On representations of the inhomogeneous de Sitter group and on equations of the Schrödinger–Foldy type

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This paper is a continuation and elaboration of our works [1] where some approach to the variable-mass problem were proposed. Here we have found a concret realization of irreducible representations of the inhomogeneous group $P(1,n)$ — the group of translations and rotations in $(1+n)$-dimensional Minkowski space in two classes (when $P_0^2-P_k^2>0$ and $P_0^2-P_k^2<0$). All the $P(1,n)$-invariant equations of the Schrödinger–Foldy type are written down. Some questions of a physical interpretation of the quantum, mechanical scheme based on the inhomogeneous de Sitter group $P(1,n)$ are discussed.

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Данная работа является продолжением и развитием работ [1], где был предложен определенный подход к проблеме переменной массы. Здесь построена конкретная реализация неприводимых представлений неоднородной группы $P(1,n)$ вращении и трансляций в $(1+n)$-мерном пространстве Минковского в двух классах (когда $P_0^2-P_k^2>0$ и $P_0^2-P_k^2<0$). Выписаны $P(1,n)$-инвариантные уравнения типа Шредингера–Фолди. Рассмотрены некоторые вопросы физической интерпретации квантовомеханической схемы, основанной на неоднородной группе де Ситтера $P(1,4)$.

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1. Introduction

Recall here the initial points of our approach to the variable mass problem proposed in ref. [1]:

A. The square of variable mass operator is defined as an independent dynamical variables

$$M^2\equiv\varkappa^2+P_4^2, \tag{1}$$

where $\varkappa$ is a fixed parameter and P_4 is an operator lice the components of three-momentum $\vec{P}$, which commutes with all the generators of the algebra $P(1,3)$ of the Poincaré group.

B. The relation between the energy P_0, three-momentum $\vec{P}$ and variable-mass M of a physical system is remained to be conventional (here everywhere $\hbar=c=1$):

$$P_0^2=\vec{P}^{\,2}+M^2\equiv P_k^2+\varkappa^2,\qquad k=1,2,3,4. \tag{2}$$

C. The spaced $p\equiv(p_0,p_1,\ldots,p_4)$ and $x\equiv(x_0,x_1,\ldots,x_4)$ are assumed to be plane and reciprocally conjugated. It follows then from A, B and C that the generalized

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relativistic group symmetry is the inhomogeneous de Sitter group[1] $P(1,4)$ — the group of translations and rotations in five-dimensional Minkowski space. This group is a minimal extention of the conventional group of relativistic symmetry — the Poincaré group $P(1,3)$.

In this paper we shall present a further studying of the approach proposed in ref. [1]. In particular, the main assertions which were briefly formulated in ref. [1], are generalized here and their detail proofs are given. In Section 2 a concrete realization of irreducible representations for the generators P_μ, $J_{\mu\nu}$ of the algebra $P(1,n)$ with arbitrary n carried out, which made it possible to give a proof of the $P(1,n)$-invariance of the Schrödinger–Foldy type equations written flown in ref. [1] for $n=4$. Some questions of a physical interpretation of quantum mechanical scheme based on the group $P(1,4)$ are considered in Section 3.

2. Realizations of the algebra representations

For the sake of generality all the considerations are made here not for the de Sitter group $P(1,4)$ but for the group $P(1,n)$ of translations and rotations in dimensional Minkowski space, which leaves unchanged, the form

$$x^2 \equiv x_0^2 - x_1^2 - \cdots - x_n^2 \equiv x_0^2 - x_k^2 \equiv x_\mu^2, \qquad \mu = 0,1,2,\ldots,n; \qquad k = 1,2,\ldots,n, \tag{3}$$

where x_μ are differences of point coordinates of this space.

Commutation relations for the generators P_μ, $J_{\mu\nu}$ of the algebra $P(1,n)$ are choosen in the form

$$[P_\mu, P_\nu] = 0, \qquad -i\,[P_\mu, J_{\nu\sigma}] = g_{\mu\nu}P_\sigma - g_{\mu\sigma}P_\nu, \tag{4a}$$

$$-i\,[J_{\mu\nu}, J_{\rho\sigma}] = g_{\mu\sigma}J_{\nu\rho} + g_{\nu\rho}J_{\mu\sigma} - g_{\mu\rho}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\rho}, \tag{4b}$$

where $g_{00}=1$, $-g_{kl}=\delta_{kl}$, P_μ is Kroneker symbol, P_μ are operators of infinitesimal displacements and $J_{\mu\nu}$ are operators infinitesimal rotations in planes $(\mu\nu)$.

Authors of refs. [2–5] have studed all the irreducible representations of the Poincaré group $P(1,3)$ and have found the concrete realization for the generators of its algebra. Their methods we generalize here for the case of group $P(1,n)$. But all the treatments are carried out in more general and compact form then it was done even for the case of $P(1,3)$.

