On two-component equations for zero mass particles

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The paper presents a detailed theoretical-group analysis of three types of two-component equations of motion which describe the particle with zero mass and spin $\frac{1}{2}$. There are studied P-, T- and C-propertias of the equations obtained.

В работе дан детальный теоретико-групповой анализ трех типов двухкомпонентных уравнений движения, описывающих частиц с нулевой массой и спином $\frac{1}{2}$. Изучены P-, T-, C-свойства найденных уравнений.

1. Introduction

In the previous paper [1] it was shown by one of the authors that starting from the four-component Dirac equation with zero mass one can obtain three types of two-component equations. One of them coincides with the Weyl equation which, as is known, is $P^{(k)}C$- and $T^{(1)}$-invariant but $P^{(k)}$-, C-noninvariant. Two other equations are noninvariant with respect to $P^{(k)}C$-transformations. For one of these two equations the PTC theorem is not valid, i.e. such an equation is noninvariant with respect to $P^{(k)}T^{(1)}C$- and $P^{(k)}T^{(2)}C$-transformations[1].

This present paper is dedicated to the detailed study of all possible (with an accuracy of the unitary equivalence) two-component and four-component (with subsidiary conditions) equations describing free notion of a particle with zero mass and spin $s = \frac{1}{2}$.

From the point of view of ideology the previous and the present papers are closely connected with the papers by Shirokov [2] and Foldy [3] in which for the first time equations of motion for a particle without antiparticle with non-zero mass and arbitrary spin were suggested. The Shirokov–Foldy equations are $P^{(k)}$- and $T^{(1)}$-invariant, but $T^{(2)}$- and C-noninvariant.

2. Three types of two-component equations

1. The helicity and energy sign [2] operators [2]

$$\Lambda = \frac{J_{12}P_3 + J_{23}P_1 + J_{32}P_1}{E}, \qquad E = \sqrt{p_1^2 + p_2^2 + p_3^2}, \qquad \hat{\varepsilon} = \frac{P_0}{E} \tag{2.1}$$

are the Casimir operators of the group $P(1,3)$ for the representations with zero mass and discrete spin.

Between the operators P, T, C and Λ, $\hat{\varepsilon}$ it is easy to establish such relations[2]:

$$P^{(k)}\Lambda = -\Lambda P^{(k)}, \qquad P^{(k)}\hat{\varepsilon} = \hat{\varepsilon}P^{(k)}, \qquad k = 1,2,3, \tag{2.2}$$

$$T^{(a)}\Lambda = \Lambda T^{(a)}, \quad a = 1,2, \qquad T^{(1)}\hat{\varepsilon} = \hat{\varepsilon}T^{(1)}, \qquad T^{(2)}\hat{\varepsilon} = -\hat{\varepsilon}T^{(2)}, \tag{2.3}$$

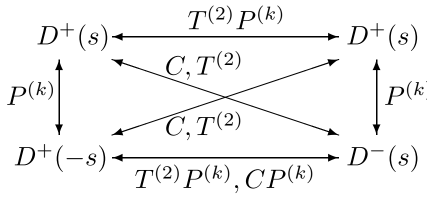
$$C\Lambda = \Lambda C, \qquad C\hat{\varepsilon} = -\hat{\varepsilon}C. \tag{2.4}$$

Препринт ИТФ–70-88Е, Киев, 1970, № 88, 22 с.

[1] Notations and definitions which are gives without explations are the same as in the paper [1].

[2] The results of this subsection are valid for the arbitrary spin.

Hence it follows such coupling scheme of irreducible representations of the proper Poincaré group by the operators P, T, C:



It is seen from the scheme (2.5) that there exist three essentially different (with respect to P-, T and C-transformations) types of two-component equations of motion on the solutions of which the following representations of the $P(1,3)$ group are realized:

$$D^+(s) \oplus D^-(-s) \qquad \text{or} \qquad D^-(s) \oplus D^+(-s), \tag{2.6}$$

$$D^+(s) \oplus D^-(s) \qquad \text{or} \qquad D^-(-s) \oplus D^+(-s), \tag{2.7}$$

$$D^+(s) \oplus D^+(-s) \qquad \text{or} \qquad D^-(s) \oplus D^-(-s). \tag{2.8}$$

Hence it follows such result:

1) the space R_1 where the representation (2.6) is realized is invariant with respect to $T^{(1)}$- and $CP^{(k)}$-transformations but noninvariant with respect to $T^{(2)}$-, $P^{(k)}$- and C-transformations;
2) the space R_2 where the representation (2.7) is realized is invariant with respect to $T^{(1)}$-, $T^{(2)}$- and C-transformations but noninvariant with respect to $P^{(k)}$- and $CP^{(k)}$-transformations;
3) the space R_3 where the representation (2.8) is realized, is invariant with respect to $P^{(k)}$- and $T^{(1)}$-transformations but noninvariant with respect to $T^{(2)}$- and C-transformations.

The two-component equations the wave functions of which are transformed according to the representations (2.6)–(2.8), have the same P-, T- and C-properties as the spaces R_1, R_2, R_3 have.

2. The Dirac equation

$$\gamma_\mu p^\mu \Psi(t,\vec{x}) = 0, \qquad \mu = 0,1,2,3 \tag{2.9}$$

is transformed to the form

$$i\frac{\partial\Phi(t,\vec{x})}{\partial t} = \gamma_0 \Phi(t,\vec{x}), \tag{2.10}$$

$$\Phi(t,\vec{x}) = U\Psi(t,\vec{x}) \tag{2.11}$$

with the help of unitary transformation [4]

$$U = \frac{1}{\sqrt{2}}\left(1 + \frac{\gamma_k p_k}{E}\right). \tag{2.12}$$

In the representation (1.2.6) for the Dirac matrices[3] where

$$\gamma_0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}$$

eq. (2.10) decomposes into two two-component system

$$i\frac{\partial \Phi_\pm(t,\vec{x})}{\partial t} = \pm\sigma_3 E \Phi_\pm(t,\vec{x}), \tag{2.13}$$

$\Phi_\pm(t,\vec{x})$ are two-component wave functions.

