A relativistically invariant mass operator

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In [1] it was shown how, for a given (discrete) mass spectrum of elementary or hypothetical particles, it was possible to construct a non-trivial algebra G containing a Poincaré algebra P as a subalgebra so that the mass operator, defined throughout the space where one of the irreducible representations G is given, is self-conjugate and its spectrum coincides with the given mass spectrum. Such an algebra was constructed in explicit form for the nonrelativistic case, i.e., the generators were written for the algebra. However, the problem of how to assign the algebra G constructively and determine an explicit form of the mass operator in the relativistic case has remained unsolved.

In the present work we present a solution of this problem, construct continuum analogs of the classical algebras $U(N)$ and $Sp(2N)$, and show that the problem of including the Poincaré algebra can be formulated in the "language" of wave function equations.

1. For simplicity, we will assume that there be given only three particles with masses m_1, m_2 and m_3. R_1, R_2 and R_3 will represent the spaces in which irreducible representations of the algebra P are realized. The operator $\left(P_\alpha^{(i)}\right)^2$ in these spaces is, as is well known, a multiple of the unit operator:

$$\left(P_\alpha^{(i)}\right)^2 R_i = m_i^2 R_i, \qquad i = 1,2,3, \quad \alpha = 0,1,2,3. \tag{1}$$

We will designate by R the linear sum of these spaces[1]. The operators of energy-momentum, angular momentum, and square of the masses of the system, which may be in various excited states in this space, take the form

$$\begin{gathered} P_\alpha = P_\alpha^{(1)} F_{11} + P_\alpha^{(2)} F_{22} + P_\alpha^{(3)} F_{33}, \\ M_{\mu\nu} = M_{\mu\nu}^{(1)} F_{11} + M_{\mu\nu}^{(2)} F_{22} + M_{\mu\nu}^{(3)} F_{33}, \end{gathered} \tag{2}$$

$$M^2 = \left(P_\alpha^{(1)}\right)^2 F_{11} + \left(P_\alpha^{(2)}\right)^2 F_{22} + \left(P_\alpha^{(3)}\right)^2 F_{33}, \tag{3}$$

where the F_{ij} designate the squared three-rowed matrices in which unit operators stand at the intersection of the i-th row and j-th column, while all other elements are zero.

It is clear that in R there are realized reducible representations of the algebra P. However, relative to certain sets of the operator G this space may not have invariant subspaces. Obviously, operators must appear in this set of the type

$$D = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}, \tag{4}$$

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[1]In the case of a real physical problem, we should have taken the linear sum with certain weights, the squares of the moduli of which could be interpreted as the probability of finding the system in one or another of the states.

where at least one of the operators d_{ij} $(i \neq j)$ is nonzero. In order to solve our problem, these operators must be constructed in explicit form.

For the determination of the explicit form of the operators use will be made of the methods of the quantum theory of fields. The vector $h_i \in R_i$ will be presented in the form [2]

$$h_i = \int d\boldsymbol{k}\, F_i(\boldsymbol{k}) a_i^+(\boldsymbol{k})|0\rangle. \tag{5}$$

For simplicity of notation, we shall assume that the distribution function $F_1(\boldsymbol{k}) = F_2(\boldsymbol{k}) = F_3(\boldsymbol{k}) = F(\boldsymbol{k})$ and all particles are without spin. The generators of the Poincaré algebra, expressed by the operators of creation and annihilation, have the form [3]

$$\begin{aligned}
&P_j^{(i)} = \int d\boldsymbol{k}\, k_j a_i^+(\boldsymbol{k}) a_i(\boldsymbol{k}),\\
&P_0^{(i)} = \int d\boldsymbol{k}\, k_0^{(i)} a_i^+(\boldsymbol{k}) a_i(\boldsymbol{k}), \qquad k_0^{(i)} = \sqrt{\boldsymbol{k}^2 + m_i^2},\\
&M_{lr}^{(i)} = \frac{i}{2}\int d\boldsymbol{k}\left(k_l f_r^{(i)}(\boldsymbol{k}) - k_r f_l^{(i)}(\boldsymbol{k})\right), \qquad M_{0l}^{(i)} = \frac{i}{2}\int d\boldsymbol{k}\, k_0^{(i)} f_l^{(i)}(\boldsymbol{k}),\\
&f_l^{(i)}(\boldsymbol{k}) = \frac{\partial a_i^+(\boldsymbol{k})}{\partial k_l} a_i(\boldsymbol{k}) - a_i^+(\boldsymbol{k}) \frac{\partial a_i(\boldsymbol{k})}{\partial k_l}.
\end{aligned} \tag{6}$$

