# On the possible types of equations for zero-mass particles

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A number of papers dedicated to the description of free particles and antiparticles with zero mass and spin $\frac{1}{2}$ has recently appeared [1–6].

A great many equations with different $C$, $P$, $T$ properties have been proposed and the impression could be formed that there are many nonequivalent theories for zero-mass particles. The purpose or this paper is to show that it is not the case and to describe all nonequivalent equations.

**1.** First we shall formulate the result [1] obtained for a particle of spin $\frac{1}{2}$ in such a form that all principal assertions will be valid for massless particles of arbitraly spin. It has been shown [1] that for a particle of spin $\frac{1}{2}$ three types of nonequivalent two-component Poincaré-invariant equations exist. These three of equations are equivalent to the Dirac equation

$$i\frac{\partial\Psi(t,\boldsymbol{x})}{\partial t}=\mathcal{H}\Psi(t,\boldsymbol{x}),\qquad \mathcal{H}=\gamma_0\gamma_a p_a,\quad a=1,2,3, \tag{1}$$

with one out of three (actually, one out of six) subsidiary conditions imposed on a wave function

$$P_1^+\Psi=0\qquad\text{or}\qquad P_1^-\Psi=0,\qquad P_1^\pm=\frac{1}{2}(1\pm i\gamma_4),\qquad \gamma_4=-\gamma_0\gamma_1\gamma_2\gamma_3, \tag{2}$$

$$P_2^+\Psi=0\qquad\text{or}\qquad P_2^-\Psi=0,\qquad P_2^\pm=\frac{1}{2}(1\pm i\gamma_4\hat{\varepsilon}),\qquad \hat{\varepsilon}=\frac{\mathcal{H}}{E}, \tag{3}$$

$$P_3^+\Psi=0\qquad\text{or}\qquad P_3^-\Psi=0,\qquad P_3^\pm=\frac{1}{2}(1\pm\hat{\varepsilon}),\qquad E=\sqrt{p_1^2+p_2^2+p_2^2}. \tag{4}$$

Conditions (2)–(4) are Poincaré invariant since the projection operators $P_a^\pm$ commute with the generators of the Poincaré group $P(1,3)$

$$\begin{gathered}P_0=\mathcal{H}=\gamma_0\gamma_a p_a,\qquad P_a=p_a=-i\frac{\partial}{\partial x_a},\qquad J_{0a}=tp_a-\frac{1}{2}(x_aP_0+P_0x_a),\\ J_{ab}=x_ap_b-x_bp_a+S_{ab},\qquad S_{ab}=\frac{i}{4}(\gamma_a\gamma_b-\gamma_b\gamma_a).\end{gathered} \tag{5}$$

It should be emphasized that only the operator $P_1^\pm$ is local in co-ordinate space. If we introduce the four-component (as a matter of fact, two-component) modes

$$\chi_a^\pm=P_a^\pm\Psi, \tag{6}$$

equations (1) with subsidiary conditions (2)–(4) can be written in the form

$$i\frac{\partial\chi_a^\pm}{\partial t}=(\gamma_0\gamma_bp_b\pm\varkappa_a\gamma_0P_a^\mp)\chi_a^\pm, \tag{7}$$

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where $\varkappa_a$ are arbitrary constants. The wave functions $\chi_a^{\pm}$ satisfy conditions (2)–(4) automatically. One of the equations (7), namely the equation for $\chi_1^{+}$ (or $\chi_1^{-}$), is equivalent, as is well known, to the two-component Weyl equation. Subsidiary conditions (2)–(4) have been generalized in [7] to massless particles of arbitrary spin starting from the $2(2s+1)$-component equation.

These results are almost evident from the group-theoretical point of view. Indeed, on the set $\{\Psi\}$ of solutions of the equation (1) the following direct sum of irreducible representations of the group $P(1,3)$ is realized:

$$D^{+}(\lambda=1)\oplus D^{-}(\lambda=-1)\oplus D^{+}(\lambda=-1)\oplus D^{-}(\lambda=1), \tag{8}$$

where $D^{\varepsilon}(\lambda)$ is the one-dimensional irreducible representation of the $P(1,3)$ group characterized by the eigenvalue $\varepsilon=\pm1$ of the sign energy operator $\hat{\varepsilon}$ and by the eigenvalue $\lambda=\pm1$ of the helicity operator

$$\hat{\Lambda}=2\frac{J_{12}P_3+J_{23}P_1+J_{01}P_2}{E}=i\gamma_4\hat{\varepsilon}. \tag{9}$$

Two-dimensional subspaces of representations

$$D^{+}(\lambda=1)\oplus D^{-}(\lambda=-1) \quad \text{or} \quad D^{+}(\lambda=-1)\oplus D^{-}(\lambda=1), \tag{10}$$

$$D^{+}(\lambda=1)\oplus D^{-}(\lambda=1) \quad \text{or} \quad D^{+}(\lambda=-1)\oplus D^{-}(\lambda=-1), \tag{11}$$

$$D^{+}(\lambda=1)\oplus D^{+}(\lambda=-1) \quad \text{or} \quad D^{-}(\lambda=1)\oplus D^{-}(\lambda=-1), \tag{12}$$

are selected by subsidiary conditions (2)–(4) from $\{\Psi\}$ in a Poincaré-invariant manner.

