On the P- and T-non-invariant two-component equation for the neutrino

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The relativistic two-component equation describing the free motion of particles with zero mass and spin $\frac{1}{2}$, which is P- and T-non-invariant but C-invariant, is found. The representation of the Poincaré group for zero mass and discrete spin is constructed. The position operator for such a particle is defined.

1. Introduction

As is known, the Dirac equation for a particle with zero mass:

$$i\frac{\partial\Psi(t,\boldsymbol{x})}{\partial t} = \gamma_0\gamma_k p_k\Psi(t,\boldsymbol{x}), \qquad k = 1,2,3, \tag{1.1}$$

is invariant with respect to the space-time reflections. If one chooses for the Dirac matrices the Weyl representation eq. (1.1) decomposes into a system of two equations

$$i\frac{\partial\Psi_\pm(t,\boldsymbol{x})}{\partial t} = \pm\sigma_k p_k\Psi_\pm(t,\boldsymbol{x}), \tag{1.2}$$

where σ_k are the Pauli matrices and $\Psi_\pm$ is a two-component spinor. The Weyl equation (1.2) for Ψ_+ (or Ψ_-) is not invariant under space reflection P and charge conjugation C but is invariant under the CP- and T-operations.

Due to the fact that the space parity in the weak interactions is not conserved it is usually assumed that neutrino is described, not by the four-component eq. (1.1), but by a two-component one (1.2). Therefore, in papers [1] an hypothesis was put forward that the weak interactions are invariant with respect to the CP operation and consequently to the T operation, if the CPT theorem is valid.

In this paper the two-component equation for a particle with zero mass and spin $\frac{1}{2}$, which is non-invariant under the time reflection of T and the CP operation, is found.

2. Equation for a neutrino with "variable mass"

On the solutions of eq. (1.1) the generators of the Poincaré group $P(1,3)$ have the form

$$\begin{aligned} &P_0^\Psi = \mathcal{H}^\Psi = \gamma_0\gamma_k p_k, \qquad P_k^\Psi = p_k, \\ &J_{kl}^\Psi = x_k p_l - x_l p_k + S_{kl}, \qquad J_{0k}^\Psi = x_0 p_k - \frac{1}{2}\left[x_k, \mathcal{H}^\Psi\right]_+, \end{aligned} \tag{2.1}$$

$$\begin{aligned} &S_{\mu\nu} = \frac{1}{4}i(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu), \qquad S_{\mu 4} = \frac{1}{4}i(\gamma_\mu\gamma_4 - \gamma_4\gamma_\mu), \\ &S_{\mu 5} = \frac{1}{2}i\gamma_\mu, \qquad S_{45} = \frac{1}{2}i\gamma_4, \qquad \mu = 0,1,2,3, \end{aligned} \tag{2.1$'$}$$

where γ_μ and γ_4 are the Dirac matrices.

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If one performs a unitary transformation [2] over eq. (1.1)

$$U_1 = \exp\left\{\frac{1}{2} i\pi S_{53} e_3\right\}, \qquad e_3 = \frac{p_3}{|p_3|}, \qquad p_3 \neq 0, \tag{2.2}$$

or

$$U_1 = \frac{1}{\sqrt{2}}(1 + \gamma_3 e_3), \tag{2.2$'$}$$

eq. (1.1) has the form

$$i\frac{\partial \chi(t,\boldsymbol{x})}{\partial t} = (\gamma_0\gamma_a p_a + \gamma_0|p_3|)\chi(t,\boldsymbol{x}), \qquad a = 1,2, \tag{2.3}$$

$$\chi = U_1\Psi, \qquad \chi \equiv \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}, \tag{2.4}$$

where $\chi_\pm$ is a two-component spinor.

The Poincaré group generators $P(1,3)$ on $\{\chi\}$ being the solution of eq. (2.3) have the form

$$\begin{aligned} &P_0^\chi = \mathcal{H}^\chi = \gamma_0\gamma_a p_a + \gamma_0|p_3|, \qquad P_k^\chi = p_k, \\ &J_{ab}^\chi = x_a p_b - x_b p_a + S_{ab}, \qquad J_{a3}^\chi = x_a p_3 - x_3 p_a - e_3 S_{a3}\gamma_3, \\ &J_{0k}^\chi = x_0 p_k - \frac{1}{2}\left[x_k, \mathcal{H}^\chi\right]_+ . \end{aligned} \tag{2.5}$$