We can now write the explicit form of the operators

$$d_{ij} = \int d\boldsymbol{k}\, d\boldsymbol{k}'\, F(\boldsymbol{k}) F'(\boldsymbol{k}') \left\{a_i^+(\boldsymbol{k}) a_j(\boldsymbol{k}') + a_j^+(\boldsymbol{k}') a_i(\boldsymbol{k})\right\}. \tag{7}$$

Obviously, the operators $D_{ij} = d_{ij} F_{ij}$ will transform vectors from space R_i to R_j. Space R is irreducible with respect to operators P_α, $M_{\mu\nu}$ and D_{ij}. This statement is a consequence of the fact that the operators D_{ij} transform a given vector from $h \in R$, to vector $h_i \in R_i$, while, since the subspace $R_i \subset R$ is noninvariant relative to these operators then, by the same token, the irreducibility of the representation G in R is shown.

The set of operators (5) and (6) (and their linear combinations) form a Lie algebra in the case where they satisfy the Jacobi identities. Calculating, for example, the commutators $[P_\alpha, D_{ij}]_-$, $[P_\beta, [P_\alpha, D_{ij}]_-]_-$ etc., it is not difficult to convince oneself that the operators derived from these are not linear combinations of the operators P_α, $M_{\mu\nu}$ and D_{ij}, i.e., the set G is an infinite-dimensional Lie algebra. All elements of the algebra G can be expressed explicitly by the operators $a_i^+(\boldsymbol{k}) a_j(\boldsymbol{k}')$, $a_j^+(\boldsymbol{k}') a_i(\boldsymbol{k})$, $[\partial a_i^+(\boldsymbol{k})/\partial k_l] a_j(\boldsymbol{k}')$, $a_l^+(\boldsymbol{k})[\partial a_r(\boldsymbol{k}')/\partial k_i']$ and all possible products of these operators. As will be shown below, these operators form a continuous Lie algebra. In R space the operators of hypercharge and isospin have the form

$$Y = j_1 F_{11} + j_2 F_{22} + j_3 F_{33}, \qquad J = i_1 F_{11} + i_2 F_{22} + i_3 F_{33}, \tag{8}$$

where j_1, j_2, j_3 and i_1, i_2, i_3 are hypercharges and isospins of particles m_1, m_2 and m_3. The formulas (8) permit the expression of the operator M^2 by the operator of hypercharge and isospin. In our case

$$M^2 = a'E + b'Y + c'J, \tag{9}$$

where E is the unit operator, and a', b', c' are arbitrary, generally speaking, constant quantities. In the triplet representations of the algebra G, which we considered, these quantities are uniquely determined by the masses m_1, m_2 and m_3. In all other representations, such uniqueness does not exist and hence formula (9) will give a mass relationship between the elementary particles. If the initial particles have spin, then

$$M^2 = aS + bY + cJ, \tag{9'}$$

where a, b, c are arbitrary numbers and S is the spin operator.

Other examples of infinite dimensional Lie algebras, containing the algebra P, are considered in [12].

Note 1. It is well known that the masses of elementary particles depend on spin, hypercharge, isospin, and other quantum numbers; hence, for determining the mass operator, one tends to express it by the operators of spin, hypercharge, and isospin. It should be noted that, generally speaking, the mass operator can always (in principle) be expressed by one operator. In fact, let

$$M^2 = f(A_1, A_2, \ldots, A_n), \tag{9''}$$

where $A_1, A_2, \ldots, A_n$ are mutually commuting self-adjoint operators, operating in a certain separable Hilbert space. In agreement with Neiman's theorem [4] concerning creation operators, one can determine such a bounded self-adjoint operator A in this space that

$$A_n = \varphi_n(A).$$

From this theorem it follows that $M^2 = f(A_1, A_2, \ldots, A_n) = \tilde{f}(A)$, i.e., the mass operator can always be represented as a function of only one operator A of a weakly closed ring. This attests to the fact that there exists one universal quantum number, with the help of which it is possible to explain the mass spectrum of elementary particles if the explicit form of the function $\tilde{f}$ is known.

Since the mass operator in the approach arises from the same sort of generator of the algebra G as, say, does the operator of isospin or hypercharge, the formula (9″) may be viewed as an equation of a hypersurface in a space of mutually commuting operators. For such an interpretation of the mass formula (9″), the generation operator A apparently plays the same role as does time in classical mechanics (where the aggregate of all trajectories lies on a certain manifold, in particular on a surface $F(x, y, z)$ for which $x = x(t)$, $y = y(t)$, $z = z(t)$).