If tor the Dirac matrices we choose the representation, where

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

then (2.10) decomposes into the system

$$i\frac{\partial \widetilde{\Phi}_\pm(t,\vec{x})}{\partial t} = \pm E \widetilde{\Phi}_\pm(t,\vec{x}), \tag{2.14}$$

$\widetilde{\Phi}_\pm(t,\vec{x})$ are two-component wave functions.

Eqs. (2.13), (2.14) in themselves (without algebra $P(1,3)$) do not unambiguously determine what particle and antiparticle they describe. Depending on the representation of the group $P(1,3)$ with respect to which its wave function is transformed under transformations from the group $P(1,3)$, the same (by the form) two-component equation of motion describes, as is seen below, different particles. In other words, it means that the equations of motion only together with the algebra $P(1,3)$ unambiguously determine what particle is described by it.

According to the results of the previous subsection for the particle with spin $s=\frac{1}{2}$ there are three essentially various two-dimension representations for the algebra $P(1,3)$. They have the following form

$$\begin{aligned}
&P_0^{\Phi_1} = \mathcal{H}^{\Phi_1} = \sigma_3 E, \qquad P_k^{\Phi_1} = p_k = -i\frac{\partial}{\partial x_k},\\
&J_{12}^{\Phi_1} = M_{12} + \frac{e_3\mathcal{H}^{\Phi_1}}{2E}, \qquad e_3 = \frac{p_3}{|p_3|}, \qquad p_3 \neq 0,\\
&J_{13}^{\Phi_1} = M_{13} - \frac{p_2\mathcal{H}^{\Phi_1}}{2E(E+|p_3|)}, \qquad J_{23}^{\Phi_1} = M_{23} + \frac{p_1\mathcal{H}^{\Phi_1}}{2E(E+|p_3|)},\\
&J_{01}^{\Phi_1} = t_0 p_1 - \frac{1}{2}\left[x_1, \mathcal{H}^{\Phi_1}\right]_+ - \frac{p_2 e_3}{2(E+|p_3|)},\\
&J_{02}^{\Phi_1} = t_0 p_2 - \frac{1}{2}\left[x_2, \mathcal{H}^{\Phi_1}\right]_+ + \frac{p_1 e_3}{2(E+|p_3|)},\\
&J_{03}^{\Phi_1} = t_0 p_3 - \frac{1}{2}\left[x_3, \mathcal{H}^{\Phi_1}\right]_+;
\end{aligned} \tag{2.15}$$

[3]See (2.6) in [1]

$$
\begin{aligned}
&P_0^{\Phi_2} = \mathcal{H}^{\Phi_2} = \sigma_3 E, \qquad P_k^{\Phi_2} = p_k, \qquad J_{12}^{\Phi_2} = M_{12} + \frac{e_3}{2E},\\
&J_{13}^{\Phi_2} = M_{13} - \frac{p_2}{2(E+|p_3|)}, \qquad J_{23}^{\Phi_2} = M_{23} + \frac{p_1}{2(E+|p_3|)},\\
&J_{01}^{\Phi_2} = t_0 p_1 - \frac{1}{2}\left[x_1, \mathcal{H}^{\Phi_2}\right]_+ - \frac{p_2 e_3 \mathcal{H}^{\Phi_2}}{2E(E+|p_3|)},\\
&J_{02}^{\Phi_2} = t_0 p_2 - \frac{1}{2}\left[x_2, \mathcal{H}^{\Phi_2}\right]_+ + \frac{p_1 e_3 \mathcal{H}^{\Phi_2}}{2E(E+|p_3|)},\\
&J_{03}^{\Phi_2} = t_0 p_3 - \frac{1}{2}\left[x_3, \mathcal{H}^{\Phi_2}\right]_+;
\end{aligned}
\tag{2.16}
$$

$$
\begin{aligned}
&P_0^{\Phi_3} = E = \mathcal{H}^{\Phi_3} \qquad P_k^{\Phi_3} = p_k, \qquad J_{12}^{\Phi_3} = M_{12} + \frac{e_3\sigma_3}{2},\\
&J_{13}^{\Phi_3} = M_{13} - \frac{p_2\sigma_3}{2(E+|p_3|)}, \qquad J_{23}^{\Phi_3} = M_{23} + \frac{p_1\sigma_3}{2(E+|p_3|)},\\
&J_{01}^{\Phi_3} = t_0 p_1 - \frac{1}{2}\left[x_1, \mathcal{H}^{\Phi_3}\right]_+ - \frac{p_2 e_3 \sigma_3}{2(E+|p_3|)},\\
&J_{02}^{\Phi_3} = t_0 p_2 - \frac{1}{2}\left[x_2, \mathcal{H}^{\Phi_3}\right]_+ + \frac{p_1 e_3 \sigma_3}{2(E+|p_3|)},\\
&J_{03}^{\Phi_3} = t_0 p_3 - \frac{1}{2}\left[x_3, \mathcal{H}^{\Phi_3}\right]_+,
\end{aligned}
\tag{2.17}
$$

where $M_{kl} = x_k p_l - x_l p_k$.

