\( P, T, C \) properties of the Poincaré invariant equations for massive particles

W.I. FUSHCHYCH

Recently [1] we have shown that for free particles and antiparticles with mass \( m > 0 \) and arbitrary spin \( s > 0 \), in the framework of the Poincaré group \( P(1,3) \), there exist three types of nonequivalent equations. In the present paper we study the \( P, T, C \) properties of these equations.

It will be convenient to investigate these properties in the canonical representation where the Hamiltonian is diagonal (as matrix) and other operators (position operator and spin operator) have adequate physical interpretation. For the transformation to this representation let us make unitary transformation [2]

\[
\mathcal{U} \left( p, s = \frac{1}{2} \right) = \exp \left[ \frac{\pi}{4} \Gamma_0 \mathcal{H}^{(8)} \right] = \frac{1}{\sqrt{2}} \left( 1 + \frac{\Gamma_0 \mathcal{H}^{(8)}}{E} \right),
\]

(1)

over the eight-component equation of the Dirac type

\[
i \frac{\partial \psi^{(8)}(t, \mathbf{x})}{\partial t} = \mathcal{H}^{(8)} \psi^{(8)}(t, \mathbf{x}).
\]

(2)

Equation (2) after the transformation (1) transfers into

\[
i \frac{\partial \Phi^{(8)}(t, \mathbf{x})}{\partial t} = \mathcal{H}^c \Phi^{(8)}(t, \mathbf{x}), \quad \mathcal{H}^c = \Gamma_0 E, \quad \Phi^{(8)} = U \psi^{(8)}.
\]

(3)

In the canonical representation the generators of the \( P(1,3) \) group have the form [2]

\[
P_0 = \mathcal{H}^c = \Gamma_0 E, \quad P_a = p_a = -i \frac{\partial}{\partial x_a}, \quad a = 1, 2, 3,
\]

\[
J_{ab} = M_{ab} + S_{ab}, \quad M_{ab} = x_a p_b - x_b p_a,
\]

\[
J_{0a} = x_0 p_a - \frac{1}{2} [x_a, \mathcal{H}^c]_+ - \Gamma_0 \frac{S_{ab} p_b + S_{04} m}{E}, \quad x_0 = t,
\]

(4)

where \( S_{ab}, S_{04} \) matrices are generators of the \( SO_4 \sim SU_2 \otimes SU_2 \) group. On the solutions \( \{ \Phi^{(8)} \} \) of eq.(2) these matrices have form

\[
S_{kl} = S^{(8)}_{kl} = \frac{i}{4} (\Gamma_k \Gamma_l - \Gamma_l \Gamma_k), \quad k, l = 1, 2, 3, 4.
\]

The representation for the generators \( P(1,3) \) in the form (4) differs from the Foldy–Shirokov [3, 4] representation. In the form (4) it is explicitly distinguished the fact that in the space where a representation of the \( P(1,3) \) group is given, also a representation

of $SO_4 \sim SU_2 \otimes SU_2$ is realized. This follows, in particular, from the fact $[H^e, S_{kl}] = 0$, i.e. it means that the matrices

$$S_a = \frac{1}{2} \left( \frac{1}{2} \epsilon_{abc} S_{bc} + S_{4a} \right), \quad T_a = \frac{1}{2} \left( \frac{1}{2} \epsilon_{abc} S_{bc} - S_{4a} \right),$$

commute with the Hamiltonian*. In other words this means that the space, where the representation of $P(1,3)$ group is realized, must be characterized (besides the mass $m$ and the sign of the energy) by pair of indices $s$ and $\tau$

$$S^2_a \Phi = s(s + 1) \Phi, \quad T^2_a \Phi = \tau(\tau + 1) \Phi, \quad s, \tau = \frac{1}{2}, 1, \frac{3}{2}, \ldots$$

we shall denote by $D^\pm(s,0)$ and $D^\pm(0,\tau)$ the irreducible representation of $P(1,3)$ group. For further understanding it should be noted that the irreducible representations $D(s,0)$ and $D(0,\tau)$ of $SO_4$ group are indistinguishable with respect to the matrices $S_{ab}$ from the $SO_3$ algebra.