From the geometrical point of view the mass equations

$$M = a + bS(S + 1)$$

for hadrons and

$$M^2 = a^2 + b^2 S(S + 1)$$

for mesons represent "trajectories" (a parabola for hadrons and hyperbola for mesons) of motion of the system, which can exist in various mass and spin states.

The mass equations of Okubo,

$$M = a + bY + c\{J(J + 1) - Y^2/4\}$$

for hadrons and

$$M^2 = a + bY + c\{J(J+1) - Y^2/4\}$$

for mesons, represent a hyperbolic paraboloid and double poled hyperboloid in an imaginary three-dimensional space (M, Y, J).

In this manner, if we quantize the general equation for a hyperbolic paraboloid:

$$c/4y^2 - cz^2 - cz - by + x - a = 0,$$

i.e., if in this equation we make the substitutions $x \to M$, $y \to Y$, $z \to J$, then we will obtain the formula of Okubo for hadrons. If with each multiplet we associate a definite hypersurface, then various transitions of one multiplet to particles of the same multiplet can be interpreted as "motion" or the given hypersurface. Transitions of particles of one multiplet to particles of another multiplet may be considered as transitions from one hypersurface to another. If to all experimentally discovered hadrons (or bosons) is assigned a single hypersurface, then all possible transitions of hadrons (bosons) to hadrons (bosons) should be interpreted as "motion" on this hypersurface, for which all quantum characteristics of the system can change.

2. The characteristic special feature of problems concerning the spectrum of atomic hydrogen and of a harmonic and anharmonic oscillator, from the group theoretic standpoint, is that all these problems can be solved by the method of embedding of the finite dimensional Lie algebra, appropriate to groups of hidden symmetry, in a broader but dimensionally finite Lie algebra [5, 6]. However, this statement does not depend on where the Hamiltonian is defined — in a Hilbert or in a vector space with indefinite metric. Thus, for example, the problem of the spectrum of an N-dimensional oscillator with complex ghosts can also be solved by the method of embedding of a finite dimensional Lie algebra in a finite dimensional Lie algebra[2].

From the above considerations (section 1) it follows that the Poincaré algebra (relativistic case) can be included by a nontrivial method only in the infinite dimensional Lie algebra (the case of non-Lie algebras are not considered here). This existing difference between the relativistic and non-relativistic problem of the embedding of the Lie algebra can be adequately explained in a natural manner. In quantum, mechanics, as is well known, we always deal with finite numbers of degrees of freedom. Transition to an infinite number of degrees of freedom, apparently, implies a transition from a finite dimensional Lie algebra to an infinite-dimensional one. We shall expiate this statement with an example.

As was shown in [6], the space of states of an N-dimensional harmonic oscillator realizes an irreducible representation of the algebra $\overline{U}(N+1) \supset U(N)$. The generators of the algebra $\overline{U}(N+1)$ satisfy the following commutation relations:

$$\left[E^{\lambda}_{\rho}, E^{\sigma}_{\varkappa}\right]_{-} = \delta_{\rho\sigma} E^{\lambda}_{\varkappa} - \delta_{\varkappa\lambda} E^{\sigma}_{\rho}, \qquad \lambda, \rho, \sigma, \varkappa = 1, \ldots, N+1, \tag{10}$$

where

$$\begin{gathered} E^{\nu}_{\mu} = \frac{1}{2}\left[a_{\mu}, a^{+}_{\nu}\right]_{+}, \qquad \mu, \nu = 1, \ldots, N, \\ E^{N+1}_{\mu} = g(H) a^{+}_{\mu}, \qquad E^{\mu}_{N+1} = f(H) a_{\mu}, \qquad E^{N+1}_{N+1} = h(H), \end{gathered} \tag{11}$$

[2]The question of inclusion of an algebra of symmetry $U(2N)$ of such an oscillator in a dynamic algebra will be considered in a subsequent paper.

$$H=\sum_{\mu=1}^{N}a_\mu^+a_\mu, \qquad \left[a_\mu,a_\nu^+\right]_-=\delta_{\mu\nu}. \tag{12}$$

If the N number of the oscillators tends toward infinity, we approach the infinite oscillator, but then the dynamic algebra of an oscillator $\overline{U}(N+1)$ and the algebra of hidden symmetry $U(N)$ go over into the infinite dimensional Lie algebra. The algebra $Sp(2N)$ may be determined by an analogous method, when $N\to\infty$.