By direct verification one can be convinced that the operators (2.15)–(2.17) satisfy the commutation relations of algebra $P(1,3)$. These three representations are not equivalent. Really the operators of energy sign and helicity have the form

$$
\begin{aligned}
&\hat\varepsilon = \frac{\mathcal{H}^c}{E} = \sigma_3, \quad \Lambda = \frac{1}{2}\hat\varepsilon \quad \text{for the representation (2.15)},\\
&\hat\varepsilon = \sigma_3, \qquad \Lambda = \frac{1}{2} \quad \text{for the representation (2.16)},\\
&\hat\varepsilon = 1, \qquad \Lambda = \frac{1}{2}\sigma_3 \quad \text{for the representation (2.17)}.
\end{aligned}
$$

Hence it is clear that the representations (2.12), (2.16), (2.17) are not equivalent and are given in the spaces R_1, R_2, R_3 respectively.

Besides two-dimensional representations given for the algebra $P(1,3)$ one can, evidently, obtain the other ones as well which however, will be unitary-equivalent to (2.15)–(2.17). If, for example, in (2.15)–(2.17) one performs the substitution

$$
e_3 \to 1, \qquad |p_3| \to p_3, \tag{2.18}
$$

then the operators obtained also realize the representations of the algebra $P(1,3)$. The explicit form for the generators of the group $P(1,3)$ obtained from (2.15)–(2.17) with the help of substitution (2.18) will be denoted in the sequal by $(2.15')$–$(2.17')$.

If in (2.15)–(2.17) the matrix σ_3 is substituted by 1 (or -1), then such operators will realize one-dimensional irreducible representations of the algebra $P(1,3)$ which are, of course, unitarily equivalent to the corresponding one-dimensional Shirokov representations [5]. The representations [5] are obtained without connection with the equations of motion and are realized on the functions $\Psi(p_1, p_2, p_3)$ not depending on the time.

Summing up all the above presented we come to the conclusion:

1) Eq. (2.13) together with algebra (2.15) (or (2.15′)) describes the particle with helicity $+\frac{1}{2}$ and the antiparticle with helicity $-\frac{1}{2}$[4];
2) Eq. (2.13) together with algebra (2.16) (or (2.16′)) describes the particle with helicity $+\frac{1}{2}$ and the antiparticle with helicity $+\frac{1}{2}$;
3) Eq. (2.14) together with algebra (2.17) (or (2.17′)) describes two particles with helicity $+\frac{1}{2}$ and $-\frac{1}{2}$[5].

If Eq. (2.13) is connected with the algebra (2.13) (or (2.15′)) the wave function of such equation is denoted by Φ_1 (or Φ_1'). The wave function in Eq. (2.13) connected with the algebra (2.16) (or (2.16′)) is denoted by Φ_2 (or Φ_2'). Similarly Φ_3 (or Φ_3') denotes the wave function in Eq. (2.14) connected with the algebra (2.17) (or (2.17′)).

3. The transition from the canonical equation (2.13) to the non-canonical one of the type (1.3.1) is realized with the help of unitary transformation [1]

$$v_1^{-1} = \frac{E + |p_3| + i(\sigma_1 p_2 - \sigma_2 p_1)}{\{2E(E + |p_3|)\}^{1/2}}. \tag{2.19}$$

Under this transformation Eq. (2.13) takes the form

$$i\frac{\partial \chi(t,\vec{x})}{\partial t} = (\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 |p_3|)\chi(t,\vec{x}), \tag{2.20}$$

where

$$\chi = \chi_1 = v_1^{-1}\Phi_1 \tag{2.21}$$

or

$$\chi = \chi_2 = v_1^{-1}\Phi_2. \tag{2.22}$$

The type of Eq. (2.14) under transformation (2.19) is not changed. The operators (2.15), (2.16) in χ-representation have the form

$$P_0^{\chi_a} = \mathcal{H} = \sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 |p_3|, \qquad P_k^{\chi_a} = p_k, \qquad a = 1, 2, \tag{2.23}$$

$$J_{\mu\nu}^{\chi_1} = J_{\mu\nu}^{\Phi_1}\left(x_k \to X_k, \mathcal{H}^{\Phi_1} \to \mathcal{H}\right), \tag{2.24}$$

$$J_{\mu\nu}^{\chi_2} = J_{\mu\nu}^{\Phi_2}\left(x_k \to X_k, \mathcal{H}^{\Phi_2} \to \mathcal{H}\right), \tag{2.25}$$

where

$$\begin{aligned} X_1 &= x_1 - \frac{\sigma_2}{2E} + \frac{\sigma_3 p_2}{2E(E + |p_3|)} - \frac{p_1(\sigma_1 p_2 - \sigma_2 p_1)}{2E^2(E + |p_3|)}, \\ X_2 &= x_2 + \frac{\sigma_1}{2E} - \frac{\sigma_3 p_1}{2E(E + |p_3|)} - \frac{p_2(\sigma_1 p_2 - \sigma_2 p_1)}{2E^2(E + |p_3|)}, \\ X_3 &= x_3 - \frac{\sigma_1 p_2 - \sigma_2 p_1}{2E^2} e_3. \end{aligned} \tag{2.26}$$

[4]The representation $D^+(s)$ corresponds to the particle and the representation $D^-(s)$ to the antiparticle.

[5]Eq. (2.14) can be interpreted as the equation of motion for one particle which can be in two states differing one from another by the helicity sign.