Choosing for the Dirac matrices somewhat unusual representation

$$\begin{aligned} &\gamma_0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \qquad \gamma_a = \begin{pmatrix} i\sigma_a & 0 \\ 0 & -i\sigma_a \end{pmatrix}, \\ &\gamma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \gamma_4 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \end{aligned} \tag{2.6}$$

eq. (2.3) decomposes into a system of two equations

$$\begin{aligned} &i\frac{\partial \chi_\pm(t,\boldsymbol{x})}{\partial t} = \left\{i\sigma_3\sigma_a p_a \pm \sigma_3|p_3|\right\}\chi_\pm(t,\boldsymbol{x}), \\ &\chi_\pm = Q_\pm\chi, \qquad Q_\pm = \frac{1}{2} \pm iS_{43} = \frac{1}{2}(1 \pm \gamma_3\gamma_4). \end{aligned} \tag{2.7}$$

Eq. (2.7) for the functions $\chi_+(t,\boldsymbol{x})$ (or $\chi_-(t,\boldsymbol{x})$) has quite the other properties relative to the discrete transformations than the Weyl equation (1.2).

We note the following:

(i) It is possible to arrive at eq. (2.3) (or (2.7)) in another way. If we “extract the square root” from the operator equation

$$\left(p_0^2 - p_a^2\right)\chi = p_3^2\chi,$$

we obtain eqs. (2.3) (or eqs. (2.7)).

(ii) The fact that the Dirac equations for zero and non-zero mass are invariant under the P-, T- and C-transformations is the consequence of the fact that they, besides being invariant with respect to the group $P(1,3)$, are invariant under the group $SU(2)\otimes SU(2)\sim O(4)$ (this question will be considered in detail in a following paper).

Eq. (2.3) coincides in form with a usual Dirac equation for zero mass if $|p_3|$ is considered as the mass of a particle. Therefore it is possible to say that eq. (2.3) describes a "flat neutrino" with variable mass $|p_3|$. Really the operator $|p_3|$ is the Casimir operator of the group $P(1,2)$ but not of the group $P(1,3)$.

Before passing to an investigation of the P-, T- and C-properties of eqs. (2.7) we shall construct the operator of the position in the space.

For eq. (2.3) the operator of the Foldy–Wouthuysen type has the form

$$U_2=\exp\left\{\frac{S_{5a}p_a}{\sqrt{p_a^2}}\operatorname{arctg}\frac{\sqrt{p_a^2}}{|p_3|}\right\}. \tag{2.8}$$

If the matrix S_{5a} have the form of (2.1′) then

$$U_2=\frac{E+|p_3|+\gamma_a p_a}{\{2E(E+|p_3|)\}^{1/2}},\qquad E=\sqrt{p_1^2+p_2^2+p_3^2}. \tag{2.9}$$

Eq. (2.3) after the transformation (2.9) transfers into

$$i\frac{\partial\Phi(t,\boldsymbol{x})}{\partial t}=\mathcal{H}^{\Phi}(t,\boldsymbol{x})=\gamma_0 E\Phi(t,\boldsymbol{x}),\qquad \Phi(t,\boldsymbol{x})=U_2\chi(t,\boldsymbol{x}). \tag{2.10}$$

The generators of the group $P(1,3)$ on $\{\Phi\}$ have the form

$$\begin{gathered}P_0^{\Phi}=\mathcal{H}^{\Phi}=\gamma_0 E,\qquad P_k^{\Phi}=p_k,\\ J_{ab}^{\Phi}=x_ap_b-x_bp_a+S_{ab},\qquad J_{a3}^{\Phi}=x_ap_3-x_3p_a-e_3\frac{S_{ab}p_b}{E+|p_3|},\\ J_{0a}^{\Phi}=x_0p_a-\frac{1}{2}\left[x_a,\mathcal{H}^{\Phi}\right]_+-\gamma_0\frac{S_{ab}p_b}{E+|p_3|},\qquad J_{03}^{\Phi}=x_0p_3-\frac{1}{2}\left[x_3,\mathcal{H}^{\Phi}\right]_+.\end{gathered} \tag{2.11}$$

It must be noted that the operators (2.11), as it can be immediately verified, satisfy the algebra $P(1,3)$ commutation relations not depending on the matrices S_{ab} explicit form, i.e. the operators (2.11), if γ_0 is substituted for 1 (or -1) and realize irreducibly the algebra $P(1,3)$ representation which is characterized by zero mass and discrete spin. The representation (2.11) differs from the corresponding Shirokov [3], Lomont–Moses [4] ones but is certainly equivalent to them.