For transition from quantum mechanics to the quantum theory of fields, it is also necessary to let the volume in which the oscillators are "contained" approach infinity [2]. For such passages to the limit, the operators a_r and a_s^+ are replaced by the general operators of annihilation $a(\boldsymbol{k})$ and creation $a^+(\boldsymbol{k})$, which satisfy the relations

$$\left[a(\boldsymbol{k}),a^+(\boldsymbol{k}')\right]_-=\delta(\boldsymbol{k}-\boldsymbol{k}'). \tag{13}$$

The dimensionally infinite algebra $\underset{N\to\infty}{U(N)}$ for this case is naturally associated with the continual algebra $U(\boldsymbol{k},\boldsymbol{k}')$, the generators of which are the operators

$$E(\boldsymbol{k},\boldsymbol{k}')=\frac{1}{2}\left[a(\boldsymbol{k}),a^+(\boldsymbol{k}')\right]_+. \tag{14}$$

It is not difficult to convince oneself that operators of the form (13) satisfy the following commutative relationships:

$$\left[E(\boldsymbol{k},\boldsymbol{k}'),E(\boldsymbol{q},\boldsymbol{q}')\right]_-=\delta(\boldsymbol{k}-\boldsymbol{q}')E(\boldsymbol{q},\boldsymbol{k}')-\delta(\boldsymbol{k}'-\boldsymbol{q})E(\boldsymbol{k},\boldsymbol{q}'). \tag{15}$$

Further, let us construct the algebras $U_N(\boldsymbol{k},\boldsymbol{k}')$ and $Sp_{2N}(\boldsymbol{k},\boldsymbol{k}')$. Consider the set of operators:

$$E_\mu^\nu(\boldsymbol{k},\boldsymbol{k}')=\frac{1}{2}\left[a_\mu(\boldsymbol{k}),a_\nu^+(\boldsymbol{k}')\right]_+, \qquad \mu,\nu=1,\ldots,N, \tag{16}$$

$$E_{\mu\nu}(\boldsymbol{k},\boldsymbol{k}')=a_\mu(\boldsymbol{k})a_\nu(\boldsymbol{k}'), \qquad E^{\mu\nu}(\boldsymbol{k},\boldsymbol{k}')=a_\mu^+(\boldsymbol{k})a_\nu^+(\boldsymbol{k}'), \tag{17}$$

where

$$\left[a_\mu(\boldsymbol{k}),a_\nu^+(\boldsymbol{k}')\right]_-=\delta_{\mu\nu}\delta(\boldsymbol{k}-\boldsymbol{k}'). \tag{18}$$

Taking into account (18), it can be shown that

$$\left[E_\mu^\nu(\boldsymbol{k},\boldsymbol{k}'),E_\alpha^\beta(\boldsymbol{q},\boldsymbol{q}')\right]_-=\delta_{\mu\beta}\delta(\boldsymbol{k}-\boldsymbol{q}')E_\alpha^\nu(\boldsymbol{q},\boldsymbol{k}')-\delta_{\nu\alpha}\delta(\boldsymbol{q}-\boldsymbol{k}')E_\mu^\beta(\boldsymbol{k},\boldsymbol{q}'), \tag{19}$$

$$\left[E_{\mu\nu}(\boldsymbol{k},\boldsymbol{k}'),E_{\alpha\beta}(\boldsymbol{q},\boldsymbol{q}')\right]_-=0, \tag{20}$$

$$\left[E_\mu^\nu(\boldsymbol{k},\boldsymbol{k}'),E_{\alpha\beta}(\boldsymbol{q},\boldsymbol{q}')\right]_-=-\delta_{\nu\beta}\delta(\boldsymbol{k}'-\boldsymbol{q}')E_{\alpha\mu}(\boldsymbol{q},\boldsymbol{k})-\delta_{\nu\alpha}\delta(\boldsymbol{q}-\boldsymbol{k}')E_{\beta\mu}(\boldsymbol{q}',\boldsymbol{k}), \tag{21}$$

$$\left[E_\mu^\nu(\boldsymbol{k},\boldsymbol{k}'),E^{\alpha\beta}(\boldsymbol{q},\boldsymbol{q}')\right]_-=\delta_{\mu\beta}\delta(\boldsymbol{k}-\boldsymbol{q}')E^{\alpha\nu}(\boldsymbol{q},\boldsymbol{k}')+\delta_{\alpha\mu}\delta(\boldsymbol{k}-\boldsymbol{q})E^{\beta\nu}(\boldsymbol{q}',\boldsymbol{k}'), \tag{22}$$