If one connects with Eqs. (2.13) and (2.14) the representations (2.15′)–(2.17′), but not the representations (2.l5)–(2.17) in this case the transition from the canonical equation to the noncanonical one can he conveniently realized with the help of the unitary transformation [6]

$$v^{-1} = \frac{E + p_3 + i(\sigma_1 p_2 - \sigma_2 p_1)}{\{2E(E + |p_3|)\}^{1/2}}. \tag{2.27}$$

Under this transformation Eq. (2.13) takes the form of the Weyl equation

$$i\frac{\partial \chi^w(t,\vec{x})}{\partial t} = (\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3)\chi^w(t,\vec{x}), \tag{2.28}$$

where

$$\chi^w \equiv \chi_1^w = v^{-1}\Phi_1' \tag{2.29}$$

or

$$\chi^w \equiv \chi_2^w = v^{-1}\Phi_2'. \tag{2.30}$$

Eq. (2.14) is unchanged by the transformation (2.27). The generators $J_{\mu\nu}^{w_2} \equiv J_{\mu\nu}^{\chi_2^w}$ coincide with (2.25) where the operator $\mathcal{H}$ has the form (2.31) and in the operators (2.26) the substitution (2.18) is performed. This algebra is denoted by (2.25′). Under transformation (2.27) the algebra (2.15′) goes into the algebra

$$\begin{aligned} &P_0^{w_1} = \mathcal{H} = \sigma_k p_k, \qquad P_k^{\chi_1^w} = p_k, \\ &J_{kl}^{w_1} = M_{kl} + \frac{\sigma_n}{2}, \qquad k,l,n \quad \text{is the cycle (1,2,3)}, \\ &J_{0k}^{w_2} = t_0 p_k - \frac{1}{2}[x_k, \mathcal{H}]_+. \end{aligned} \tag{2.31}$$

The algebra (2.17′) is transformed into the algebra

$$\begin{aligned} &P_0^{w_3} = \mathcal{H} = E, \qquad P_k^{w_3} = p_k, \\ &J_{kl}^{w_3} = M_{kl} + \frac{\sigma_n}{2}, \qquad k,l,n \quad \text{is the cycle (1,2,3)}, \\ &J_{0k}^{w_3} = t_0 p_k - \frac{1}{2}[x_k, \mathcal{H}]_+ + \frac{\sigma_n p_l - \sigma_l p_n}{E}. \end{aligned} \tag{2.32}$$

The operators $P_\mu^{w_k}$, $J_{\mu\nu}^{w_k}$ are defined on the corresponding sets $\{\chi_k^w\}$, $k = 1, 2, 3$.

From the above given analysis it follows such a result:

1) Eq. (2.20) together with the algebra (2.24) (or Eq. (2.28) together with the algebra (2.31)) describes the particle with helicity $+\frac{1}{2}$ and the antiparticle with helicity $-\frac{1}{2}$;
2) Eq. (2.20) together with the algebra (2.25) (or eq. (2.28)) together with the algebra (2.25′)) describes the particle with helicity $+\frac{1}{2}$ and the antiparticle with helicity $+\frac{1}{2}$.

Thus, the same (by the form) Eq.(2.20) (or the Weyl equation (2.28)) describes different types of particles and antiparticles depending on the representation of group $P(1,3)$ with respect to which the wave functions χ (or χ^w) are transformed under transformations from the group $P(1,3)$.

Not only the equations invariant with respect to the group $P(1,3)$ have such a dual nature but also the equations invariant with respect to the inhomogeneous de Sitter group $P(1,4)$. Within the frameworks of the $P(1,4)$ group the same (by the form) equation of the Dirac type describes the various types of particles and antiparticles [7].

§ 3. *P*-, *T*- and *C*-properties of two-component equations

In studying P-, T- and C-properties of the equations of motion one does not indicate as a rule, with what algebra $P(1,3)$ the given equation is connected. Such an approach, as it follows from the results of the previous section, is not quite correct for the studying P-, T, C-properties of Eqs. (2.13), (2.20), (2.28), since the same equation connected with various algebras $P(1,3)$ can have various properties with respect to the space-time reflections.

In order the equation, invariant with respect to the proper group $P(1,3)$, be P-, T- and C-invariant it is necessary and sufficient to satisfy such relations:

$$\begin{aligned}
&\big[P^{(k)},\mathcal{H}\big]_-=0, && k=1,2,3,\\
&\big[P^{(k)},P_l\big]_-=0 && \text{for}\quad k\neq l,\\
&\big[P^{(k)},P_l\big]_+=0 && \text{for}\quad k=l,\\
&\big[P^{(k)},J_{lr}\big]_+=0 && \text{for}\quad k=l,\ k=r,\\
&\big[P^{(k)},J_{lr}\big]_-=0 && \text{for}\quad k\neq l,\ k\neq r,\\
&\big[P^{(k)},J_{0l}\big]_-=0 && \text{for}\quad k\neq l,\\
&\big[P^{(k)},J_{0l}\big]_+=0 && \text{for}\quad k=l;
\end{aligned}\qquad(3.1)$$

$$\Big[T^{(1)},\mathcal{H}\Big]_-=\Big[T^{(1)},J_{0l}\Big]_-=0,\qquad \Big[T^{(1)},P_k\Big]_+=\Big[T^{(1)},J_{kl}\Big]_+=0,\qquad(3.2)$$

$$\Big[T^{(2)},\mathcal{H}\Big]_+=\Big[T^{(2)},J_{0l}\Big]_+=0,\qquad \Big[T^{(2)},P_k\Big]_-=\Big[T^{(2)},J_{kl}\Big]_-=0,\qquad(3.3)$$

$$[C,\mathcal{H}]_+=[C,P_k]_+=[C,J_{\mu\nu}]_+=0.\qquad(3.4)$$

Hence it follows that the equation of motion is invariant with respect to P-transformation if all the conditions (3.1) are satisfied. Usually when studying P-properties of the equations one verifies only the first relation from (3.1) that, evidently, is not sufficient for the correct conclusion.