The position operator on a set $\{\chi\}$ looks as

$$\begin{gathered}X_a^{\chi}=U_2^{-1}x_aU_2=x_a-\frac{S_{5a}}{E}+\frac{S_{5c}p_cp_a}{E^2(E+|p_3|)}+\frac{S_{ac}p_c}{E(E+|p_3|)},\\ X_3^{\chi}=U_2^{-1}x_3U_2=x_3+e_3\frac{S_{5c}p_c}{E^2},\qquad S_{5c}=-\frac{1}{2}i\gamma_c.\end{gathered} \tag{2.12}$$

The position operator on a set of solution $\{\Psi\}$ of eq. (1.1) looks as follows

$$\begin{gathered}X_a^{\Psi}=U_1^{-1}X_a^{\chi}U_1=x_a+e_3\frac{\gamma_3S_{5a}}{E}-e_3\frac{\gamma_3S_{5c}p_cp_a}{E^2(E+|p_3|)}+\frac{S_{ac}p_c}{E(E+|p_3|)},\\ X_3^{\Psi}=U_1^{-1}X_3^{\chi}U_1=x_3-\frac{\gamma_3S_{5c}p_c}{E^2}.\end{gathered} \tag{2.13}$$

(iii) If one performs a transformation on eq. (1.1)

$$\widetilde{U}_1 = \frac{1}{\sqrt{2}}(1+\gamma_3) \tag{2.14}$$

and then a transformation

$$\widetilde{U}_2 = \frac{E+p_3+\gamma_a p_a}{\{2E(E+|p_3|)\}^{1/2}}, \tag{2.15}$$

it will transform into the equation

$$i\frac{\partial\widetilde{\Phi}(t,\boldsymbol{x})}{\partial t} = \gamma_0\widetilde{\Phi}(t,\boldsymbol{x}), \qquad \widetilde{\Phi} = \widetilde{U}_2\widetilde{U}_1\Psi. \tag{2.16}$$

The generators of the group $P(1,3)$ on $\{\widetilde{\Phi}\}$ coincide with (2.11) where the substitution was made $e_3 \to 1$, $|p_3| \to p_3$.

3. P-, T- and C-properties of two-component equation

Here we shall study the properties of one of the two-component eqs. (2.7)[1]

$$i\frac{\partial\chi(t,\boldsymbol{x})}{\partial t} = (i\sigma_3\sigma_a p_a + \sigma_3|p_3|)\chi(t,\boldsymbol{x}), \tag{3.1}$$

under the discrete transformations.

We shall denote through $P^{(k)}$ $(k=1,2,3)$ the space inversion operator of one axis which is determined as

$$P^{(1)}\chi(t,x_1,x_2,x_3) = r^{(1)}\chi(t,-x_1,x_2,x_3). \tag{3.2}$$

Analogously $P^{(2)}$ and $P^{(3)}$ are determined.

As is well known, two non-equivalent definitions of the time-reflection operator exist. According to Wigner the time-inversion operator is

$$T^{(1)}\chi(t,\boldsymbol{x}) = \tau^{(1)}\chi^*(-t,\boldsymbol{x}). \tag{3.3}$$

According to Pauli it is:

$$T^{(2)}\chi(t,\boldsymbol{x}) = \tau^{(2)}\chi(-t,\boldsymbol{x}). \tag{3.4}$$

The operator of the charge conjugation can be defined as the product of the operators $T^{(1)}$, $T^{(2)}$ or as

$$C\chi(t,\boldsymbol{x}) = \tau^{(3)}\chi^*(t,\boldsymbol{x}), \tag{3.5}$$

where $r^{(k)}$, $\tau^{(k)}$ are the 2×2 matrices.

The operators P, T, C with the group $P(1,3)$ generators satisfy the usual commutation relations.

The generators of the group $P(1,3)$ on the solutions $\{\chi\}$ of eq. (3.1) have the form of eq. (2.5) where

$$\begin{gathered}\mathcal{H}^{\chi} \to i\sigma_3\sigma_a p_a + \sigma_3|p_3| = -\sigma_2 p_1 + \sigma_2 p_2 + \sigma_3|p_3|,\\ S_{ab} \to \frac{1}{4}i(\sigma_b\sigma_a - \sigma_a\sigma_b), \qquad S_{a3}\gamma_3 \to -\frac{1}{2}\sigma_a,\end{gathered} \tag{3.6}$$

and the matrix γ_0 is substituted for the matrix σ_3.