$$\begin{gathered}\left[E_{\mu\nu}(\boldsymbol{k},\boldsymbol{k}'),E^{\alpha\beta}(\boldsymbol{q},\boldsymbol{q}')\right]_-=\delta_{\nu\alpha}\delta(\boldsymbol{k}'-\boldsymbol{q})E_\mu^\beta(\boldsymbol{k},\boldsymbol{q}')+\delta_{\mu\alpha}\delta(\boldsymbol{k}-\boldsymbol{q})E_\nu^\beta(\boldsymbol{k}',\boldsymbol{q}')+\\ +\delta_{\nu\beta}\delta(\boldsymbol{k}'-\boldsymbol{q}')E_\mu^\alpha(\boldsymbol{k},\boldsymbol{q})+\delta_{\mu\beta}\delta(\boldsymbol{k}-\boldsymbol{q})E_\nu^\alpha(\boldsymbol{k}',\boldsymbol{q}),\end{gathered} \tag{23}$$

$$\left[E^{\mu\nu}(\boldsymbol{k},\boldsymbol{k}'),E^{\alpha\beta}(\boldsymbol{q},\boldsymbol{q}')\right]_- = 0. \tag{24}$$

The set of operators $\left\{E_\mu^\nu(\boldsymbol{k},\boldsymbol{k}')\right\}$, satisfying the relations (19) form a continuous Lie algebra $U_N(\boldsymbol{k},\boldsymbol{k}')$. The set of operators $\left\{E_\mu^\nu(\boldsymbol{k},\boldsymbol{k}'),E^{\mu\nu}(\boldsymbol{q},\boldsymbol{q}')\right\}$, satisfying the relationships (19)–(24), form the continuous Lie algebra $Sp_{2N}(\boldsymbol{k},\boldsymbol{k}')$.

Utilizing the commutating relations (19)–(24) it is possible to show that the elements from $Sp_{2N}(\boldsymbol{k},\boldsymbol{k}')\supset U_N(\boldsymbol{k},\boldsymbol{k}')$ satisfy the Jacobi identity. Since elements of the algebra $U_N(\boldsymbol{k},\boldsymbol{k}')$ depend continuously on the variables $\boldsymbol{k}$ and $\boldsymbol{k}'$, it is then possible to formally determine the derivative

$$\frac{\partial E_\mu^\nu(\boldsymbol{k},\boldsymbol{k}')}{\partial k_i}\equiv A^i_{\mu\nu}(\boldsymbol{k},\boldsymbol{k}'),\qquad \frac{\partial E_\mu^\nu(\boldsymbol{k},\boldsymbol{k}')}{\partial k'_j}\equiv B^j_{\mu\nu}(\boldsymbol{k},\boldsymbol{k}'),\qquad i,j=1,2,3. \tag{25}$$

Taking into account (19), it is not difficult to establish that

$$\begin{aligned}&\left[A^i_{\mu\nu}(\boldsymbol{p},\boldsymbol{p}'),A^j_{\alpha\beta}(\boldsymbol{q},\boldsymbol{q}')\right]_- =\\ &\qquad=\delta_{\mu\beta}\frac{\partial\delta(\boldsymbol{p}-\boldsymbol{q}')}{\partial p_i}A^i_{\alpha\nu}(\boldsymbol{q},\boldsymbol{p}')-\delta_{\alpha\nu}\frac{\partial\delta(\boldsymbol{q}-\boldsymbol{p}')}{\partial q_j}A^i_{\mu\beta}(\boldsymbol{p},\boldsymbol{q}'),\end{aligned} \tag{26}$$

$$\begin{aligned}&\left[B^i_{\mu\nu}(\boldsymbol{p},\boldsymbol{p}'),B^j_{\alpha\beta}(\boldsymbol{q},\boldsymbol{q}')\right]_- =\\ &\qquad=\delta_{\mu\beta}\frac{\partial\delta(\boldsymbol{p}-\boldsymbol{q}')}{\partial q'_j}B^i_{\alpha\nu}(\boldsymbol{q},\boldsymbol{p}')-\delta_{\alpha\nu}\frac{\partial\delta(\boldsymbol{q}-\boldsymbol{p}')}{\partial p'_i}B^j_{\mu\beta}(\boldsymbol{p},\boldsymbol{q}'),\end{aligned} \tag{27}$$

$$\begin{aligned}&\left[E_\mu^\nu(\boldsymbol{p},\boldsymbol{p}'),A^i_{\alpha\beta}(\boldsymbol{q},\boldsymbol{q}')\right]_- =\\ &\qquad=\delta_{\mu\beta}\delta(\boldsymbol{p}-\boldsymbol{q}')A^i_{\alpha\nu}(\boldsymbol{q},\boldsymbol{p}')-\delta_{\alpha\nu}\frac{\partial\delta(\boldsymbol{q}-\boldsymbol{p}')}{\partial q_i}E_\mu^\beta(\boldsymbol{p},\boldsymbol{q}'),\end{aligned} \tag{28}$$