From the canonical eight-component equation (3) we can obtain the following three types of nonequivalent four-component equations

$$i \frac{\partial \Phi_a(t, x)}{\partial t} = H_a \Phi_a(t, x), \quad a = 1, 2, 3,$$

$$H_1 = H_2 = \epsilon \gamma_0 E, \quad H_3 = \epsilon E, \quad \epsilon = \pm 1,$$

where $\gamma_0$ is the hermitian and diagonal $4 \times 4$ matrix**. Under a transformation of the $P(1,3)$ group the four-component wave functions $\Phi_1, \Phi_2, \Phi_3$ transform on the representations (for the sake of brevity we consider only case $\epsilon = +1$)

$$D^+(s,0) \oplus D^-(0,\tau), \quad s = \tau = \frac{1}{2},$$

$$D^+(s,0) \oplus D^-(s,0), \quad s = \frac{1}{2}, \quad \tau = 0,$$

$$D^+(s,0) \oplus D^+(0,\tau), \quad s = \tau = \frac{1}{2}.$$  

On the manifolds $\{\Phi_1\}, \{\Phi_2\}, \{\Phi_3\}$ the generators $P_\mu, J_{\alpha\beta}$ have the forms

$$P_{0}^{(1)} = H_1, \quad P_{a}^{(1)} = p_a, \quad J_{ab}^{(1)} = M_{ab} + S_{ab},$$

$$J_{ba}^{(1)} = x_0 p_a - \frac{1}{2} [x_a, H_1]_+ - \gamma_0 \frac{S_{ab} p_b + S_{4a} m}{E};$$

$$P_{0}^{(2)} = H_2, \quad P_{a}^{(2)} = p_a, \quad J_{ab}^{(2)} = M_{ab} + S_{ab},$$

$$J_{ba}^{(2)} = x_0 p_a - \frac{1}{2} [x_a, H_2]_+ - \gamma_0 \frac{S_{ab} p_b + \frac{1}{2} \epsilon_{abc} S_{bc} m}{E};$$

*In fact, eq(2) or (3) is invariant with respect to $SO_6 \supset SO_4$ group [2]. A relativistic equation of motion for particle with spin $\frac{3}{2}$ is invariant also with respect to the $SO_6$ group.

**The fact that the $H_1$ and $H_2$ have identical forms in two eqs.(5) must not lead into confusion since the equation of motion is defined completely if only we determine both the Hamiltonian and the representation of $P(1,3)$ group.
that quantities $r_1, r_2, t_1, t_2$ are the matrices which do not depend on the momentum. If the conditions (21), (22) are not imposed, then the operators $P$ and $T$ may be nonlocal (in this case the quantities depend on the momentum).

In addition to the discrete operators $P$ and $T$ we shall introduce some more discrete operators:

$$M\Phi(t, x, m) = r_m \Phi(t, x, -m), \quad M^2 \sim 1,$$

(23)
it is equal to the operator tors and the projections this it is necessary to analyse the commutation relations between the discrete opera-

\[ \begin{align*}
M_t \Phi(t, x, m) &= m_t \Phi(-t, x, -m), \quad M_t^2 \sim 1, \\
M_x \Phi(t, x, m) &= m_x \Phi(t, -x, -m), \quad M_x^2 \sim 1, \\
[M, P_\mu] &= 0 = [M, J_{\mu\nu}], \quad \mu, \nu = 0, 1, 2, 3, \\
[M_t, P_0] &= 0 = [M_t, J_{0a}], \\
[M_x, P_0] &= 0 = [M_x, J_{ab}], \\
[M, P_a] &= 0 = [M, J_{ab}], \\
[M_x, P_a] &= 0 = [M_x, J_{0a}],
\end{align*} \]

where \( r_m, m_t, m_x \) are the \( 4 \times 4 \) matrices.

There is no need to define specially the operator of the charge conjugation \( C \) since it is equal to the operator \( T^{(1)} \cdot T^{(2)} \) (or \( P^{(1)} \cdot P^{(2)} \)).