How we give the explicit expressions for the operators $r^{(k)}$, $\tau^{(i)}$ (see formulas (1.3.2)–(1.3.5)) determining the operators of discrete transformations.

On the sets $\{\Phi_1\}$ and $\{\Phi_1'\}$ the operators $P^{(k)}$, $T^{(2)}$ and C cannot be determined since the range of values of these operators does not belong to the sets $\{\Phi_1\}$ and $\{\Phi_1'\}$. The operator $T^{(1)}$ on $\{\Phi_1\}$, $\{\Phi_1'\}$ can be defined and it is determined by such operators

$$\tau^{(1)}=1\qquad\text{or}\quad\sigma_3\qquad\text{on}\quad\{\Phi_1\},\qquad(3.5)$$

$$\tau^{(1)}=\frac{\sigma_3p_2+ip_1}{\sqrt{p_1^2+p_2^2}}\qquad\text{on}\quad\{\Phi_1'\}.\qquad(3.6)$$

$T^{(1)}$, $T^{(2)}$ and C on the sets $\{\Phi_2\}$, $\{\Phi_2'\}$ are given by the operators $\tau^{(k)}$, $k = 1, 2, 3$

$$\tau^{(2)} = \sigma_1 \quad \text{or} \quad \sigma_2 \quad \text{on} \quad \{\Phi_2\}, \tag{3.7}$$

$$\tau^{(3)} = \sigma_1 \quad \text{or} \quad \sigma_2 \quad \text{on} \quad \{\Phi_2\}. \tag{3.8}$$

The operator $\tau^{(1)}$ on the sets $\{\Phi_1\}$ and $\{\Phi_2\}$ has the form (3.5)

$$\tau^{(1)} = \frac{p_2 + ip_1}{\sqrt{p_1^2 + p_2^2}} \quad \text{or} \quad \frac{\sigma_3(p_2 + ip_1)}{\sqrt{p_1^2 + p_2^2}} \quad \text{on} \quad \{\Phi_2'\}, \tag{3.9}$$

$$\tau^{(2)} = \sigma_1 \quad \text{or} \quad \sigma_2 \quad \text{on} \quad \{\Phi_2'\}, \tag{3.10}$$

$$\tau^{(3)} = \sigma_1 \frac{p_2 + ip_1}{\sqrt{p_1^2 + p_2^2}} \quad \text{on} \quad \{\Phi_2'\}. \tag{3.11}$$

The operators $P^{(k)}$ are not determined on $\{\Phi_2\}$ and $\{\Phi_2'\}$. $P^{(k)}$ and $T^{(1)}$ on the sets $\{\Phi_3\}$, $\{\Phi_3'\}$ are given by:

$$\tau^{(1)} = 1 \quad \text{or} \quad \sigma_3 \quad \text{on} \quad \{\Phi_3\}, \tag{3.12}$$

$$r^{(k)} = \sigma_1 \quad \text{or} \quad \sigma_2 \quad \text{on} \quad \{\Phi_3\}, \tag{3.13}$$

$$\tau^{(1)} = \frac{p_2 + i\sigma_3 p_1}{\sqrt{p_1^2 + p_2^2}} \quad \text{on} \quad \{\Phi_3'\}, \tag{3.14}$$

$$\begin{gathered} r^{(1)} = \sigma_1 \qquad r^{(2)} = \sigma_2 \quad \text{on} \quad \{\Phi_3'\}, \\ r^{(3)} = \frac{\sigma_a p_a}{\sqrt{p_1^2 + p_2^2}} \quad \text{on} \quad \{\Phi_3'\}. \end{gathered} \tag{3.15}$$

The operators $r^{(k)}$ and $\tau^{(k)}$ on the sets $\{\chi\}$ and $\{\chi^w\}$ have the form

$$\begin{gathered} \tau^{(1)} = 1 - \frac{p_1^2}{E(E + |p_3|)} + \frac{ip_1 p_2 \sigma_3}{E(E + |p_3|)} - \frac{i\sigma_2 p_1}{E} \quad \text{or} \\ \tau^{(1)} = \sigma_3 \left(1 - \frac{p_2^2}{E(E + |p_3|)}\right) - \frac{ip_1 p_2}{E(E + |p_3|)} + \frac{\sigma_2 p_2}{E} \quad \text{on} \quad \{\chi_1\},\ \{\chi_2\}, \end{gathered} \tag{3.16}$$

$$\tau^{(1)} = \sigma_2 \quad \text{on} \quad \{\chi_1^w\}, \tag{3.17}$$

$$\begin{gathered} \tau^{(2)} = \left(1 - \frac{p_1^2}{E(E + |p_3|)}\right)\sigma_1 - \frac{\sigma_2 p_1 p_2}{E(E + |p_3|)} - \frac{\sigma_3 p_1}{E} \quad \text{or} \\ \tau^{(2)} = \left(1 - \frac{p_2^2}{E(E + |p_3|)}\right)\sigma_2 - \frac{\sigma_1 p_1 p_2}{E(E + |p_3|)} - \frac{\sigma_3 p_2}{E} \quad \text{on} \quad \{\chi_2\}, \end{gathered} \tag{3.18}$$

$$\tau^{(3)} = \sigma_1 \quad \text{or} \quad \tau^{(3)} = \frac{\sigma_2 |p_3|}{E} - \frac{p_2 \sigma_3 + ip_1}{E} \quad \text{on} \quad \{\chi_2\}; \tag{3.19}$$