For representations of the class I ($P^2 \equiv P_0^2 - P_k^2 > 0$) when the group $O(n)$ of rotations in a n-dimensional Euclidean space is the little group of the group $P(1,n)$, the generators P_μ, $J_{\mu\nu}$ are of the form

$$P = p \equiv (p_0, p_1, \ldots, p_n) \equiv (p_0, p_k), \qquad J_{kl} = x_{[k}p_{l]} + S_{kl}, \qquad J_{0k} = x_{[0}p_{k]} - \frac{S_{kl}p_l}{\sqrt{p^2+p_k^2}+\sqrt{p^2}}, \tag{5}$$

where

$$P^2 \equiv p^2 \equiv p_0^2 - p_k^2 > 0, \qquad x_{[\mu}p_{\nu]} \equiv x_\mu p_\nu - x_\nu p_\mu,$$

[1] The algebras and groups connected with them are designated here with the same symbols.

operators x_μ, p_μ are defined by relations

$$[x_\mu, p_\mu] = -ig_{\mu\nu}, \qquad [x_\mu, x_\nu] = [p_\mu, p_\nu] = 0, \tag{6}$$

and S_{kl} are matrices realizing irreducible representations $D(s, t, \ldots)$ of the algebra $O(n)$ which have been completely studied in ref. [6] (here the numbers $s, t, \ldots$ are numbers which identify a correspondence irroducible representations of the algebra $O(n)$). Using (6) and relations for the generators S_{kl} (which are not written down here), one can immediately verify that (5) actually satisfy the relations (4). Since in this class the little group of the group $P(1, n)$ coincides with the compact group $O(n)$, all the irreducible representations of the group $P(1, n)$ are here unitary and finite-dimensional (concerning a set of "spin" indexes $s_3, t_3, \ldots$).

A concrete form of the operators P_μ, $J_{\mu\nu}$ which are defined by Eqs. (5), depends on a choice of concrete form of matrices S_{kl} and operators x_μ, p_μ which are defined by relations (6). The concrete form of the operators x_μ, p_μ and S_{kl} depends on what of operators, constituting a complete set of commuting dynamical variables are operators of multiplicationd ("diagonal operators"). The sets $(P_0, P_1, \ldots, P_n, S_3, T_3, \ldots)$ or $(x_0, x_1, \ldots, x_n, S_3, T_3, \ldots)$ are examples of such a complete sets where $S_3, T_3, \ldots$ are all the independent commuting generators of the algebra $O(n)$. In the general case a complete set of dynamical variables is constructed from the corresponding number of commuting combinations of operators x_μ, p_μ and S_{kl}. Different concrete forms of operators P_μ, $J_{\mu\nu}$ which are defined by the choice of other complete set as diagonal, are connected by unitary transformations. The form (5) for the generators is the most general in the sense that it is not bound to the choice of concrete complete set as diagonal.

A few words about a apace of vectors Ψ, in which the operators (5) are defined. It is an Hilbert space of vector-functions depending on the eigenvalues of operators of a diagonal complete sel. For instance, in the x-representation where the operators x_μ are diagonal (i.e., are operators of multiplication) and, as it follows from relations (6), $p_\mu = ig_{\mu\nu}\partial_\nu$, $\partial_\nu \equiv \partial/\partial x_\nu$ the operators (5) are defined in the Hilbert apace of the vector-function $\Psi = \Psi(x) = \Psi(x_0, x_1, \ldots, x_n)$ of $(1+n)$ independent variables x_μ. The components of a vector Ψ are functions not only of $x_0, x_1, \ldots, x_n$ but also of auxiliary variables $s_3, t_3, \ldots$, i.e., are functions $\Psi(x_0, x_1, \ldots, x_n, s_3, t_3, \ldots)$, where $s_3, t_3, \ldots$ are eigenvalues of "spin" operators $S_3, T_3, \ldots$ and, as it is known, take discrete values. In p-representation where p_μ are operators of multiplication and, according to (6), $x_\mu = ig_{\mu\nu}\partial/\partial p_\nu$ vector-functions are $\Psi = \widetilde{\Psi}(p) \equiv \widetilde{\Psi}(p_0, p_1, \ldots, p_n)$ and their components are $\widetilde{\Psi}(p_0, p_1, \ldots, p_n, s_3, t_3, \ldots)$. The scalar product of vectors Ψ is defined as

$$\begin{gathered}
(\Psi, \Psi') \equiv \int d^{1+n}x \; \Psi^+(x)\Psi'(x) = \\
= \int d^{1+n}x \sum_{s_3, t_3, \ldots} \Psi^*(x, s_3, t_3, \ldots)\Psi'(x, s_3, t_3, \ldots) = \\
= \int d^{1+n}p \; \widetilde{\Psi}^+(p)\widetilde{\Psi}'(p) = \int d^{1+n}p \sum_{s_3, t_3, \ldots} \widetilde{\Psi}^*(p, s_3, t_3, \ldots)\widetilde{\Psi}'(p, s_3, t_3, \ldots),
\end{gathered} \tag{7}$$

where $d^{1+n}x = dx_0 dx_1 \ldots dx_n$, Ψ and $\widetilde{\Psi}$ being connected by Fourier-transformations.

For representations of the class III ($P^2 = P_0^2 - P_k^2 < 0$) when the little group of the group $P(1,n)$ is already uncompact group $O(1,n-1)$ of rotations in $1+(n-1)$-dimensional pseudo-Euklidean space, the generators P_μ, $J_{\mu\nu}$ are of the form

$$
\begin{gathered}
P = p \equiv (p_0, p_k) = (p_0, p_a, p_n), \\
J_{ab} = x_{[a}p_{b]} + S_{ab}, \qquad J_{an} = x_{[a}p_{n]} - \frac{S_{ab}p_b - S_{a0}p_0}{\sqrt{-p^2 - p_a^2 + p_0^2} + \sqrt{-p^2}}, \\
J_{0a} = x_{[0}p_{a]} + S_{0a}, \qquad J_{0n} = x_{[0}p_{n]} - \frac{S_{0a}p_a}{\sqrt{-p^2 - p_a^2 + p_0^2} + \sqrt{-p^2}},
\end{gathered}
\tag{8}
$$

where $a, b = 1, 2, \ldots, n-1$ the operators x_μ, p_μ are defined by relations (6) as before and the operators (S_{0a}, S_{ab}) are generators of the algebra $O(1,n-1)$ in corresponding irreducible representations which have been well studied by Gelfand and Grayev [7].