If we use the explicit forms (10)–(28) for the generators \( P_\mu \) and \( J_{\alpha\beta} \) and carrying out the analysis of the conditions (13)–(28) we come to the following results:

1) Equation (5) for the function \( \Phi_1 \) (taking into consideration the representation (10)) is \( C, M_x, M_t, P^{(1)}T^{(2)} \) invariant, but \( P^{(1)}, P^{(2)}, T^{(2)}, M \) noninvariant;

2) Equation (5) for the function \( \Phi_2 \) (taking into consideration the representation (11)) is \( P^{(2)}, T^{(1)}, M_x, P^{(1)}T^{(2)} \) invariant, but \( P^{(1)}, T^{(2)}, C, M, M_t \) noninvariant;

3) Equation (5) for the function \( \Phi_3 \) (taking into consideration the representation (12)) is \( P^{(1)}, T^{(2)}, M, M_x, P^{(1)}T^{(2)} \) invariant, but \( T^{(1)}, C, P^{(2)}, M_t \) noninvariant.

These assertions may be proved also starting from eight-component equation (2) (or (3)) in which constraints have been imposed on the wave function [1]. To establish this it is necessary to analyse the commutation relations between the discrete operators and the projections \( P^+_1, P^+_2, P^+_3 \).

**Note 1.** It can be easily checked that

\[ P^{(1)}S_a = T_aP^{(1)}, \quad MS_a = T_aM, \quad T^{(1)}S_a = S_aT^{(1)}. \]

The transformation connecting the canonical representations (10)–(12) and the Foldy–Shirokov representation has the form

\[ U_1 = \frac{m + E + \gamma_4\gamma_a p_a}{(2E(E + m))^{1/2}}. \]

**Note 2.** If we put \( m = 0 \) in the reducible representation (4), then it reduces into the following direct sum of the irreducible representation of the \( P(1, 3) \) algebra

\[ D^+ \left( \frac{1}{2}, 0 \right) \oplus D^- \left( 0, \frac{1}{2} \right) \oplus D^- \left( \frac{1}{2}, 0 \right) \oplus D^+ \left( 0, \frac{1}{2} \right) \rightarrow \]

\[ \rightarrow D^+ \left( \frac{1}{2}, 0 \right) \oplus D^+ \left( -\frac{1}{2}, 0 \right) \oplus D^- \left( 0, \frac{1}{2} \right) \oplus D^- \left( 0, -\frac{1}{2} \right) \oplus \]

\[ \oplus D^- \left( \frac{1}{2}, 0 \right) \oplus D^- \left( -\frac{1}{2}, 0 \right) \oplus D^+ \left( 0, \frac{1}{2} \right) \oplus D^+ \left( 0, -\frac{1}{2} \right) \]

\[ \text{In the coupling scheme, brought in ref. [1], the correction } D^+ (s, 0) \sim D^- (0, s) \text{ should be done.} \]
where members $\frac{1}{2}$ and $-\frac{1}{2}$ are the eigenvalues of the operators $S_α p_α/E$ and $T_α p_α/E$. These operators commute with the generators $P_μ$, $J_{αβ}$ when $m = 0$. From (31) follows that there exist 28 types of mathematical nonequivalent two-component equations for massless particles.

**Note 3.** In order that Poincaré-invariant equation $m \neq 0$ was totally $P$, $T$, $C$ invariant it is necessary and sufficient that the wave function was transformed on the following direct sum of representation of $P(1, 3)$

\[
D^+(s, τ)\oplus D^-(s, τ)\oplus D^+(τ, s)\oplus D^-(τ, s), \quad \text{if} \quad τ \neq s, \\
D^+(s, τ)\oplus D^-(s, τ), \quad \text{if} \quad τ = s.
\]

The representation $D^+(s, τ)$ is in general reducible with respect to the $P(1, 3)$ algebra, therefore the wave function describes a multiplet of particles with variable-spin, but fixed mass. The spin of the multiplet can take the values from $(s − τ)$ to $(s + τ)$. The equations of motion describing a physical system with variable-mass and variable-spin were considered in ref. [5].