$$\begin{gathered} \tau^{(1)} = \frac{1}{E}(p_2 - ip_1)(p_2^2 - i\sigma_2 p_1 E - i\sigma_1 p_2 p_3 + i\sigma_1 p_1 p_2) \quad \text{or} \\ \tau^{(1)} = \frac{1}{E}(-ip_1 p_2 + \sigma_2 p_2 E - \sigma_1 p_1 p_3 + \sigma_3 p_1^2)(p_2 - ip_1) \quad \text{on} \quad \{\chi_2^w\}; \end{gathered} \tag{3.20}$$

$$\tau^{(2)} = \sigma_1 - \frac{\sigma_3 p_1}{E} - \frac{p_1\sigma_a p_a}{E(E+|p_3|)} \qquad \text{on} \quad \{\chi_2^w\}, \quad \text{or}$$
$$\tau^{(2)} = \sigma_2 - \frac{\sigma_3 p_2}{E} - \frac{p_2\sigma_a p_a}{E(E+|p_3|)} \qquad \text{on} \quad \{\chi_2^w\}; \tag{3.21}$$

$$\tau^{(3)} = \frac{p_2 - ip_1}{E}\left\{\sigma_1(p_1^2+p_2^2) - ip_2p_3 + \sigma_3p_1p_3\right\} \qquad \text{on} \quad \{\chi_2^w\}; \tag{3.22}$$

$$\tau^{(1)} = \frac{1}{E}(p_2 - i\sigma_1 p_3 + i\sigma_3 p_1) \qquad \text{on} \quad \{\chi_3^w\}; \tag{3.23}$$

$$r^{(k)} = \sigma_k, \qquad k=1,2,3 \qquad \text{on} \quad \{\chi_3^w\}. \tag{3.24}$$

The operators (3.16)–(3.24) are obtained from (3.5)–(3.11), (3.14),(3.17) with the help of transformations (2.19), (2.27). The transformation law of these operators is given in (D.10)–(D.16).

Summing up all the above said we come to the final conclusion:

1) Eq. (2.13) for the function Φ_1 (or Φ_1') is $T^{(1)}$- and $P^{(k)}C$-invariant, but $P^{(k)}$-, $T^{(2)}$- and C-noninvariant;
2) Eq. (2.13) for the function Φ_2 (or Φ_2') is $T^{(1)}$-, $T^{(2)}$- and C-invariant, but $P^{(k)}$- and $CP^{(k)}$-noninvariant;
3) Eq. (2.14) is $P^{(k)}$- and $T^{(1)}$-invariant, but $T^{(2)}$-, C- and $CP^{(k)}$-noninvariant.

Evidently Eqs. (2.20), (2.28) have these properties as well.

Note 1. In [1] we established P-, T- and C-properties of Eq. (2.20) starting from the assumption that $r^{(k)}$, $\tau^{(k)}$ on the set $\{\chi\}$ are the 2×2 matrices. As is seen from the previous such an assumption is limited. On the set $\{\chi\}$ $r^{(k)}$, $\tau^{(k)}$ are the operator functions depending on the momentum components of the particle.

Note 2. Under the four-dimensional rotations in Minkovski space the wave functions Φ_1, χ_1, Φ_2, χ_2^w, Φ_3, χ_3, χ_2 are transformed nonlocally.

In conclussion of this section we give some corrollaries immediately following from the previous, which can be useful for the construction of weak interaction models on the basis of the equations obtained.

Corrollary 1. *Any (one-component or two-component) equation of motion for the particles with zero mass is Invariant with respect to the Wigner reflection of time* $T^{(1)}$.

Corrollary 2. *Eq. (2.20) for the function* χ_2 *(or (2.28) for the function* χ_2^w*) is* $T^{(1)}C$*- and* $T^{(2)}C$*-invariant, but* PC*-,* $PT^{(1)}$*-,* $PT^{(2)}$*-,* $PT^{(1)}C$*- and* $PT^{(2)}C$*-noninvariant. It means that for such equation neither hypothesis of combined parity conservation, nor hypothesis of* PTC*-invariance conservation is valid.*

Corrollary 3. *Eq. (2.14) is* $PT^{(1)}$*-,* $T^{(2)}C$*- and* $PT^{(2)}C$*-invariant, but* $PT^{(2)}$*-,* PC*- and* $PT^{(2)}C$*-noninvariant.*

§ 4. CP-noninvariant subsidiary conditions

The results of previous sections can be rather simply and briefly formulated if one describes the zero mass particle with the help of four-component wave function. In this case the wave function has the redundant (nonphysical) components which ran be invariantly separated with the help of relativistic-invariant subsidiary conditions. From § 2, 3 it follows that there are three types of subsidiary conditions. One of them is well known and has the form

$$\mathcal{P}_1^+\Psi = 0 \qquad \text{or} \qquad \mathcal{P}_1^-\Psi = 0, \tag{4.1}$$

$$\mathcal{P}_1^\pm = \frac{1}{2}(1 \pm \gamma_4). \tag{4.2}$$

Eq. (2.9) together with the condition (4.1) is equivalent to Eq. (2.28) for the function χ_1^w (or (2.20) for the function χ_1).