Components of vector-functions, in the space of which the operators (8) are defined, are the functions of variables $s_3, t_3, \ldots$ (besides of variables x_μ or p_μ, of course) which are the eigenvalues of the corresponding independent commute generators of the algebra $O(1,n-1)$. In contrast to the case I, in this case the variables $s_3, t_3, \ldots$ may take both discrete and continual valuea. Remind (see ref. [7]) that the group $O(1,n-1)$ has both unitary and nonunitary representations, all the unitary representations being infinite-dimensional (in the last case the "spin" variables $s_3, t_3, \ldots$ take continual values). In accordance with this, among the representations of the group $P(1,n)$ in the class III there will be both unitary (only infinite-dimensional) and nonunitary (finite- and infinite-dimensional) irreducible representations.

Now we shall give here a recipe of constructing the representations of the class III from those of the class I.

Note first that if operators P, J realize representation of the algebra $P(1,n)$, then operators $\widetilde{P}$, $\widetilde{J}$ being defined by means of

$$(P_0, P_a, P_n) = (-i\widetilde{P}_n, \widetilde{P}_a, i\widetilde{P}_0), \tag{9a}$$

$$\left(\begin{array}{c|c} J_{0a} & J_{0n} \\ \hline J_{ab} & J_{an} \end{array}\right) = \left(\begin{array}{c|c} -i\widetilde{J}_{na} & \widetilde{J}_{n0} \\ \hline \widetilde{J}_{ab} & i\widetilde{J}_{a0} \end{array}\right), \tag{9b}$$

realize a representation of the algebra $P(1,n)$ too. To proof this assertion, it is enough to verify that from the commutation relations (4) for P, J and from definitions (9), it follows that the operators $\widetilde{P}$, $\widetilde{J}$satisfy the commutation relations (4) too.

Define, further, the operators $\widetilde{x}$, $\widetilde{p}$ and $\widetilde{s}$ by means of the following relations

$$(x_0, x_a, x_n) = (-i\widetilde{x}_n, \widetilde{x}_a, i\widetilde{x}_0), \tag{10a}$$

from

$$(p_0, p_a, p_n) = (-i\widetilde{p}_n, \widetilde{p}_a, i\widetilde{p}_0), \tag{10b}$$

$$(S_{ab}, S_{an}) = (\widetilde{S}_{ab}, i\widetilde{S}_{a0}). \tag{10c}$$

From (6) and (10a), (10b) it follows that operators $\widetilde{x}$, $\widetilde{p}$ satisfy the relations (6) too, whereas the operators $(\widetilde{S}_{0a}, \widetilde{S}_{ab})$ defined by Eqs. (10c) satisfy the commutation relations for the generators of the algebra $O(1,n-1)$, as soon as the S_{kl} satisfy the commutation relations for the generators of the algebra $O(n)$.

Rewrite now the operators (3) in the form

$$P=(p_0,p_a,p_n),\qquad J_{ab}=x_{[a}p_{b]}+S_{ab},\qquad J_{0n}=x_{[a}p_{n]}+S_{an},$$
$$J_{0a}=x_{[0}p_{a]}-\frac{S_{ab}p_b+S_{an}p_n}{p_0+\sqrt{p_0^2-p_a^2-p_n^2}},\qquad J_{0n}=x_{[0}p_{n]}-\frac{S_{na}p_a}{\sqrt{p_0^2-p_a^2-p_n^2}}.\tag{5'}$$

Using the definitions (9) and (10) corresponding to the schematic substitution "$i0\Leftrightarrow n$" when the operators with the symbol 2 "$\sim$" are getting from the operators without the symbol "$\sim$", we obtain from (5):

$$\begin{aligned}
&P\equiv(\widetilde{p}_0,\widetilde{p}_a,\widetilde{p}_n)=(-ip_n,p_a,ip_0),\\
&\widetilde{J}_{ab}=\widetilde{x}_{[a}\widetilde{p}_{b]}+\widetilde{S}_{ab},\qquad i\widetilde{J}_{a0}=\widetilde{x}_{[a}\widetilde{p}_{0]}+i\widetilde{S}_{a0},\\
&-i\widetilde{J}_{na}=-i\widetilde{x}_{[n}\widetilde{p}_{a]}-\frac{\widetilde{S}_{ab}\widetilde{p}_b-\widetilde{S}_{a0}\widetilde{p}_0}{-i\widetilde{p}_n+\sqrt{-\widetilde{p}_n^2-\widetilde{p}_a^2+\widetilde{p}_0^2}},\\
&\widetilde{J}_{n0}=\widetilde{x}_{[n}\widetilde{p}_{0]}-\frac{i\widetilde{S}_{0a}\widetilde{p}_a}{-i\widetilde{p}_n+\sqrt{-\widetilde{p}_n^2-\widetilde{p}_a^2+\widetilde{p}_0^2}}.
\end{aligned}\tag{8'}$$

By virtue of Eqs. (10a), we have $\widetilde{P}^2=-P^2<0$, so that (8′) realizes a representations of the class III as soon as (5) realizes a representation of the class I. Omitting in (8′) the symbol "$\sim$", we obtain (8).