$$\begin{aligned}&\left[E_\mu^\nu(\boldsymbol{p},\boldsymbol{p}'),B^j_{\alpha\beta}(\boldsymbol{q},\boldsymbol{q}')\right]_- =\\ &\qquad=\delta_{\mu\beta}\frac{\partial\delta(\boldsymbol{p}-\boldsymbol{q}')}{\partial q'_j}E_\alpha^\nu(\boldsymbol{q},\boldsymbol{p}')-\delta_{\alpha\nu}\delta(\boldsymbol{q}-\boldsymbol{p}')B^j_{\mu\beta}(\boldsymbol{p},\boldsymbol{q}'),\end{aligned} \tag{29}$$

$$\begin{aligned}&\left[A^i_{\mu\nu}(\boldsymbol{p},\boldsymbol{p}'),B^j_{\alpha\beta}(\boldsymbol{q},\boldsymbol{q}')\right]_- =\\ &\qquad=\delta_{\mu\beta}\frac{\partial^2\delta(\boldsymbol{p}-\boldsymbol{q}')}{\partial p_i\partial q'_j}E_\alpha^\nu(\boldsymbol{q},\boldsymbol{p}')-\delta_{\alpha\nu}\delta(\boldsymbol{q}-\boldsymbol{p}')\frac{\partial^2E_\mu^\beta(\boldsymbol{p},\boldsymbol{q}')}{\partial p_i\partial q'_j}.\end{aligned} \tag{30}$$

Analogously, the relation may be established also for the derivatives

$$\frac{\partial E_{\mu\nu}(\boldsymbol{k},\boldsymbol{k}')}{\partial k_i},\qquad \frac{\partial E_{\mu\nu}(\boldsymbol{k},\boldsymbol{k}')}{\partial k'_j},\qquad \frac{\partial E^{\mu\nu}(\boldsymbol{k},\boldsymbol{k}')}{\partial k'_i},\qquad \frac{\partial E^{\mu\nu}(\boldsymbol{k},\boldsymbol{k}')}{\partial k_j}.$$

From the relations (26), (28) and (27), (29) it can be seen that the set of operators $\left\{E_\mu^\nu(\boldsymbol{p},\boldsymbol{p}'),A^i_{\mu\nu}(\boldsymbol{q},\boldsymbol{q}')\right\}$ and $\left\{E_\mu^\nu(\boldsymbol{p},\boldsymbol{p}'),B^i_{\mu\nu}(\boldsymbol{q},\boldsymbol{q}')\right\}$ also form a continuous Lie algebra.

For consideration of the continuous Lie algebras, we may introduce, by analogy to the classical Lie algebra theory, the concepts of the universal enveloping Lie algebra, the center, Casimir operators, etc. It is clear that all these concepts require refinement from the mathematical point of view since, so far as we know, such Lie algebras are not considered in the mathematical literature. As regards the problem of classification and formulation of all irreducible representations of the algebra $Sp_{2N}(\boldsymbol{k},\boldsymbol{k}')$, it leads, as can be seen from relations (16) and (17), to the problem of the description of all unitary non-equivalent commutation relations (18). This last problem, as is known, has not been solved up to the present time.

With the operators $E^\nu_\mu(\boldsymbol{p},\boldsymbol{p}')$, apparently, one cannot directly associate certain physical quantities (energy, momentum, angular momentum, etc.). However, the integral operators derived from these operators, i.e., operators of the type

$$E^\nu_\mu=\int d\boldsymbol{p}\,d\boldsymbol{p}'\,f^\nu_\mu(\boldsymbol{p},\boldsymbol{p}')E^\nu_\mu(\boldsymbol{p},\boldsymbol{p}'),$$

as can be seen from Sec. 1, can be assigned definite physical meanings.

It is possible to display other continuous Lie algebras. Thus, for example, the operators

$$\left\{E^\nu_\mu(\boldsymbol{k},\boldsymbol{k}'),E_{\mu_1\mu_2\ldots\mu_n}(\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n)=a_{\mu_1}(\boldsymbol{p}_1)a_{\mu_2}(\boldsymbol{p}_2)\ldots a_{\mu_n}(\boldsymbol{p}_n)\right\}$$

or

$$\left\{E^\nu_\mu(\boldsymbol{k},\boldsymbol{k}'),E^{\mu_1\mu_2\ldots\mu_n}(\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n)=a^+_{\mu_1}(\boldsymbol{p}_1)a^+_{\mu_2}(\boldsymbol{p}_2)\ldots a^+_{\mu_n}(\boldsymbol{p}_n)\right\}$$

also form continuous Lie algebras.

