, x_5, x_6 to the new variables ξ, θ, φ. It is to be emphasized, however, that the supplementary condition of the type (5) breaks down the invariance with respect to the whole group $P_{1,6}$ but conserves the invariance relative to its subgroup $P_{1,3}\subset P_{1,6}$.

Note 1. On the set $\{\Phi\}$ besides the representations of the Poincaré algebra $P_{1,3}$ (the external algebra), we may construct one more algebra of Poincaré $K_{1,3}$ (the internal algebra). The representation of the algebra $K_{1,3}$ has the following form:

$$K_0 = \frac{1}{2}M, \qquad K_a = k_a = -i\frac{\partial}{\partial \xi_a}, \qquad L_{ab} = m_{ab} + S_{ab},$$
$$m_{ab} = \xi_a k_b - \xi_b k_a, \qquad L_{0a} = -\frac{1}{2}(\xi_a K_0 + K_0 \xi_a) - \frac{S_{ab}k_b}{K_0 + m}. \tag{18}$$

This algebra describes an intrinsic relative motion of the two-particle system with respect to the centre of mass. The algebra $P_{1,3}$ describes a motion of the centre of mass. Equations (7), (13) are not invariant in respect to the whole algebra $K_{1,3}$.

Note 2. We note that the results obtained do not contradict the O'Raifeartaigh's theorem [6] since the operators (2) of the algebra $P_{1,3}$ together with the operators (18) of the algebra $K_{1,3}$ form the infinite-dimensional Lie algebra.

Note 3. Equation (13) jointly with tin condition (14) for the case $s = \frac{1}{2}$ is equivalent to the ordinary four-component Dirac equation for the particle with the spin $s = \frac{1}{2}$.

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