Since all the representations of the class I are finite-dimensional, such a recipe allows to obtain only finite-dimensional representations of the class III (i.e., not all the representations of this class). If, however, getting (8) from (8′), the operators $(\widetilde{S}_{0a},\widetilde{S}_{ab})$ will be substituted by operators (S_{0a},S_{ab}) realizing an infinite-dimensional representation of the algebra $O(1,n-1)$, we obtain the corresponding infinite-dimensional representation of the algebra $P(1,n)$. Thus it is shown that the formula (8) defines all the representations of the class III of the algebra $P(1,n)$.

The representations of the class II ($P^2=0$, $P\neq0$) requires a special treatment. However, in the case when one of invariants of the algebra $P(1,n)$, namely, the invariant

$$W\equiv\frac{1}{2}P_\mu J_{\nu\alpha}^2-P_\mu P_\nu J_{\mu\sigma}J_{\nu\sigma},$$

vanishes, the representations of the class II are particular cases of representations of the class I, and formulaes for the generators P_μ, $J_{\mu\nu}$ are obtained from (5) by the limit procedure $p^2\to0$. The detailed discussion of all the representations of the class II is not given here. As to the class IV ($P=0$), in this case the group $P(1,n)$ reduces to the group $O(1,n)$, therefore the problem of classification and realization of representations of the algebra $P(1,n)$ reduces to the problem of classification and realization of representations of the algebra $O(1,n)$ already studied in ref. [7].

Let us discuss now a role of the variable x_0. If we mean possibility to Interpret vectors Ψ constituting the representation space for the group $P(1,n)$, as state vectors of the physical system (see below Section 3), we must interpret x_0 as the time, i.e., as a parameter which is not an operator and which therefore is not to be included in a complete set dynamical variables. It means that, for instance, in the x-representation a vector-function Ψ is a function of only n dynamical variables: $\Psi = \Psi(x_1, \ldots, x_n)$. If the condition C of the Section 1 is not to be violated, the number of independent dynamical variables in p-representation coincides with those in x-representation, i.e., not all the dynamical variables $p_0, p_1, \ldots, p_n$ are independent. For the representations in which the invariant P^2 is a fixed constant, the latter are connected by the relation

$$p^2 \equiv p_0^2 - p_k^2 = \varkappa^2 > 0, \qquad p_0^2 - p_k^2 = -\eta^2 < 0 \tag{11}$$

for the class I and III respectively. One can, for example, chose

$$p_0 = \varepsilon\sqrt{p_k^2 + \varkappa^2} \qquad \text{and} \qquad p_0 = \varepsilon\sqrt{p_k^2 - \eta^2}, \qquad \varepsilon = \frac{p_0}{|p_0|} \tag{12}$$

for I and III. Then in p-representation $\Psi = \widetilde{\Psi}(p_1, \ldots, p_n)$. Of course, one can accept that $\Psi = \varphi(p_0, p_1, \ldots, p_n)$, but under the condition (11), so that

$$\Psi = \varphi(p_0, p_1, \ldots, p_n) = \sqrt{2p_0}\,\widetilde{\Psi}(p_1, \ldots, p_n)\delta(p^2 - a^2), \qquad a = \varkappa^2, -\eta^2. \tag{13}$$

In the space of vector-functions Ψ discussed the scalar product can be defined by $P(1,n)$-invariant way:

$$\begin{aligned}(\Psi, \widetilde{\Psi}) &= \int d^{1+n}p\ \varphi^+(p_0, p_1, \ldots, p_n)\varphi'(p_0, p_1, \ldots, p_n) = \\ &= \int d^n p\ \widetilde{\Psi}^+(p_1, \ldots, p_n)\widetilde{\Psi}'(p_1, \ldots, p_n).\end{aligned} \tag{14}$$

The operators P_μ, $J_{\mu\nu}$ defined in this space of vector-functions Ψ, have the form (5) and (8) where the sudstitution

$$x_{[0}p_{k]} \to -\frac{1}{2}(x_k p_0 + p_0 x_k), \tag{15}$$

is made, p_0 is defined by (12), x_k and p_k are defined by relations

$$[x_k, p_l] = i\delta_{kl}, \qquad [x_k, x_l] = [p_k, p_l] = 0, \tag{6$'$}$$

while S_{kl} and (S_{0a}, S_{ab}) are the same as in the formulae (5) and (8).

Thus, the “quantum mechanical” representation (of the Foldy–Shirokov [3, 5] type) of the generators P_μ, $J_{\mu\nu}$ of the algebra $P(1,n)$ is of the form:

For the class I

$$\begin{aligned}&P = (p_0, p_k), \qquad p_0 \equiv \varepsilon\sqrt{p_k^2 + \varkappa^2}, \\ &J_{kl} = x_{[k}p_{l]} + S_{kl}, \qquad J_{0k} = -\frac{1}{2}(x_k p_0 + p_0 x_k) - \frac{S_{kl}p_l}{p_0 + \varkappa};\end{aligned} \tag{16}$$

For the class III

$$\begin{aligned}&P = (p_0, p_k), \qquad p_0 \equiv \varepsilon\sqrt{p_k^2 - \eta^2}, \\ &J_{ab} = x_{[a}p_{b]} + S_{ab}, \qquad J_{an} = x_{[a}p_{n]} - \frac{S_{ab}p_b - S_{a0}p_0}{p_n + \eta}, \\ &J_{0n} = -\frac{1}{2}(x_n p_0 + p_0 x_n) - \frac{S_{0a}p_a}{p_n + \eta}, \qquad J_{0a} = -\frac{1}{2}(x_a p_0 + p_0 x_a) + S_{0a}.\end{aligned} \tag{17}$$

Since operators $Q = x_k, p_\mu, S, P_\mu, J_{\mu\nu}$ of (16) and (17) are defined on the apace of vectors Ψ not depending on the time x_0, the representations (16) and (17) are, in fact, the representations of the algebra $P(1,n)$ in the Heisenberg picture where for the operators Q as functions of the time x_0, the equation of motion

$$i\partial_0 Q = [Q, P_0] \tag{18}$$

is postulated.

