P, T, C properties of the Poincaré invariant equations for massive particles

W.I. FUSHCHYCH

Recently [1] we have shown that for free particles and antiparticles with mass $m > 0$ and arbitrary spin $s > 0$, in the framework of the Poincaré group $P(1,3)$, there exist three types of nonequivalent equations. In the present paper we study the P, T, C properties of these equations.

It will be convinient to investigate these properties in the canonical representation where the Hamiltonian is diagonal (as matrix) and other operators (position operator and spin operator) have adequate physical interpretation. For the transformation to this representation let us make unitary transformation [2]

$$\mathcal{U}\left(p, s=\frac{1}{2}\right) = \exp\left[\frac{\pi}{4}\frac{\Gamma_0\mathcal{H}^{(8)}}{E}\right] = \frac{1}{\sqrt{2}}\left(1+\frac{\Gamma_0\mathcal{H}^{(8)}}{E}\right),$$
$$\mathcal{H}^{(8)} \equiv \Gamma_0\Gamma_k p_k, \qquad p_4 \equiv m, \qquad k=1,2,3,4, \tag{1}$$

over the eight-component equation of the Dirac type

$$i\frac{\partial\Psi^{(8)}(t,\boldsymbol{x})}{\partial t} = \mathcal{H}^{(8)}\Psi^{(8)}(t,\boldsymbol{x}). \tag{2}$$

Equation (2) after the transformation (1) transfers into

$$i\frac{\partial\Phi^{(8)}(t,\boldsymbol{x})}{\partial t} = \mathcal{H}^c\Phi^{(8)}(t,\boldsymbol{x}), \qquad \mathcal{H}^c = \Gamma_0 E, \qquad \Phi^{(8)} = U\Psi^{(8)}. \tag{3}$$

In the canonical representation the generators of the $P(1,3)$ group have the form [2]

$$P_0 = \mathcal{H}^c = \Gamma_0 E, \qquad P_a = p_a = -i\frac{\partial}{\partial x_a}, \qquad a=1,2,3,$$
$$J_{ab} = M_{ab} + S_{ab}, \qquad M_{ab} = x_a p_b - x_b p_a, \tag{4}$$
$$J_{0a} = x_0 p_a - \frac{1}{2}[x_a, \mathcal{H}^c]_+ - \Gamma_0\frac{S_{ab}p_b + S_{04}m}{E}, \qquad x_0 \equiv t,$$

where S_{ab}, S_{04} matrices are generators of the $SO_4 \sim SU_2 \otimes SU_2$ group. On the solutions $\{\Phi^{(8)}\}$ of eq.(2) thhese matrices have form

$$S_{kl} = S_{kl}^{(8)} = \frac{i}{4}(\Gamma_k\Gamma_l - \Gamma_l\Gamma_k), \qquad k,l=1,2,3,4.$$

The representation for the generators $P(1,3)$ in the form (4) differs from the Foldy–Shirokov [3, 4] representation. In the form (4) it is explicity distinguished the fact that in the space where a representation of the $P(1,3)$ group is given, also a representation

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of $SO_4 \sim SU_2 \otimes SU_2$ is realized. This follows, in particular, from the fact $[\mathcal{H}^c, S_{kl}]_- = 0$, i.e. it means that the matrices

$$S_a = \frac{1}{2}\left(\frac{1}{2}\varepsilon_{abc}S_{bc} + S_{4a}\right), \qquad T_a = \frac{1}{2}\left(\frac{1}{2}\varepsilon_{abc}S_{bc} - S_{4a}\right),$$

commute with the Hamiltonian*. In other words this means that the space, where the representation of $P(1,3)$ group is realized, must be characterized (besides the mass m and the sign of the energy) by pair of indices s and τ

$$S_a^2\Phi = s(s+1)\Phi, \qquad T_a^2\Phi = \tau(\tau+1)\Phi, \qquad s,\tau = \frac{1}{2}, 1, \frac{3}{2}, \ldots .$$

we shall denote by $D^{\pm}(s,0)$ and $D^{\pm}(0,\tau)$ the irreducible representation of $P(1,3)$ group. For futher understanding it should be noted that the irreducible representations $D(s,0)$ and $D(s,0)$ of SO_4 group are indistinguishable with respect to the matrices S_{ab} from the SO_3 algebra.