[1]In what follows, under χ we shall understand the two-component spinor χ_+.

Using the definitions (3.2)–(3.5) it is not difficult to verify that eq. (3.1) is $P^{(3)}$-, C-invariant but $P^{(1)}$-, $P^{(2)}$-, $T^{(1)}$-, $T^{(2)}$-non-invariant.

Thus, eq. (3.1) is $P^{(3)}C$-, $P^{(1)}P^{(2)}P^{(3)}C$- and $P^{(a)}CT^{(a)}$-invariant but $P^{(3)}CT^{(a)}$- and $P^{(a)}C$-non-invariant.

We note the following:

(i) The result obtained is a consequence of the fact that the projection operators $Q_\pm$, with the operators of the discrete transformations, satisfy the following relations

$$\begin{aligned} &P^{(a)}Q_\pm = Q_\mp P^{(a)}, \qquad T^{(a)}Q_\pm = Q_\mp T^{(a)},\\ &P^{(3)}Q_\pm = Q_\pm P^{(3)}, \qquad CQ_\pm = Q_\pm C. \end{aligned} \tag{3.7}$$

(ii) The two-component equations for the functions χ_+ and χ_- are equivalent to the four-component one (2.3) with the subsidiary relativistic-invariant conditions

$$Q_-\chi = \left(\frac{1}{2} - iS_{43}\right)\chi = \frac{1}{2}(1-\gamma_3\gamma_4)\chi = 0, \tag{3.8}$$

$$Q_+\chi = \left(\frac{1}{2} + iS_{43}\right)\chi = \frac{1}{2}(1+\gamma_3\gamma_4)\chi = 0, \tag{3.9}$$

respectively. For eq. (1.1) these conditions look like

$$\left(\frac{1}{2} + ie_3S_{45}\right)\Psi = \frac{1}{2}(1-e_3\gamma_4)\Psi = 0, \tag{3.8$'$}$$

$$\left(\frac{1}{2} - ie_3S_{45}\right)\Psi = \frac{1}{2}(1+e_3\gamma_4)\Psi = 0. \tag{3.9$'$}$$

Eq. (1.1) with the subsidiary conditions (3.8′) and (3.9′) can be joined and can be written in the form of two $P^{(a)}$- and $T^{(b)}$-non-invariant but $P^{(3)}$- and C-invariant equations

$$\{\gamma_\mu p^\mu + \varkappa(1+e_3\gamma_4)\}\,\Psi_1(t,\boldsymbol{x}) = 0, \qquad \{\gamma_\mu p^\mu + \varkappa(1-e_3\gamma_4)\}\,\Psi_2(t,\boldsymbol{x}) = 0,$$

where $\varkappa$ is some constant value. The four-component equations for the neutrino, which are the union of eq. (1.1) and the usual subsidiary condition, were recently considered in ref. [6]. These equations, as well as the Weyl equations (1.2), are P- and C-non-invariant but $T^{(1)}$-invariant.

The unitary operator of type U_2 for the two-component eq. (3.1) has the form

$$V_1 = \exp\left\{ i\frac{S_ap_a}{\sqrt{p_a^2}} \operatorname{arctg}\frac{\sqrt{p_a^2}}{|p_3|}\right\}, \qquad S_k = \frac{1}{2}\varepsilon_{kln}S_{ln}, \tag{3.10}$$

or

$$V_1 = \frac{E + |p_3| + i\sigma_ap_a}{\{2E(E+|p_3|)\}^{1/2}}. \tag{3.11}$$

The position operator on the set of solutions $\{\chi\}$ of eqs. (3.1) looks as follows

$$\begin{aligned} &X_a^{\chi_+} = V_1^{-1}x_aV_1 = x_a - \frac{\sigma_a}{2E} + \frac{\sigma_cp_cp_a}{2E^2(E+|p_3|)} - i\frac{(\sigma_a\sigma_c - \sigma_c\sigma_a)p_c}{4E(E+|p_3|)},\\ &X_3^{\chi_+} = V_1^{-1}x_3V_1 = x_3 + e_3\frac{\sigma_bp_b}{2E^2}. \end{aligned} \tag{3.12}$$