In the Schrödinger picture vectors Ψ depends explicitly on the time x_0 as on a parameter (but not as on a dynamical variable!) and for this dependence the equation of the Schrödinger–Foldy type is postulated

$$i\partial_0 \Psi(x_0) = P_0 \Psi(x_0), \tag{19}$$

where P_0 is defined by (12) and in x-representation $\Psi(x_0) = \Psi(x_0, x_1, \ldots, x_n)$, in p-representation $\Psi(x_0) = -\varphi(x_0, p_0, p_1, \ldots, p_n)$ under the condition (11) or $\Psi(x_0) = \Psi(x_0, p_1, \ldots, p_n)$ etc. These functions are vector-functions, the manifold of which constitutes the representation space for irreducible representations of the group $P(1,n)$ in the Schrödinger picture. It is clear therefore that their components are functions not only on $x_0, x_1, \ldots, x_n$ (or $x_0, p_1, \ldots, p_n$ etc.) but also on "spin" variables $s_3, t_3, \ldots$ discussed above in connectian with representations of homogeneous group $O(n)$ and $O(1, n-1)$. In accordance with the domain of definition of "spin" variables $s_3, t_3, \ldots$ in different classes, the equation (19) is finite-component or infinite-component. In the class I, where "spin" variables $s_3, t_3, \ldots$ take only discrete and finite values, all the equations (19) are finite-component and their solutions Ψ realises the unitary representations (i.e., vectors Ψ are normalizable). In the class III we have both finite-component and infinite-component equations, but unitary representations can be realized only on the solutions of the infinite-component equations.

One can suspect that owing to standing out of the time x_0 in the equation (19), the last is not invariant under the group $P(1,n)$ discussed. For the equation (19) the conventional demand of invariance under the given group is equivalent to the demand that the manifold of its solutions is invariant under this group (i.e., that any of its solution under transformations from $P(1,n)$ remains a solution of it but, generally speaking, another one). The mathematical formulation of this requirement is to satisfy the condition

$$[(i\partial_0 - P_0), Q]\Psi = 0, \tag{20}$$

where Ψ is any of solutions of Eq. (19) and Q is any generator of $P(1,n)$ or any linear combination of them, i.e., any element of the algebra $P(1,n)$. Therefore the generators $Q = P_\mu, J_{\mu\nu}$ must have such a form that both the relations (4) and the condition (20) must be satisfied. One can immediately verify that such operators P_μ, $J_{\mu\nu}$ are given by formulas (5) and (8) where, however, the substitution

$$x_{[0}p_{k]} \to x_0 p_k - \frac{1}{2}(x_k p_0 + p_0 x_k) \tag{15$'$}$$

is made and operators x_k, p_μ are defined by (6′) and (12).

Thus, the "quantum mechanical" representation of the generators P_μ, $J_{\mu\nu}$ of the algebra $P(1,n)$ in the Schrödinger picture have the form:

For the class[2] I

$$
\begin{aligned}
&P = (p_0, p_k), \qquad p_0 = \varepsilon\sqrt{p_k^2 + \varkappa^2},\\
&J_{kl} = x_{[k}p_{l]} + S_{kl}, \qquad J_{0k} = x_0 p_k - \frac{1}{2}(x_k p_0 + p_0 x_k) - \frac{S_{kl}p_l}{p_0 + \varkappa};
\end{aligned}
\tag{16'}
$$

For the class III

$$
\begin{aligned}
&P = (p_0, p_k), \qquad p_0 = \varepsilon\sqrt{p_k^2 - \eta^2},\\
&J_{ab} = x_{[a}p_{b]} + S_{ab}, \qquad J_{an} = x_{[a}p_{n]} - \frac{S_{ab}p_b - S_{a0}p_0}{p_n + \eta},\\
&J_{0a} = x_0 p_a - \frac{1}{2}(x_a p_0 + p_0 x_a) + S_{0a},\\
&J_{0n} = x_0 p_n - \frac{1}{2}(x_n p_0 + p_0 x_n) - \frac{S_{0a}p_a}{p_n + \eta}.
\end{aligned}
\tag{17'}
$$

It should be emphasized that in the Schrödinger picture the operators do not depend on the time x_0, except of the operators J_{0k}. These last depend on the time x_0 only by due to the presence of the term x_0p_k; it is just the presence of the term x_0p_k to satisfy the invariance condition (20) of the equation (19).

Note in the end of this section that last years the problem of using in physics some groups like $P(m,n)$, $O(m,n)$ etc. as groups of generasized symmetry, was repeatedly arised (see, for instance, ref. [8] and refs. in ref. [9]). The consequent physical analysis of a quantumscheme based on either unificated group, is in fact possible only after a mathematical analysis of representations of this group and equations connected with it, like the analysis made here for the group $P(1,n)$. The method used here for studying the representations of the group $P(1,n)$, is extend on the groups $P(m,n)$ without special difficulties. Thus the problem of classification of representations and realization of an inhomogeneous group $P(m,n)$ is in fact reduced to the problem of classification and realization of homogeneous groups of the type $O(m',n')$ already studied in ref. [7].