3. In [7] it was shown that the set of infinitesimal operators of homogeneous Lorentz group $O(3,1)$ and operators L_μ, entering into the relativistic equation

$$\left(L_\mu\frac{\partial}{\partial x_\mu}+\varkappa\right)\Phi(x_0,\boldsymbol{x})=0,\qquad \mu=0,1,2,3 \tag{31}$$

form a Lie algebra, which is an isomorphous set of infinitesimal operators of the de Sitter group $O(4,1)$. The function $\Psi(x_0,\boldsymbol{x})$ for a Lorentz transformation is transformed according to the representation $R=\sum\limits_{i=1}^{n}\oplus R_i^{l_0^i,l_1^i}$, where $\left(l_0^i,l_1^i\right)$ are pairs of numbers to which are given the irreducible representations of $O(3,1)$. Since the generators of group $O(3,1)$ and operators L_μ transform one solution of Eq.(31) to another solution, it is clear that in all solution sets of (31) there are realized irreducible representations of group $O(4,1)$. Since $\Phi(x_0,\boldsymbol{x})$ pertains to a apace which is a linear sum of spaces in which is realized the irreducible representation $O(3,1)$, then, obviously, the spectrum of the Casimir operators,

$$K_1=-\frac{1}{2}M_{\mu\nu}M_{\mu\nu},\qquad K_2=-\frac{1}{4}\varepsilon_{\mu\nu\rho\sigma}M_{\mu\nu}M_{\rho\sigma},\qquad \mu,\nu,\rho,\sigma=0,1,2,3$$

in this space will be discrete.

On the basis of the above it is natural to propose the following problem: to formulate an equation for the wave function Ψ which would be invariant relative to the Poincaré group and in all sets of solutions (solution space) of this equation of the spectrum of Casimir operators,

$$P^2 = P_\mu P_\mu, \qquad W^2 = W_\alpha W_\alpha, \qquad W_\alpha = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta} P_\beta M_{\gamma\delta} \tag{32}$$

would be discrete.

For the solution of this problem we will use one of the results of Foldy [8]. In [8] it was shown that with each irreducible unitary representation of the Poincaré group with mass m and spin s there can be associated a Schrödinger equation

$$H\Psi(x_0, \boldsymbol{x}) = i\frac{\partial\Psi(x_0, \boldsymbol{x})}{\partial t}, \tag{33}$$

where $H = (\boldsymbol{P}^2 + m^2)^{1/2}$ and $\Psi(x_0, \boldsymbol{x})$ is the $(2s+1)$-component wave function, quadratically integrable over the space variables. The question of the uniqueness of such correspondence (i.e., the question of possible existence of another equation which would also express the free motion of a relativistic particle with mass m and spin s) is left open in [8].

The single ambiguity, which apparently arises from the establishment of this correspondence, is tied to the extraction of the square root of the operator $\boldsymbol{P}^2 + m^2$. Actually there is no such ambiguity, since the operator $\boldsymbol{P}^2 + m^2$ is positive, and by virtue of theorems [10] the square root of a positive self-adjoint operator is uniquely determined. This is proof in itself that the stated correspondence is isomorphic.

If the Hamiltonian in Eq.(34) is expressed in the form

$$\widetilde{H} = \sqrt{\boldsymbol{P}^2 + M^2},$$

where $\boldsymbol{P}^2 = P_1^2 + P_2^2 + P_3^2$, and M^2 is the operator determined by formula (3) it can then be seen that (34) is a natural generalization of the relativistic Eq.(33) (in which the constant value m^2 is replaced by the operator M^2) in the case where the particle can take on various mass states.

In this manner, every relativistic equation expressing a free particle of mass m and spin s is unitarily equivalent to Eq.(33) ($H > 0$).

Since the Casimir operators P^2 and W^2 enter the theory on equal terms, then we may use the operator W^2 to obtain the equation of motion of a free particle. In this case, the equation which, generally speaking, is unitarily equivalent to Eq.(33), has the form

$$\sqrt{W^2 + m^2 s(s+1)}X(\boldsymbol{x}, t) = W_0 X(\boldsymbol{x}, t), \tag{33'}$$

i.e., between X and Ψ, there exists the coupling $X = V\Psi$, where V is the isometric operator.

