On a motion equation for two particles in relativistic quantum mechanics

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Breit [1] was the first who proposed to describe the motion for two relativistic particles by means of a semi-relativistic Dirac-type equation. The wave function of this equation has sixteen components. The possibility of covariant description of a system of particles interacting m quantum mechanics was proved by Thomas and Bakamjian [2] and Foldy [3]. In quantum field theory the two-body problem is described by means of the Bethe–Salpeter equation or the Logunov–Tavkhelidze–Kadyshevsky equations [4].

The purpose of the present note is to propose, in the framework of relativistic quantum mechanics, a new Poincaré-invariant equation for two particles with masses m_1, m_2 and spin $s_1 = s_2 = \frac{1}{2}$. It is a first-order linear differential equation for the eight-component wave function. With the help of this equation the description of the motion of two-particle systems is reduced to the description of one-particle systems in the $(1+6)$-dimensional Minkowski space which can be in two spin states ($s = 0$ or $s = 1$).

At first we derive the equation for two noninteracting particles. To this end we shall pass from the momenta of two particles $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ to the new canonical variables

$$\boldsymbol{P} = (P_1, P_2, P_3) = \boldsymbol{p}_1 + \boldsymbol{p}_2, \qquad \boldsymbol{K} = (K_1, K_2, K_3).$$

The connection between the variables $\boldsymbol{K}$ and $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ is rather complicated (see, e.g., [5, 6]) and we do not equate it here. The total energy of the two-particle system in the variables $\boldsymbol{P}$ and $\boldsymbol{K}$ has for our discussion a very convenient structure [5, 6]

$$E = (\boldsymbol{P}^2 + M^2)^{1/2}, \qquad M = \left(m_1^2 + \boldsymbol{K}^2\right)^{1/2} + \left(m_2^2 + \boldsymbol{K}^2\right)^{1/2}. \tag{1}$$

The square energy for the case when $m_1 = m_2 \equiv \frac{1}{2}m$ takes the very simple form

$$E^2 = p_a^2 + p_{a+3}^2 + m^2, \qquad p_a \equiv P_a, \qquad p_{a+3} \equiv 2K_a, \qquad a = 1, 2, 3. \tag{2}$$

The square root from this expression is the equation for two particles

$$i\frac{\partial \Psi(t, x_1, x_2, \dots, x_6)}{\partial t} = \mathcal{H}(\hat{p}_1, \hat{p}_2, \dots, \hat{p}_6)\Psi(t, x_1, x_2, \dots, x_6), \tag{3}$$

where

$$\begin{gathered}\mathcal{H}(\hat{p}_1, \hat{p}_2, \dots, \hat{p}_6) = \Gamma_0\Gamma_a\hat{p}_a + \Gamma_0\Gamma_{a+3}\hat{p}_{a+3} + \Gamma_0 m, \\ \hat{p}_a = -i\frac{\partial}{\partial x_a}, \qquad \hat{p}_{a+3} = -i\frac{\partial}{\partial x_{a+3}},\end{gathered} \tag{4}$$

the 8×8 matrices Γ_0, Γ_a, Γ_{a+3} obey a Clifford algebra, and has such a representation:

$$\Gamma_0 = \sigma_3 \otimes 1, \qquad \Gamma_a = 2i\sigma_2 \otimes s_a, \qquad \Gamma_{a+3} = 2i\sigma_1 \otimes \tau_a, \tag{5}$$

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$$s_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \qquad s_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \\ 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},$$

$$s_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \tau_1 = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix},$$

$$\tau_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \\ -1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \qquad \tau_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

The σ_a are the Pauli matrices.