3. Physical interpretation

Last years many attempts of using different groups like $O(m,n)$, $P(m,n)$ as relativizing internal symmetry groups like $SU(n)$, were undertaken. The problem of a relativistic generalization of an internal symmetry group is in fact connected with finding a total symmetry group G containing non-trivially the Poincaré group $P(1,3)$ (the group of "external" symmetry) and a group of "internal" symmetry like $SU(n)$. As it is shown in refs. [10], it is impossible non-trivially to unity the algebra $P(1,n)$ and the algebra of "internal" symmetries in the framework of a finite-dimensional algebra Lie G, if the spectrum of the mass operator $M^2 \equiv P_0^2 - \vec{P}^{\,2}$ is discrete. In ref. [11] a non-trivial example of the algebra $G \supset P(1,3)$ was constructed for the case when the spectrum of the mass operator is already stripe; but the algebra G was found to be infinite-dimensional in this case too. The consideration of the infinite-dimensional algebras for the physical purposes is difficult both owing to the absence

[2]Our formulae (16′) for generators P_μ, $J_{\alpha\beta}$ in the case $P^2 > 0$ coinside with the corresponding formulae (B.25–28) in ref. [5] if the last are rewritten in the tensor form.

of developed mathematical apparatus of such algebras and owing to the necessity of solving a very difficult problem of physical interpretation of all the commuting generators, the number of which is infinite. Do not speak about that the question of writing down equations of motions, invariant under such an algebras, is quite not clear. All this circumstances prompt that, to find a finite-dimensional algebra $G \subset P(1,3)$ of a total symmetry group, we have to refuse from the demand of the discreticity or even stripiticity of the mass spectrum. In this case one can propose many groups as total symmetry groups (the groups of the type $P(m,n)$). However, in a $G = P(m,n), O(m,n)$, as like as in cases of other groups which are groups of coordinate transformations in spaces of great dimensions, it still arises a difficult problem of necessity to give a physical interpretation to the great number of commuting operators.

Below we deal only with the inhomogeneous de Sitter group $P(1,4)$ which is a minimal extention of the Poincaré group $P(1,3)$. Here we discuss a main topics of physical interpretation of a quantum mechanical scheme based on this group. The group $P(1,4)$ is the most attractive because of in this case it is a success to give a clear physical meaning to a complete set of commuting variables.

In p-representation a component of the wave function Ψ — the a solution of the equation (19) with $n = 4$ is a function of six dynamical variables of corresponding complete set:

$$\Psi(x_0, \vec{p}, p_4, s_3, t_3).$$

As usually, this component is interpreted as the probability amplitude of finding (by measuring at the given moment of the time $t = x_0$) the indicated values $\vec{p}$, p_4, s_3, t_3 of the complete set $\vec{P}$, P_4, S_3, T_3. The physical meaning of the operators $\vec{P}$ and P_4 is given in Section 1. We discuss below the definition and physical meaning of the operators S_3, T_3 in the class I.

Remind that in the case of $P(1,3)$ the operators S_{kl} $k,l = 1,2,3$ in (16′) which constitute the spin vector $\vec{S} = (S_{23}, S_{31}, S_{12})$, are generators of the group $O(3)$ (the little group of the group $P(1,3)$ in the class I) and they are interpreted as an angular momenta of proper rotations. More exactly they should be interpreted as an angular momenta which are connected with intrinsic (internal) motion because when $\vec{P} = 0$, the angular momenta J_{kl} do not vanish but reduce to the spin angular monenta S_{kl}.

In the case of $P(1,4)$ there are six angular momentum operators, which describe the internal motion of particle (i.e., the motion when $\vec{p} = p_4 = 0$): $J_{kl/\vec{p}=p_4} = S_{kl}$, $k,l = 1,\ldots,4$. The operators S_{kl} are generators of the group $O(4)$ (the little group of the group $P(1,4)$) in the class I). They can be combined into two 3-dimensional vectors $\vec{S}$ and $\vec{T}$ defined by components

$$S_a \equiv \frac{1}{2}(S_{bc} + S_{4a}), \qquad T_a \equiv \frac{1}{2}(S_{bc} - S_{4a}), \tag{21}$$

where $(a,b,c) = \text{cycl}(1,2,3)$. These components satisfy the relations

$$\begin{aligned} &[S_a, S_b] = iS_c, \qquad [T_a, T_b] = iT_c, \\ &\left[S_a, \vec{S}^{\,2}\right] = \left[T_a, \vec{T}^{\,2}\right] = [S_a, T_b] = 0. \end{aligned} \tag{22}$$

Remined that $\vec{S}^{\,2}$ and $\vec{T}^{\,2}$ are the invariants of the algebra $O(4)$ being for irreducible representations $D(s,t)$ of this algebra

$$\vec{S}^{\,2} = s(s+1)\hat{1}, \qquad \vec{T}^{\,2} = t(t+1)\hat{1}, \tag{23}$$

where $s,t = 0, \frac{1}{2}, 1, \ldots$ and $\hat{1}$ is the $(2s+1)(2s+1)$-dimensional unit matrix. It was just the relations (22) and (23) to allow us [1] to interpret 3-vectors $\vec{S}$ and $\vec{T}$ as the spin and isospin operators.