From the canonical eight-component equation (3) we can obtain the following three types of nonequivalent four-component equations

$$i\frac{\partial\Phi_a(t,\boldsymbol{x})}{\partial t} = \mathcal{H}_a\Phi_a(t,\boldsymbol{x}), \qquad a = 1,2,3, \tag{5}$$

$$\mathcal{H}_1 = \mathcal{H}_2 = \varepsilon\gamma_0 E, \qquad \mathcal{H}_3 = \varepsilon E, \qquad \varepsilon = \pm 1, \tag{6}$$

where γ_0 is the hermitian and diagonal 4×4 matrix**. Under a transformation of the $P(1,3)$ group the four-component wave functions Φ_1, Φ_2, Φ_3 transform on the representations (for the sake of brevity we consider only case $\varepsilon = +1$)

$$D^+(s,0) \oplus D^-(0,\tau), \qquad s = \tau = \frac{1}{2}, \tag{7}$$

$$D^+(s,0) \oplus D^-(s,0), \qquad s = \frac{1}{2},\ \tau = 0, \tag{8}$$

$$D^+(s,0) \oplus D^+(0,\tau), \qquad s = \tau = \frac{1}{2}. \tag{9}$$

On the manifolds $\{\Phi_1\}$, $\{\Phi_2\}$, $\{\Phi_3\}$ the generators P_μ, $J_{\alpha\beta}$ have the forms

$$\begin{aligned} &P_0^{(1)} = \mathcal{H}_1, \qquad P_a^{(1)} = p_a, \qquad J_{ab}^{(1)} = M_{ab} + S_{ab}, \\ &J_{0a}^{(1)} = x_0 p_a - \frac{1}{2}[x_a, \mathcal{H}_1]_+ - \gamma_0\frac{S_{ab}p_b + S_{a4}m}{E}; \end{aligned} \tag{10}$$

$$\begin{aligned} &P_0^{(2)} = \mathcal{H}_2, \qquad P_a^{(2)} = p_a, \qquad J_{ab}^{(2)} = M_{ab} + S_{ab}, \\ &J_{0a}^{(2)} = x_0 p_a - \frac{1}{2}[x_a, \mathcal{H}_2]_+ - \gamma_0\frac{S_{ab}p_b + \frac{1}{2}\varepsilon_{abc}S_{bc}m}{E}; \end{aligned} \tag{11}$$

*In fact, eq.(2) or (3) is invariant with respect to $SO_6 \supset SO_4$ group [2]. A relativistic equation of motion for particle with spin $\frac{3}{2}$ is invariant also with respect to the SO_6 group.

**The fact that the $\mathcal{H}_1$ and $\mathcal{H}_2$ have identical forms in two eqs.(5) must not lead into confusion since the equation of motion is defined completly if only we determine both the Hamiltonian and the representation of $P(1,3)$ group.

$$
\begin{aligned}
&P_0^{(3)} = \mathcal{H}_3 = E, \qquad P_a^{(3)} = p_a, \qquad J_{ab}^{(3)} = M_{ab} + S_{ab}, \\
&J_{0a}^{(3)} = x_0 p_a - \frac{1}{2}[x_a, E]_+ - \frac{S_{ab}p_b + S_{a4}m}{E} \equiv x_0 p_a - x_a E + S_{0a}\frac{\mathcal{H}}{E},
\end{aligned}
\tag{12}
$$

where

$$
\mathcal{H} = \gamma_0\gamma_k p_k, \qquad S_{\mu\nu} = \frac{i}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu), \qquad \mu = 0, 1, 2, 3, 4.
$$

It should be noted that only in the last representation (12) the Hamiltonian $\mathcal{H}_3 = E$ is the positive-definite operator. If we add to the algebra (12) an operator of the change $Q = \gamma_0$, then such algebra (in the quantum mechanics framework) has the same properties as the corresponding Poincaré algebra, obtained by the procedure of the Dirac equation quantization.