To complete our treatment, we find the position operator for the neutrino which is described by the Weyl equation (1.2), for example for the function Ψ_+. This equation under a transformation

$$V = \frac{E + |p_3| + i\sigma_k\xi_k}{2\sqrt{\xi_k p_k}}, \tag{3.13}$$

where the vector ξ has the following components

$$\xi_k \equiv \{p_1 - p_2e_3, p_2 + e_3p_1, e_3(E + |p_3|)\},$$

takes a canonical form

$$i\frac{\partial\Phi_+(t,\boldsymbol{x})}{\partial t} = \sigma_3' E\Phi_+(t,\boldsymbol{x}), \qquad \sigma_3' = \sigma_3 e_3, \qquad \Phi_+(t,\boldsymbol{x}) = V\Psi_+(t,\boldsymbol{x}). \tag{3.14}$$

The position operator for a neutrino which is described by the Weyl equation (1.2) (for Ψ_+) looks like

$$X_a^W = V^{-1}x_aV = x_a + ie_3\frac{\sigma_3\sigma_a}{2E} - i\frac{e_3\sigma_3\sigma_cp_cp_a}{2E^2(E+|p_3|)} - i\frac{(\sigma_a\sigma_c - \sigma_c\sigma_a)p_c}{4E(E+|p_3|)},$$

$$X_3^W = V^{-1}x_3V = x_3 - i\frac{\sigma_3\sigma_bp_b}{2E^2}.$$

The other definitions of the operators X_k and V for the neutrino are given in ref. [5].

(iii) From Dirac eq. (1.1) one can, generally speaking, obtain three types of non-equivalent two-component equations. On the set of solutions of eq. (1.1) a direct sum of four irreducible representations $D^\varepsilon(s)$ of the group $P(1,3)$

$$D^{\varepsilon=1}\left(s=\frac{1}{2}\right) \oplus D^{\varepsilon=-1}\left(s=-\frac{1}{2}\right) \oplus D^{\varepsilon=1}\left(s=-\frac{1}{2}\right) \oplus D^{\varepsilon=-1}\left(s=\frac{1}{2}\right) \tag{3.15}$$

is realized, where ε is an energy sign, s is a helicity. Hence it follows that there exist three types of two-component equations on the set of which the following representation of the group $P(1,3)$

$$D^{\varepsilon=1}\left(s=\frac{1}{2}\right) \oplus D^{\varepsilon=-1}\left(s=-\frac{1}{2}\right),$$

or

$$D^{\varepsilon=1}\left(s=-\frac{1}{2}\right) \oplus D^{\varepsilon=-1}\left(s=\frac{1}{2}\right), \quad D^{\varepsilon=1}\left(s=\frac{1}{2}\right) \oplus D^{\varepsilon=-1}\left(s=\frac{1}{2}\right), \tag{3.16}$$

or

$$D^{\varepsilon=1}\left(s=-\frac{1}{2}\right) \oplus D^{\varepsilon=-1}\left(s=-\frac{1}{2}\right), \quad D^{\varepsilon=1}\left(s=\frac{1}{2}\right) \oplus D^{\varepsilon=-1}\left(s=-\frac{1}{2}\right), \tag{3.17}$$

or

$$D^{\varepsilon=1}\left(s=\frac{1}{2}\right) \oplus D^{\varepsilon=-1}\left(s=-\frac{1}{2}\right) \tag{3.18}$$

are realized. If on the solutions of two-component equation there realizes the representation (3.16) then this equation will be $T^{(1)}$-invariant but C-, P-, $T^{(2)}$-non-invariant, if the representation (3.17) does then it will be $T^{(1)}$-, $T^{(2)}$-, C-invariant but P-non-invariant, and if the representation (3.18) it will be $T^{(1)}$-, P-invariant but C-, $T^{(2)}$-non-invariant. This problem will be considered in more detail in another paper.

4. Equation for a flat neutrino

The motion group in the Minkovski three-space is the $P(1,2)$ group of rotations and translations conserving the form

$$x^2 = x_0^2 - x_1^2 - x_2^2.$$

In this case the simplest spinor equation is

$$i\frac{\partial\chi_\pm(t,x_1,x_2)}{\partial t} = (i\sigma_3\sigma_a p_a \pm \sigma_3 m)\chi_\pm(t,x_1,x_2), \tag{4.1}$$

$\chi_\pm$ is the two-component spinor and m is the eigenvalue of the operator $\sqrt{P_\mu^2}$.