It is clear from (16′) that in the representations of the class I the generators of the algebra $P(1,4)$ are constructed not only from the spin operators but also from the isospin operators (and, of course, of x_k and p_k). In this sense in quantum mechanic scheme based on the group $P(1,4)$ the spin and isospin are presented on the same foot, unlike from the case of conventional theory. Furthermore, unlike from to the latter, in our case both the spin and isospin are entered dynamically. Indeed, in the conventional approach the group $P(1,3)\otimes SU(2)_T$ is taken as the total symmetry group, so that the generators of $SU(2)_T$ commute with the generators of $P(1,3)$ (even in the presence of interactions). The group $P(1,4)$ which we taken as a total symmetry group, is not isomorphic to the group $P(1,3)\otimes SU(2)_T$ furthermore, as in can be seen from (21) and (16′), $SU(2)_T \subset O(4) \subset P(1,4)$ as like as $SU(2)_S \subset O(4) \subset P(1,4)$, and the isospin operators (as like as the spin operators) do not commute with $P(1,3) \subset P(1,4)$.

The manifold of solutions of the equation (19) realizes in the case discussed the irreducible representation $D^{\pm}(s,t)$ of the algebra $P(1,4)$, where the sings "$\pm$" refer to the values $\varepsilon = \pm 1$ of the invariant — the sign of energy, the numbers as s and t determine the eigenvalues of the invariants

$$\vec{S}^{\,2} = \frac{W}{4p^2} + \frac{V}{2\sqrt{p^2}}, \qquad \vec{T}^{\,2} = \frac{W}{4p^2} - \frac{V}{2\sqrt{p^2}}, \tag{24}$$

which are invariants both of $P(1,4)$ and $O(4)$.

In quantum scheme based on $P(1,4)$, possible states of an "elementary particle" (when $\varepsilon = +1$) or "antiparticle" (when $\varepsilon = -1$) with given values of s, t and $p^2 = \varkappa^2$ are states which constitute the representation space for an irreducible representation $D^{\pm}(s,t)$ of the group $P(1,4)$. This is just the definition of the elementary particle in the $P(1,4)$-quantum scheme. The simplest states of this particle are identified by eigenvalues of complete set of comuting variables. It is clear that the representation $D^{\pm}(s,t)$, irreducible with respect to $P(1,4)$, is reducible with respect to $P(1,3) \subset P(1,4)$ therefore the "elementary particle" defined here, is not elementary in the conventional sense (i.e., with respect to the group $P(1,3)$). Indeed, a solution Ψ of Eq. (19) with given s and t contains componets identified not only by values of the 3-component s_3 of spin but also by values of the 3-components t_3, of isospin, so that the vector Ψ describes in fact the whole multiplet – the set of states with different values of t_3, $-t \leq t_3 \leq t$ (and, of course, of s_3, $s \leq s_3 \leq s$). For example, the vector $\Psi^{\pm}$ with $\varepsilon \pm 1$, $s = 0$ and $t = \frac{1}{2}$ describes a meson isodublet like

$$\Psi^{+} \equiv \begin{pmatrix} \Psi^{+}_{(0,\frac{1}{2})} \\ \Psi^{+}_{(0,-\frac{1}{2})} \end{pmatrix} = \begin{pmatrix} K^{+} \\ K^{0} \end{pmatrix}, \qquad \Psi^{-} = \begin{pmatrix} \widetilde{K}^{0} \\ K^{-} \end{pmatrix}.$$

The parameter $\varkappa$ (the threshold value of the free state energy or the "bare" rest mass) is the same for all the members of the given multiplet. Of course, the introduction of a sutable interaction into the equation (19) leads to the mass splitting within a multiplet.

The approach proposed may be found successful for a consequent description of unstable systems (resonances, particles or systems with non-fixed mass) already in the framework of the quantum mechanics[3]. As it is known, the conventional quantum mechanical approach deals a with finding complex eigenvalues of energy operators which must be Hermitian in a Hilbert space of wave functions, i.e., in fact, one must go put of the framework of Hilbert space; the latter is connected with breaking of such a fundamental principles as unitarity, hermiticity etc. [12].

There are no similar difficulties in the quantum mechanical approach proposed. Indeed, here the mass operator is an independent dynamical variable (1), it is Hermitian, defined in the Hilbert space; therefore one can find its eigenvalues m^2 and distributions $\rho(m^2)$ in the same Hilbert space, as like as they find eigenvalues and distributions for operators of energy, momentum and other dynamical variables. For example, if we have a stationary wave function $\Psi = \{\Psi(\vec{x}, x_4, s_3, t_3)\}$ of, generally speaking, an unstable multiplet (we meant: a solution of an equation of the type (19) with an interaction not depending on the time x_0) then

$$\rho(m^2, s_3, t_3) = \int d^3x \left| \int dx_4 \; e^{-i\sqrt{m^2-\varkappa^2}x_4} \Psi(\vec{x}, x_4, s_3, t_3) \right|^2 . \tag{25}$$

If the distribution (25) with the given s_3, t_3 has one maximum, the experimentally observed mass of the particle with given s_3, t_3 is defined either by the position of the maximum or form

$$\bar{m}^2 = \int d^3x \; dx_4 \; \Psi^*(\vec{x}, x_4, s_3, t_3)(p_4^2 + \varkappa^2)\Psi(\vec{x}, x_4, s_3, t_3) \tag{26}$$

and its mean lifetime τ is defined from

$$\bar{m}^2\bar{\tau}^2 = 1. \tag{27}$$

If the distribution (25) has more than one maximum, the position of them defines an experimentally observed masses of unstable particles and the semi-widths of the distribution (25) in the regions of maximums define the lifetimes of corresponding unstable particles. If, finally, $\rho(m^2, s_3, t_3)$ has a δ-like singularity in a point $m^2 = m_0^2$, the m_0 is identified with the mass of a stable particle.