It is well known [3] that there exist two nonequivalent definitions of the space-reflection operator P:

$$
P^{(1)}\Phi(t, \boldsymbol{x}, m) = r_1\Phi(t, -\boldsymbol{x}, m), \qquad \left(P^{(1)}\right)^2 \sim 1, \tag{13}
$$

$$
P^{(2)}\Phi(t, \boldsymbol{x}, m) = r_2\Phi^*(t, -\boldsymbol{x}, m), \qquad \left(P^{(2)}\right)^2 \sim 1, \tag{14}
$$

$$
[P^{(1)}, P_0]_- = 0 = [P^{(1)}, J_{ab}]_-, \qquad [P^{(1)}, P_a]_+ = 0 = [P^{(1)}, J_{0a}]_+, \tag{15}
$$

$$
[P^{(2)}, P_0]_+ = 0 = [P^{(2)}, J_{ab}]_+, \qquad [P^{(2)}, P_a]_- = 0 = [P^{(2)}, J_{0a}]_-. \tag{16}
$$

Also there exist two nonequivalent definitions of the time-reflection T:

$$
T^{(1)}\Phi(t, \boldsymbol{x}, m) = t_1\Phi(-t, \boldsymbol{x}, m), \qquad \left(T^{(1)}\right)^2 \sim 1, \tag{17}
$$

$$
T^{(2)}\Phi(t, \boldsymbol{x}, m) = t_2\Phi^*(-t, \boldsymbol{x}, m), \qquad \left(T^{(2)}\right)^2 \sim 1, \tag{18}
$$

$$
[T^{(1)}, P_0]_+ = 0 = [T^{(1)}, J_{0a}]_+, \qquad [T^{(1)}, P_a]_- = 0 = [T^{(1)}, J_{ab}]_-, \tag{19}
$$

$$
[T^{(2)}, P_0]_- = 0 = [T^{(2)}, J_{0a}]_-, \qquad [T^{(2)}, P_a]_+ = 0 = [T^{(2)}, J_{ab}]_+. \tag{20}
$$

Besides these conditions usually imposed on the discrete operators P and T we shall require also the subsidiary conditions

$$
[\hat{X}_a, P^{(1)}]_+ = 0 = [P^{(2)}, \hat{X}_a]_+, \tag{21}
$$

$$
[T^{(1)}, \hat{X}_a]_- = 0 = [T^{(2)}, \hat{X}_a]_- \tag{22}
$$

to be satisfied where $\hat{X}_a$ is a position operator. The conditions (21) and (22) guarantee that quantities r_1, r_2, t_1, t_2 are the matrices which do not depend on the momentum. If the conditions (21), (22) are not imposed, then the operators P and T may be nonlocal (in this case the quantities depend on the momentum).

In addition to the discrete operators P and T we shall introduce some more discrete operators:

$$
M\Phi(t, \boldsymbol{x}, m) = r_m\Phi(t, \boldsymbol{x}, -m), \qquad M^2 \sim 1, \tag{23}
$$

$$M_t\Phi(t,\boldsymbol{x},m)=m_t\Phi(-t,\boldsymbol{x},-m),\qquad M_t^2\sim 1,\tag{24}$$

$$M_x\Phi(t,\boldsymbol{x},m)=m_x\Phi(t,-\boldsymbol{x},-m),\qquad M_x^2\sim 1,\tag{25}$$

$$[M,P_\mu]_-=0=[M,J_{\mu\nu}]_-,\qquad \mu,\nu=0,1,2,3,\tag{26}$$

$$[M_t,P_0]_+=0=[M_t,J_{0a}]_+,\qquad [M_t,P_a]_-=0=[M_t,J_{ab}]_-,\tag{27}$$

$$[M_x,P_0]_-=0=[M_x,J_{ab}]_-,\qquad [M_x,P_a]_+=0=[M_x,J_{0a}]_+,\tag{28}$$

where r_m, m_t, m_x are the 4×4 matrices.

There is no need to define specially the operator of the charge conjugation C since it is equal to the operator $T^{(1)}\cdot T^{(2)}$ (or $P^{(1)}\cdot P^{(2)}$).