Eq. (4.1) for χ_+ (or χ_-) like eq. (3.1) is invariant under the $P^{(1)}P^{(2)}$- and C-operations but non-invariant under the $P^{(a)}$ and $T^{(b)}$-operations.

Thus, eq. (4.1) for the wave function χ_+ (or χ_-) is $P^{(1)}P^{(2)}C$-, $T^{(a)}P^{(b)}$- and $P^{(a)}CT^{(b)}$-invariant but $P^{(a)}C$- and $CT^{(a)}$-non-invariant.

It should be noted that the equation being the “direct sum” of the equation for $\chi_+(t,x_1,x_2)$ and $\chi_-(t,x_1,x_2)$ is invariant under the P-, T- and C-transformations [7].

Finally, we quote one more example of the P- and C-non-invariant equation which is invariant with respect to the inhomogeneous De Sitter group. Such is the Dirac equation:

$$i\frac{\partial\Psi(t,\boldsymbol{x},x_4)}{\partial t} = (\gamma_0\gamma_k p_k + \gamma_0\varkappa)\Psi(t,\boldsymbol{x},x_4), \qquad k = 1,2,3,4. \tag{4.2}$$

This equation as is shown in refs. [2, 7] is $T^{(1)}$-, $T^{(2)}C$-invariant but $P^{(k)}$-, $T^{(2)}$- and C-non-invariant.

All the results obtained in this paper can be generalized for the arbitrary spin s case, if one uses for this the purpose the equation (ref. [2]):

$$i\frac{\partial\Psi(t,\boldsymbol{x})}{\partial t} = \lambda S_{0l}p_l\Psi(t,\boldsymbol{x}), \qquad l = 1,2,3, \tag{4.3}$$

where λ is some fixed parameter (for the Dirac equation $\lambda = -2i$), and $S_{\mu\nu}$, $S_{\mu4}$, S_{45} are the matrices (not 4×4 ones) realizing the algebra $O(1,5)$ representation.

(i) If we transform the usual Dirac equation describing the motion of the non-zero mass particle m with a spin $\frac{1}{2}$ as

$$V_2 = \frac{\gamma_3 p_3 + q_3 + m}{\{2q_3(q_3+m)\}^{1/2}}, \qquad q_3 \equiv \sqrt{p_3^2+m^2}, \tag{4.4}$$

it has the form

$$i\frac{\partial\Psi'(t,\boldsymbol{x})}{\partial t} = H'\Psi'(t,\boldsymbol{x}), \tag{4.5}$$

$$H' = \gamma_0\gamma_a p_a + \gamma_0 q_3, \qquad \Psi' = V_2\Psi, \qquad a = 1,2. \tag{4.6}$$

Choosing the representation (2.6) for the Dirac matrices eq. (4.5) is decomposed into the set of two independent equations

$$i\frac{\partial \Psi'_{+}(t,\boldsymbol{x})}{\partial t} = (-\sigma_2 p_1 + \sigma_1 p_2 + \sigma_3 q_3)\Psi'_{+}(t,\boldsymbol{x}), \tag{4.7}$$

$$i\frac{\partial \Psi'_{-}(t,\boldsymbol{x})}{\partial t} = (-\sigma_2 p_1 + \sigma_1 p_2 - \sigma_3 q_3)\Psi'_{-}(t,\boldsymbol{x}), \tag{4.8}$$

where Ψ'_{+} and Ψ'_{-} are two-component wave functions.

Eq. (4.7) or (4.8) describes a free motion of spinless particle and antiparticle with the mass m. Thus besides of the Klein–Gordon equation there exist the other equations of the type (4.7) and (4.8) which are also relativistically invariant and describe the spinless particle motion with non-zero mass. The two-component eq. (4.7) is equivalent to the four-component Dirac equation

$$i\frac{\partial \Psi(t,\boldsymbol{x})}{\partial t} = (\gamma_0\gamma_k p_k + \gamma_0 m)\Psi(t,\boldsymbol{x}), \qquad k = 1,2,3 \tag{4.9}$$

with such subsidiary condition

$$\left(1 - \frac{\gamma_3\gamma_4 m + \gamma_4 p_3}{q_3}\right)\Psi(t,\boldsymbol{x}) = 0. \tag{4.10}$$

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