It is important to emphasize that in accord with our interpretation, the particles experimentally observed are described not by the free equation (19), but by an equation of the type (19) with a sutable interaction which may breakdown the $P(1,4)$-invariance, but, of course conserves the $P(1,3)$-invariance[4]. As for solutions of the free equation (19) they are some hypothetical ("bare") states which may not correspond, to any real particles. From view point of this interpretation there are two types of interactions: interactions which cause a "dressing" of particles and are inhierent even in asymptotical states, and usual interactions which cause a scattering processes of real ("dressed") particles. Therefore, in particular, the 5-dimensional

[3]The consequent consideration of such problems demands, obviously, the quantum field approach, but a quantum mechanical approach can be regarded as a half-fenomenology.

[4]In this sense the consideration of $P(1,4)$-symmetry here presented is only a base for its sutable violation — analogously to considerations and violations of $SU(n)$-symmetries.

conservation law following from the free $P(1,4)$-invariant scheme; may have not a real sense.

In the interpretation of the $P(1,4)$-scheme proposed we automatically have the $SU(2)_T$-systematic of particles. In contrast to the conventional systematics, our one is a dynamical in the sense that for compaund model like those of Fermi–Yang, Goldhaber–Györgyi–Kristy and others we can write down an equation in which spin and isospin variables are mixed non-trivially.

Emphasize, that the interpretation of the $P(1,4)$-scheme proposed and, in particular, of the complete set of commuting variables mentioned above, was mainly based on the definition (1) of the varlable-mass operator as an independent dynamical variable. This interpretation does not pretend, of course, to be the only one and complete. In particular, the problem of giving the "fifth coordinate" x_4 the more immediate physical sense than that one laying under its definition as a dynamical variable canonically conjugated to the mass variable p_4, and the same problem refers to operators like J_{04}, J_{a4}, $a = 1,2,3$ we do not discusse here. The more detail discussion of these problem is possible only in connection with considerations of solutions of equations like Eq. (19) with sutable interactions what is not a subject of this article.

Here we have considered the $P(1,n)$-invariant equations of the Schrödinger–Foldy type in an arbitrary dimensional Minkowski space, in which the differential operators $\partial_k \equiv \partial/\partial x_k$ of the "space" variables are presented under square root. This equations describing the positive and negative states separatelly, are suitable for quasirelativistic quantum mechanical considerations (e.g., for calculations of spin-isospin effects in $P(1,4)$-invariant equations with interactions included). Theoretic-field considerations are usually based on equations of first order on ∂_μ. The general form of $P(1,n)$-invariant linear on ∂_μ equation is

$$(B_\mu \partial_\mu \pm \varkappa)\Phi^{\pm} = 0, \qquad \mu = 1,2,\ldots,n,n+1, \tag{28}$$

where the operators B_μ are defined by the relations

$$[B_\mu, J_{\rho\sigma}] = \delta_{\mu\rho} B_\sigma - \delta_{\mu\sigma} B_\rho, \qquad (\mu,\rho,\sigma = 1,\ldots,n+1). \tag{29}$$

For the representation of the class I the operators B_μ are finite-dimensional; for those of the class III the operators B_μ can be both finite- and infinite-dimensional. Concrete forms of operators B_μ can be founed by the method proposed in ref. [13].

In this paper we have not considered the problem of invariance of the equation (28) as to discrete transformations, that is relatively to

$$x'_k = -x_k, \qquad x'_0 = -x_0. \tag{30}$$

This problem has been investigated by one of us [14]. As it is shown in [14] the equation (28) for $n = 2m$, $m = 1,2,3,\ldots$, is neither invariant as to transformations (30) nor

$$x'_0 = -x_0, \qquad x'_k = x_k, \qquad k = 1,2,\ldots,2m. \tag{31}$$

Thus, in the field theory constructed on the basis of the groups $P(1,2)$, $P(1,4)$, $P(1,6)$ and so on the theorem CPT may be broken down. It should be emphasized, however, that the direct of the manifold of solutions $\{\Phi^+\}$ and $\{\Phi^-\}$ T-, CPT-invariant.

1. Fushchych W.I., Krivsky I.Yu., *Nucl. Phys. B*, 1968, **7**, 79;
Фущич В.И., Кривский И.Ю., Препринт ИТФ-68-72, Киев, 1968.
2. Wigner E.P., *Ann. Math.*, 1939, **40**, 149.
3. Широков Ю.М., *ЖЭТФ*, 1957, **33**, 1196.
4. Joos H., *Fortschr. Phys.*, 1962, **10**, 65.
5. Foldy L., *Phys. Rev.*, 1956, **102**, 568.
6. Гельфанд И.М., Цейтлин М., *ДАН СССР*, 1950, **71**, 1017.
7. Гельфанд И.М., Граев М.И., *Труды Московского математического общества*, 1959, **8**; *Изв. АН СССР. Сер. матем.*, 1965, **29**, 5.
8. Hederfledt G.C., Henning J., *Fortschr. Phys.*, 1968, **16**, 491.
9. Соколик Г.А., Групповые методы в теории элементарных частиц, Атомиздат, М., 1965.
10. O'Raifeartaigh L., *Phys. Rev. Lett.*, 1965, **14**, 575;
Jost R., *Helv. Phys. Acta*, 1966, **39**, 369.
11. Фущич В.И., *Укр. физ. журн.*, 1968, **13**, 362.
12. Mathews P.T., Salam A., *Phys. Rev.*, 1958, **112**, 283.
13. Фущич В.I., *Укр. фiз. журн.*, 1966, **8**, 907.
14. Фущич В.И., *Письма ЖЭТФ*, 1969.