The operators $P$, $T$, $C$ (their definitions see e.g. in [8]) and $\hat{\Lambda}$, $\hat{\varepsilon}$ satisfy the relations

$$[P^{(1)},\hat{\Lambda}]_{+}=[P^{(1)},\hat{\varepsilon}]_{-}=[T^{(2)},\hat{\Lambda}]_{-}=[T^{(2)},\hat{\varepsilon}]_{-}=[C,\hat{\Lambda}]_{-}=[C,\hat{\varepsilon}]_{+}=0. \tag{13}$$

Taking into account (13) one obtains the relations

$$P^{(1)}P_j^{\pm}=P_j^{\mp}P^{(1)},\quad P^{(1)}P_3^{\pm}=P_3^{\pm}P^{(1)},\quad T^{(2)}P_a^{\pm}=P_a^{\pm}T^{(2)},\quad j=1,2, \tag{14}$$

$$CP_1^{\pm}=P_1^{\mp}C,\qquad CP_2^{\pm}=P_2^{\pm}C,\qquad CP_3^{\pm}=P_3^{\pm}C. \tag{15}$$

From (14), (16) it follows that

1) the system of equations (1), (2) is $T^{(2)}$, $P^{(1)}$, $C$-invariant but $P^{(1)}$, $C$-noninvariant,
2) the system of equations (1), (3) is $T^{(2)}$, $C$-invariant but $P^{(1)}$-noninvariant,
3) the system of equations (1), (4) is $T^{(2)}$, $P^{(1)}$-invariant but $C$-noninvariant.

To obtain these result we have used only the relations (13) which are valid for massless particles of arbitrary spin. The above discussion is followed by tins conclusion: *if the particle (and antiparticle) of zero mass is characterized by helicity and by the sign of energy only (without additional quantum numbers) three and only three types of two-component Poincaré-invariant essentially different (in respect to C, P, T propertis) equations exist*. It is interesting to note that the hypothesis of

Lee and Yang and Landau on $CP$-parity conservation is not valid for the equations (1), (3); (1), (4). Moreover the system of equations (1), (3) is $CP^{(1)}T^{(2)}$- and $CP^{(1)}T^{(1)}$-noninvariant.

**Note 1.** Equation (1) with subsidiary conditions

$$P_2^{\varepsilon}P_3^{\varepsilon'}\Psi = 0, \qquad \varepsilon, \varepsilon' = \pm 1, \tag{16}$$

$$P_2^{\varepsilon}P_3^{\varepsilon'}\Psi = \Psi, \tag{17}$$

is equivalent to three- and one-component equations

$$(\gamma_\mu p_\mu + \bar{\varkappa}_0 P_2^{\varepsilon}P_3^{\varepsilon'})\varphi^{\varepsilon\varepsilon'} = 0, \qquad \varphi^{\varepsilon\varepsilon'} = \frac{1}{2}(1 - P_2^{\varepsilon}P_3^{\varepsilon'})\Psi, \qquad \mu = 0, 1, 2, 3, \tag{18}$$

$$(\gamma_\mu P_\mu + \bar{\varkappa}_1 P_2^{\varepsilon}P_3^{-\varepsilon'} + \bar{\varkappa}_2 P_2^{-\varepsilon}P_3^{\varepsilon'} + \varkappa_3 P_2^{-\varepsilon}P_3^{-\varepsilon'})\bar{\varphi}^{\varepsilon\varepsilon'} = 0, \qquad \bar{\varphi}^{\varepsilon\varepsilon'} = P_2^{\varepsilon}P_3^{\varepsilon'}\Psi, \tag{19}$$

respectively, where $\bar{\varkappa}_\mu$ are arbitary constants. It is not difficult to calculate that there are fifteen equations (2)–(4), (16), (17) exhausting all possible nonequivalent Poincaré-invariant subsidiary conditions which can be imposed on $\{\Psi\}$.