The two-particle equation (3) will be defined completely in that case if we determine both the Hamiltonian and the Poincaré generators [7]. The generators of the $P_{1,3}$ group on $\{\Psi\}$ have such a form:

$$\begin{aligned} &P_0 = \mathcal{H}(\hat{p}_1, \ldots, \hat{p}_6) = \Gamma_0\Gamma_A\hat{p}_A + \Gamma_0 m, \qquad P_a = p_a, \qquad A = 1, 2, \ldots, 6, \\ &J_{ab} = M_{ab} + m_{ab} + S_{ab}, \qquad a, b = 1, 2, 3, \\ &J_{0a} = tp_a - \frac{1}{2}(x_a\mathcal{H} + \mathcal{H}x_a) - \frac{\mathcal{H}}{\sqrt{\mathcal{H}^2}}\frac{(S^{(2)}_{ab} + m_{ab})p_b}{\sqrt{\mathcal{H}^2} + M}, \end{aligned} \tag{6}$$

where

$$\begin{aligned} &M_{ab} \equiv \hat{x}_a\hat{p}_b - \hat{x}_b\hat{p}_a, \qquad m_{ab} \equiv \hat{x}_{a+3}\hat{p}_{b+3} - \hat{x}_{b+3}\hat{p}_{a+3}, \qquad S_{ab} = S^{(1)}_{ab} + S^{(2)}_{ab}, \\ &S^{(1)}_{ab} = \frac{i}{4}(\Gamma_a\Gamma_b - \Gamma_b\Gamma_a), \qquad S^{(2)}_{ab} = \frac{i}{4}(\Gamma_{a+3}\Gamma_{b+3} - \Gamma_{b+3}\Gamma_{a+3}), \\ &[\hat{x}_a, \hat{p}_b]_- = i\delta_{ab}, \qquad [\hat{x}_{a+3}, \hat{p}_{b+3}]_- = i\delta_{ab}, \\ &[\hat{x}_a, \hat{x}_b]_- = [\hat{x}_a, \hat{x}_{a+3}]_- = [\hat{x}_{a+3}, \hat{x}_{b+3}]_- = 0, \qquad [\hat{x}_a, \hat{p}_{b+3}]_- = [\hat{x}_{a+3}, \hat{p}_b]_- = 0. \end{aligned} \tag{7}$$

It can be immediately verified that the operators (6) satisfy the Poincaré algebra. It follows that eq. (3) is Poincaré invariant. If we perform the unitary transformation

$$U = \frac{(E + M + \Gamma_c p_c)(M + m + \Gamma_{c+3}p_{c+3})}{2\{ME(E + m)(M + m)\}^{1/2}} \tag{8}$$

on the operators (6), then we obtain

$$\begin{aligned} &P^c_0 = UP_0U^\dagger = \Gamma_0 E, \qquad P^c_a = p_a, \qquad J^c_{ab} = UJ_{ab}U^\dagger = J_{ab}, \\ &J^c_{0a} = tp_a - \frac{1}{2}(x_aP^c_0 + P^c_o x_a) - \Gamma_0\frac{m_{ab}p_b + S_{ab}p_b}{E + M}. \end{aligned} \tag{9}$$

The transformed generators (9) have canonical form [2, 3]. The position operators X_a and X_{a+3} on a set $\{\Psi\}$ look like

$$X_a = U^\dagger x_a U = x_a + \frac{S^{(1)}_{ab}p_b}{E(E + M)} + i\left(\frac{\Gamma_a}{2E} - \frac{p_a\Gamma_c p_c}{2E^2(E + M)}\right)\frac{m + \Gamma_{c+3}p_{c+3}}{M}, \tag{10}$$

$$X_{a+3} = U^\dagger x_{a+3} U = x_{a+3} + \frac{S^{(2)}_{a+3\,b+3} p_{b+3}}{M(M+m)} + \frac{i\Gamma_{a+3}}{2M} - \\ -i\frac{p_{a+3}\Gamma_{c+3}p_{c+3}}{2M^2(M+m)} - i\frac{p_{a+3}}{2E^2M^2}\Gamma_c p_c(m + \Gamma_{c+3}p_{c+3}). \quad (11)$$