If we use the explicit forms (10)–(28) for the generators P_μ and $J_{\alpha\beta}$ and carrying out the analysis of the conditions (13)–(28) we come to the following results:

1) Equation (5) for the function Φ_1 (taking into consideration the representation (10)) is C, M_x, M_t, $P^{(1)}T^{(2)}$ invariant, but $P^{(1)}$, $P^{(2)}$, $T^{(2)}$, M noninvariant;
2) Equation (5) for the function Φ_2 (taking into consideration the representation (11)) is $P^{(2)}$, $T^{(1)}$, M_x, $P^{(1)}T^{(2)}$ invariant, but $P^{(1)}$, $T^{(2)}$, C, M, M_t noninvariant*;
3) Equation (5) for the function Φ_3 (taking into consideration the representation (12)) is $P^{(1)}$, $T^{(2)}$, M, M_x, $P^{(1)}T^{(2)}$ invariant, but $T^{(1)}$, C, $P^{(2)}$, M_t noninvariant.

These assertions may be proved also starting from eight-component equation (2) (or (3)) in which constraints have been imposed on the wave function [1]. To establish this it is necessary to analyse the commutation relations between the discrete operators and the projections $P_1^{\pm}$, $P_2^{\pm}$, $P_3^{\pm}$.

Note 1. It can be easily checked that

$$P^{(1)}S_a=T_aP^{(1)},\qquad MS_a=T_aM,\qquad T^{(1)}S_a=S_aT^{(1)}.\tag{29}$$

The transformation connecting the cannonical representations (10)–(12) and the Foldy–Shirokov representation has the form

$$U_1=\frac{m+E+\gamma_4\gamma_ap_a}{\{2E(E+m)\}^{1/2}}.\tag{30}$$

Note 2. If we put $m=0$ in the reducible representation (4), then it reduces into the following direct sum of the irreducible representation of the $P(1,3)$ algebra

$$\begin{aligned}&D^+\left(\frac12,0\right)\oplus D^-\left(0,\frac12\right)\oplus D^-\left(\frac12,0\right)\oplus D^+\left(0,\frac12\right)\to\\ &\quad\to D^+\left(\frac12,0\right)\oplus D^+\left(-\frac12,0\right)\oplus D^-\left(0,\frac12\right)\oplus D^-\left(0,-\frac12\right)\oplus\\ &\qquad\oplus D^-\left(\frac12,0\right)\oplus D^-\left(-\frac12,0\right)\oplus D^+\left(0,\frac12\right)\oplus D^+\left(0,-\frac12\right),\end{aligned}\tag{31}$$

*In the coupling scheme, brought in ref. [1], the correction $D^+(s,0)\overset{C}{\leftrightarrow}D^-(0,s)$ should be done.

where members $\frac{1}{2}$ and $-\frac{1}{2}$ are the eigenvalues of the operators $S_a p_a/E$ and $T_a p_a/E$. These operators commute with the generators P_μ, $J_{\alpha\beta}$ when $m = 0$. From (31) follows that there exist 28 types of mathematical nonequivalent two-component equations for massless particles.

Note 3. In order that Poincaré-invariant equation $m \neq 0$ was totally P, T, C invariant it is necessary and sufficient that the wave function was transformed on the following direct sum of representation of $P(1,3)$

$$D^+(s,\tau) \oplus D^-(s,\tau) \oplus D^+(\tau,s) \oplus D^-(\tau,s), \qquad \text{if} \quad \tau \neq s, \tag{32}$$

$$D^+(s,\tau) \oplus D^-(s,\tau), \qquad \text{if} \quad \tau = s. \tag{33}$$

The representation $D^+(s,\tau)$ is in general reducible with respect to the $P(1,3)$ algebra, therefore the wave function describes a multiplet of particles with variable-spin, but fixed mass. The spin of the multiplet can take the values from $(s-\tau)$ to $(s+\tau)$. The equations of motion describing a physical system with variable-mass and variable-spin were considered in ref. [5].

1. Fushchych W.I., *Lett. Nuovo Cimento*, 1972, **4**, 344.
2. Fushchych W.I., *Theor. Math. Phys.*, 1971, **7**, 3; Preprint ITF-70-32, Kyiv, 1970.
3. Foldy L.L., *Phys. Rev.*, 1956, **102**, 568.
4. Shirokov Yu.M., *Zurn. Eksp. Teor. Fiz.*, 1957, **33**, 1196.
5. Fushchych W.I., *Theor. Math. Phys.*, 1970, **4**, 360; Preprint ITF-70-4, Kyiv, 1970; Fushchych W.I., Krivsky I.Yu., *Nucl. Phys. B*, 1969, **14**, 573.