**Note 2.** If a zero-mass particle is characterized by two (but not by one) quantum numbers, there exist more than three types of nonequivalent two-component equations. Theoretically such a possibility exists due to commutativity of Dirac's Hamiltonian for a particle of spin $\frac{1}{2}$ with $SO_4 \sim SU_2 \otimes SU_2$ algebra. It means that besides the mass two conserved quantum numbers $s$ and $\tau$ exist. For the zero-mass case the eigenvalues of helicity-type operators

$$\hat{\Lambda}_1 = \frac{S_a p_a}{p}, \qquad \hat{\Lambda}_2 = \frac{\tau_a p_a}{p},$$
$$S_a = \frac{1}{2}\left(\frac{1}{2}\varepsilon_{abc}S_{bc} + S_{4a}\right), \qquad \tau_a = \frac{1}{2}\left(\frac{1}{2}\varepsilon_{abc}S_{bc} - S_{4a}\right) \tag{20}$$

are conserved. If the massless particle is characterized by cigenvalues of operators (20), the number of theoretically possible equations increases. This follows from the fact that the two-dimensional irreducible representation of the group $P(1,3)$ for $m \neq 0$ is reduced in the case $m = 0$ to the following direct sum of one-dimensional irreducible representations:

$$D^{\pm}\left(0, \frac{1}{2}\right) \to D^{\pm}\left(0, +\frac{1}{2}\right) \oplus D^{\pm}\left(0, -\frac{1}{2}\right),$$
$$D^{\pm}\left(\frac{1}{2}, 0\right) \to D^{\pm}\left(+\frac{1}{2}, 0\right) \oplus D^{\pm}\left(-\frac{1}{2}, 0\right). \tag{21}$$

We shall not analyze all possible equations in this case (it is difficult to do this using the results of paper [8]) because it is not clear from the physical point of view how one can distinguish, say, the representations $D^{\pm}\left(0, -\frac{1}{2}\right)$ and $D^{\pm}\left(-\frac{1}{2}, 0\right)$.

**2.** Let us now show that four- and two-component equations obtained in [4, 5] are isometrically equivalent to the Dirac equation (1) and to the Weyl equation.

Consider the four-component equation of the type [4]

$$i\frac{\partial\Phi(t,\boldsymbol{x})}{\partial t} = \mathcal{H}_\Phi\Phi(t,\boldsymbol{x}) = (\alpha_a p_a + \Lambda)\Phi(t,\boldsymbol{x}), \qquad \alpha_a = \gamma_a\gamma_a, \tag{22}$$

where $\Lambda$ is an operator satisfying the condition

$$\alpha_a p_a \Lambda = -\Lambda \alpha_a p_a, \qquad \Lambda^2 = 0. \tag{23}$$

Equation (22) can be obtained from (1) with the help of the isometric transformation

$$\Psi \to \Phi = V_1 \Psi, \qquad \mathcal{H} \to \mathcal{H}_\Phi = V_1 \mathcal{H} V_1^{-1}, \tag{24}$$

where

$$V_1 = 1 - \frac{1}{2}\frac{\alpha_a p_a}{E^2}\Lambda, \qquad V_1^{-1} = 1 + \frac{1}{2}\frac{\alpha_a p_a}{E^2}\Lambda. \tag{25}$$

The Hamiltonian $\mathcal{H}_\Phi$ is Hermitian in respect of the following scalar product:

$$(\Phi_1, \Phi_2) = \int d^3\boldsymbol{x}\ \Phi_1^\dagger(t, \boldsymbol{x})(V_1^{-1})^\dagger V_1^{-1} \Phi_2(t, \boldsymbol{x}). \tag{26}$$

To draw the correct conclusion about the $C$, $P$, $T$ propertios equation (22) it is necessary to write the algebra (5) in the $\Phi$-representation. We shall not do this here. We shall remark only that due to the invariance of equation (1) under $P^{(1)}$, $T^{(2)}$, $C$ transformations equation (22) is invariant with respect to the transformations

$$P_\Phi^{(1)} = V_1 P^{(1)} V_1^{-1}, \qquad C_\Phi = V_1 C V_1^{-1}, \qquad T_\Phi^{(2)} = V_1 T^{(2)} V_1^{-1}. \tag{27}$$

One can show in an analogous manner that the two-component equation of the type [5]

$$\begin{gathered} i\frac{\partial \chi(t, \boldsymbol{x})}{\partial t} = (\sigma_a p_a + B)\chi(t, \boldsymbol{x}), \\ B\sigma_a p_a = -\sigma_a p_a B, \qquad B^2 = 0, \end{gathered} \tag{28}$$

can be obtained from the Weyl equation with the help of the operator

$$V_2 = 1 - \frac{1}{2}\frac{\sigma_a p_a}{E^2}B, \qquad V_2^{-1} = 1 + \frac{1}{2}\frac{\sigma_a p_a}{E^2}B. \tag{29}$$

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