An interaction Hamiltonian for two particles, in the absence of external fields, can have the form

$$\mathcal{H} = \Gamma_0\Gamma_A p_A + \Gamma_0\{m^2 + V(r)\}^{1/2}, \quad (12)$$

where $V(r)$ is an arbitrary function depending on $r \equiv \sqrt{x^2_{c+3}}$. In the special case when $V(r) = e^4/r^2$ the interaction Hamiltonian can be written as

$$\mathcal{H} = \Gamma_0^{(16)}\Gamma_A^{(16)} p_A + \frac{e^2}{r}\Gamma_0^{(16)}\Gamma_7^{(16)} + \Gamma_0 m, \quad (13)$$

where the 16×16 matrices $\Gamma_0^{(16)}$, $\Gamma_A^{(16)}$, $\Gamma_7^{(16)}$ satisfy a Cifford algebra. An external electromagnetic field is introduced in eq. (3) in the following way:

$$p_a \to \pi_a = p_a - e\mathcal{A}_a(t, x_1, x_2, x_3), \quad p_{a+3} \to \pi_{a+3} = p_{a+3} - e\mathcal{A}_{a+3}(t, x_4, x_5, x_6).$$

An extraction of the positive solutions from eq. (3) is realized by means of the subsidiary condition

$$\left(1 - \frac{\mathcal{H}}{\sqrt{\mathcal{H}^2}}\right)\Psi = 0 \quad \text{or} \quad \left(1 - \frac{\Gamma_\mu p^\mu}{\sqrt{p_\mu^2}}\right)\Psi = 0, \quad \mu = 0, 1, 2, \ldots, 6.$$

It is evident that these conditions are invariant under the Poincaré group.

It should be noted that the function $V(r)$ may be of arbitrary form, therefore the relative velocity $\mathcal{V}_{a+3}$,

$$\hat{\mathcal{V}}_{a+3}\Psi \equiv -i[X_{a+3}, \mathcal{H}]\Psi = \mathcal{V}_{a+3}\Psi, \quad (14)$$

with respect to the centre-of-mass may be arbitrary. To do $\mathcal{V}_{a+3}$ smaller than the photon velocity it is necessary to impose the condition

$$\mathcal{V}^2_{a+3} = \mathcal{V}_4^2 + \mathcal{V}_5^2 + \mathcal{V}_6^2 < 1.$$

These questions will be considered in more detail in another paper.

Finally we shall find the equation for two particles with mass $m_1 \neq m_2$. Let us, with Kadyshevsky et al. [8], represent M in such a form

$$M = \frac{m_1 + m_2}{\sqrt{m_1 m_2}}(m_1 m_2 + \boldsymbol{K}'^2)^{1/2}, \quad (15)$$

where

$$\boldsymbol{K}'^2 = -m_1 m_2 + \frac{m_1 m_2}{(m_1 + m_2)^2}\left(\sqrt{m_1^2 + \boldsymbol{K}^2} + \sqrt{m_2^2 + \boldsymbol{K}^2}\right)^2. \quad (16)$$

In the variables $\boldsymbol{P}$ and $\boldsymbol{K}'$ formula (2) can be rewritten as

$$E^2 = \boldsymbol{P}^2 + \frac{(m_1+m_2)^2}{m_1 m_2}\boldsymbol{K}'^2 + (m_1+m_2)^2. \tag{17}$$

It follows that the equation of motion for the two particles is

$$i\frac{\partial\Psi(t,x_1,\ldots,x_6)}{\partial t} = \left\{\Gamma_0\Gamma_a\hat{p}_a + \frac{m_1+m_2}{\sqrt{m_1 m_2}}\Gamma_0\Gamma_{a+3}\hat{p}_{a+3} + \right. \\ \left. +(m_1+m_2)\Gamma_0\right\}\Psi(t,x_1,\ldots,x_6), \tag{18}$$

$$\hat{p}_a = -i\frac{\partial}{\partial x_a}, \qquad \hat{p}_{a+3} \equiv \hat{\boldsymbol{K}}'_a = -i\frac{\partial}{\partial x_{a+3}}.$$

In this equation Ψ is also an eight-component function.

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