Now we find two other relativistic-invariant subsidiary conditions. Besides the matrix γ_4 the energy sign operator commutes with the algebra (1.2.1). Hence it is clear that the operators

$$\mathcal{P}_2^\pm = \frac{1}{2}\left(1 \pm \gamma_4\frac{\mathcal{H}}{E}\right), \qquad \mathcal{H} = \gamma_0\gamma_k p_k, \tag{4.3}$$

$$\mathcal{P}_3^\pm = \frac{1}{2}\left(1 \pm \frac{\mathcal{H}}{E}\right) \tag{4.4}$$

commute with the algebra (1.2.1). The operators $\mathcal{P}_2^\pm$, $\mathcal{P}_3^\pm$ are the projection operators. We show then that the conditions

$$\mathcal{P}_2^+\Psi = 0 \qquad \text{or} \qquad \mathcal{P}_2^-\Psi = 0, \tag{4.5}$$

$$\mathcal{P}_3^+\Psi = 0 \qquad \text{or} \qquad \mathcal{P}_3^-\Psi = 0, \tag{4.6}$$

can be considered as subsidiary conditions.

Between the operators $\mathcal{P}_2^\pm$, $\mathcal{P}_3^\pm$ and P, T, C it is easy to establish the following relations:

$$\begin{aligned} &P^{(k)}\mathcal{P}_2^\pm = \mathcal{P}_2^\mp P^{(k)}, \qquad && T^{(1)}\mathcal{P}_2^\pm = \mathcal{P}_2^\pm T^{(1)}, \\ &T^{(2)}\mathcal{P}_2^\pm = \mathcal{P}_2^\pm T^{(2)}, && C\mathcal{P}_2^\pm = \mathcal{P}_2^\pm C, \end{aligned} \tag{4.7}$$

$$\begin{aligned} &P^{(k)}\mathcal{P}_3^\pm = \mathcal{P}_3^\pm P^{(k)}, \qquad && T^{(1)}\mathcal{P}_3^\pm = \mathcal{P}_3^\pm T^{(1)}, \\ &T^{(2)}\mathcal{P}_3^\pm = \mathcal{P}_3^\mp T^{(2)}, && C\mathcal{P}_3^\pm = \mathcal{P}_3^\mp C. \end{aligned} \tag{4.8}$$

From (4.7), (4.8) it follows that the condition (4.5) is $T^{(1)}$-, $T^{(2)}$- and C-invariant, but $P^{(k)}$- and $CP^{(k)}$-noninvariant, and the condition (4.6) is $P^{(k)}$- and $T^{(1)}$-invariant, but $T^{(2)}$- and C-noninvariant. It means that the representation (2.7) is realized on the set $\mathcal{P}_2^\pm\{\Psi\}$, and the representation (2.8) is realized on the set $\mathcal{P}_3^\pm$.

Thus we came to the following result:

1) Eq. (2.9) with subsidiary condition (4.5) is $T^{(1)}$-, $T^{(2)}$- and C-invariant, but $P^{(k)}$-, $CP^{(k)}$-, $P^{(k)}T^{(1)}C$- and $P^{(k)}T^{(2)}C$-noninvariant;
2) Eq. (2.9) with subsidiary condition (4.6) is $P^{(k)}$-, $T^{(1)}$- and $P^{(k)}T^{(2)}C$-invariant, but $T^{(2)}$-, C-, $P^{(k)}T^{(1)}C$- and $P^{(k)}C$-noninvariant.

Eq. (2.9) with subsidiary conditions (4.1), (4.5), (4.6) can be written in the form of three equations

$$\left(\gamma_\mu p^\mu + \varkappa_k \mathcal{P}_k^+\right) \mathcal{P}_k^- \Psi(t,x) = 0, \qquad k = 1,2,3, \tag{4.9}$$

where $\varkappa_k$, $k = 1,2,3$ are the arbitrary constant numbers. For eqs. (4.9) the conditions (4.1), (4.5), (4.6) are satisfied automatically.

Appendix

In this appendix we present the main formulas according to which the operators r, $r^{(k)}$, $\tau^{(k)}$ in representations $\{\Phi\}$ and $\{\chi\}$ were calculated (see (3.5)–(3.25)).

To make it complete we give a definition to the combined parity

$$CP\Phi(t,\vec{x}) = \theta\Phi^*(t,-\vec{x}), \qquad CP^{(k)}\Phi(t,\vec{x}) = \theta^{(k)}\Phi^*(t,-x_k), \tag{D.1}$$

$$P\Phi(t,\vec{x}) = r\Phi(t,-\vec{x}), \qquad P \equiv P^{(1)}P^{(2)}P^{(3)}. \tag{D.2}$$

From (3.1)–(3.4) and from the definitions (D.1), (D.2), (1.3.2)–(1.3.5) we obtain such relations

$$\begin{aligned} &[r,p_k]_+ = 0, \qquad r\mathcal{H}(-\vec{p}) - \mathcal{H}(\vec{p})r = 0,\\ &rJ_{kl}(-\vec{x}) - J_{kl}(\vec{x})r = 0, \qquad rJ_{0k}(-\vec{x}) + J_{0k}(\vec{x})r = 0, \end{aligned} \tag{D.3}$$

$$\begin{aligned} &r(\vec{p})r(-\vec{p}) = 1, \qquad [r^{(k)},p_n]_\pm = 0, \qquad r^{(k)}\mathcal{H}(-p_k) - \mathcal{H}(-p_k)r^{(k)} = 0,\\ &r^{(k)}J_{nl}(-x_k) \pm J_{nl}(x_k)r^{(k)} = 0, \qquad r^{(k)}J_{0n}(-x_k) \pm J_{0n}(x_k)r^{(k)} = 0, \end{aligned} \tag{D.4}$$

where “+” is taken if $k = n$ or $k = l$;

$$\begin{aligned} &r^{(k)}(p_k)r^{(k)}(-p_k) = 1, \qquad \tau^{(1)}\mathcal{H}^* - \mathcal{H}\tau^{(1)} = 0, \qquad \tau^{(1)}p_k^* + p_k\tau^{(1)} = 0,\\ &\tau^{(1)}J_{kl}^* + J_{kl}\tau^{(1)} = 0, \qquad \tau^{(1)}J_{0k}^*(-t_0) - J_{0k}(t_0)\tau^{(1)} = 0, \end{aligned} \tag{D.5}$$

$$\begin{aligned} &\tau^{(2)}p_k - p_k\tau^{(2)} = 0, \qquad \tau^{(2)}\mathcal{H} + \mathcal{H}\tau^{(2)} = 0,\\ &\tau^{(2)}J_{kl} - J_{kl}\tau^{(2)} = 0, \qquad \tau^{(2)}J_{0k}(-t_0) + J_{0k}(t_0)\tau^{(2)} = 0, \end{aligned} \tag{D.6}$$

$$\begin{aligned} &\tau^{(3)}p_k^* + p_k\tau^{(3)} = 0, \qquad \tau^{(3)}\mathcal{H} + \mathcal{H}\tau^{(3)} = 0,\\ &\tau^{(3)}J_{kl}^* + J_{kl}\tau^{(3)} = 0, \qquad \tau^{(3)}J_{0k}^* + J_{0k}\tau^{(3)} = 0, \end{aligned} \tag{D.7}$$

$$\begin{aligned} &[\theta,p_n]_- = 0, \qquad \theta\mathcal{H}^*(-\overline{p}) + \mathcal{H}(\overline{p})\theta = 0,\\ &\theta J_{nl}^*(-\overline{x}) + J_{nl}(\overline{x})\theta = 0, \qquad \theta J_{0n}^*(-\overline{x}) - J_{0n}(\overline{x})\theta = 0, \qquad \theta\theta^*(-\overline{p}) = 1, \end{aligned} \tag{D.8}$$

$$\begin{aligned} &[\theta^{(n)},p_m]_\pm = 0, \qquad \theta^{(n)}\mathcal{H}^*(-p_k) + \mathcal{H}(p_k)\theta^{(n)} = 0,\\ &\theta^{(n)}J_{ml}^*(-x_n) \pm J_{ml}(x_n)\theta^{(n)} = 0, \qquad \theta^{(n)}J_{0m}^*(-x_n) \pm J_{0m}(x_n)\theta^{(n)} = 0,\\ &\theta^{(n)}(p_n)\left(\theta^{(n)}(-p_n)\right)^* = 1, \end{aligned} \tag{D.9}$$

where “$-$” is taken if $k = m$ or $k = l$.

With the help of definition (1.3.2)–(1.3.5), (D.1), (D.2) we find tee connection between the operators $r^{(k)}$, $\tau^{(k)}$, θ defined on the sets $\{\chi\}$ and $\{\Phi\}$

$$\begin{aligned} &\left\{r^{(n)}\right\}^\Phi = U\left\{r^{(n)}\right\}^\chi U^{-1}(\ldots,-p_n),\\ &\left\{r^{(n)}\right\}^\chi = U^{-1}\left\{r^{(n)}\right\}^\Phi U(\ldots,-p_n); \end{aligned} \tag{D.10}$$

$$\left\{\tau^{(1)}\right\}^{\Phi} = U\left\{\tau^{(1)}\right\}^{\chi} U^{-1*}, \qquad \left\{\tau^{(1)}\right\}^{\chi} = U^{-1}\left\{\tau^{(1)}\right\}^{\Phi} U^{*}; \tag{D.11}$$

$$\left\{\tau^{(2)}\right\}^{\Phi} = U\left\{\tau^{(2)}\right\}^{\chi} U^{-1}, \qquad \left\{\tau^{(2)}\right\}^{\chi} = U^{-1}\left\{\tau^{(2)}\right\}^{\Phi} U; \tag{D.12}$$

$$\left\{\tau^{(3)}\right\}^{\Phi} = U\left\{\tau^{(3)}\right\}^{\chi} U^{-1*}, \qquad \left\{\tau^{(3)}\right\}^{\chi} = U^{-1}\left\{\tau^{(3)}\right\}^{\Phi} U^{*}; \tag{D.13}$$

$$\{\theta\}^{\Phi} = U\{\theta\}^{\chi} U^{-1*}(-\overline{p}), \qquad \{\theta\}^{\chi} = U^{-1}\{\theta\}^{\Phi} U^{*}(-\overline{p}); \tag{D.14}$$

$$\begin{aligned} &\left\{\theta^{(n)}\right\}^{\Phi} = U\left\{\theta^{(n)}\right\}^{\chi} U^{-1*}(\ldots, -p_n), \\ &\left\{\theta^{(n)}\right\}^{\chi} = U^{-1}\left\{\theta^{(n)}\right\}^{\Phi} U^{*}(\ldots, -p_n); \end{aligned} \tag{D.15}$$

$$\{r\}^{\Phi} = U\{r\}^{\chi} U^{-1}(-\overline{p}), \qquad \{r\}^{\chi} = U^{-1}\{r\}^{\Phi} U(-\overline{p}), \tag{D.16}$$

where, for example, $\{r^{(n)}\}^{\Phi}$ $\left(\{r^{(n)}\}^{\chi}\right)$ denotes the operator $r^{(k)}$, defined on the set $\{\Phi\}$ $(\{\chi\})$.

From these relations it is seen that in a general case r, $r^{(k)}$, $\tau^{(k)}$, θ, $\theta^{(k)}$ are the operator functions dependent on p_l and σ_μ.

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