Establishment of isomorphism between the Schrödinger equations and the irreducible unitary representations of the Poincaré group permits the writing of the equation which would have the above state properties. This equation has the form

$$\widetilde{H}\widetilde{\Psi}_+ = i\frac{\partial\widetilde{\Psi}_+}{\partial t}, \tag{34}$$

where

$$\widetilde{H}=\begin{pmatrix}\sqrt{\boldsymbol{P}^2+m_1^2} & 0 & 0\\ 0 & \sqrt{\boldsymbol{P}^2+m_2^2} & 0\\ 0 & 0 & \sqrt{\boldsymbol{P}^2+m_3^2}\end{pmatrix},\qquad \widetilde{\Psi}_+=\begin{pmatrix}\Psi_+^{m_1,s_1}(x_0,\boldsymbol{x})\\ \Psi_+^{m_2,s_2}(x_0,\boldsymbol{x})\\ \Psi_+^{m_3,s_3}(x_0,\boldsymbol{x})\end{pmatrix}.$$

The plus sign means that the sign value of the supplementary Casimir operator (the sign of the energy) for the Poincaré group [9] for these solutions is equal to $+1$.

The Schrödinger equation which would also be invariant under time reflection has the form

$$H'X=i\frac{\partial X}{\partial t}, \tag{35}$$

where $H'=\begin{pmatrix}\widetilde{H} & 0\\ 0 & \widetilde{H}\end{pmatrix}$, $X=\begin{pmatrix}\widetilde{\Psi}_+\\ \widetilde{\Psi}_-\end{pmatrix}$.

In conclusion, let us note that, in agreement with the theorems of O'Raifeltaigh [13] in the space of the solutions of Eq.(34) one cannot realize an irreducible representation of a finite dimensional Lie algebra which would contain the Poincaré algebra as a subalgebra.

If Eqs. (31) and (33) are considered equivalent (the unitary equivalence is constructed only for equations describing particles with spin 1/2), then the formula for $\Phi'(x_0,\boldsymbol{x})$, expressing motion of a particle which may be in various mass states, has the same formal appearance as the equations for elementary particles. However, the quantity $\varkappa$ is then not a constant but a variable, taking on the following values:

$$\varkappa=\pm m_1\lambda_1,\pm m_2\lambda_2,\pm m_3\lambda_3,\ldots, \tag{36}$$

where $m_i^2=p_0^2-\boldsymbol{p}^2$ and λ_i is some real nonzero eigenvalue of the operator L_0. In [15] it is shown that only for such values of $\varkappa$ do Eqs.(31) have nonzero plane wave solutions.

The relation (36) can be written in the form of the mass formula:

$$M=\varkappa L_0^{-1}. \tag{36$'$}$$

The operators which transform solutions of Eq.(31) with fixed mass to solutions which have a different mass are constructed from creation and annihilation operators by an analogous method (as in Sec. 1).

Note 2. Equation (31), as was shown in [11], excluding the Dirac equation, cannot be reduced by the unitary representations of the Foldy–Woythysen type to a Schrödinger equation. Consequently, the function $\Phi(x_0,\boldsymbol{x})$, strictly speaking, is not a wave function of a particle with fixed mass m and spin s.

The construction of a non-trivial theory of interaction based on Eq.(35), i.e., the introduction of potential in (35), by excluding those theories which with the help of unitary representations reduce to free particles (or as is generally stated, to the theory of free quasiparticles) [3], meets with difficulties in practice [14].

From the previous considerations, with every elementary particle there is associated a space R_i, in which is realized an irreducible representation of algebra P.

A particle which can be found in various excited states is associated with space R which is a linear sum of the spaces R_i. The inadequacy of such an approach lies in the fact that all elementary particles are considered as stable, and consequently possessing definite mass. Actually, a definite mass to these resonances cannot be ascribed, since particles are then nonstable.

To account for this fact, it is sufficient in the above mentioned considerations to change the linear sum to the linear integral:

$$R = \int \oplus R(m) g(m), \tag{37}$$

where the metric $g(m)$ is concentrated on the set composed of one or more points (depending on how many stable particles) and nonoverlapping intervals $[m_i', m_i'']$.

A more expanded formulation of equation (37) has the appearance

$$R = R^{s_0}(m_0) \oplus \sum_{i=1}^{m} R_i, \tag{38}$$

$$R_i = \int \oplus R^{s_i}(m) f^{s_i}(m) dm, \tag{39}$$

where $R^{s_i}(m)$ is the space in which is realized the irreducible representation of algebra P with mass m and spin s_i; the function $f^{s_i}(m)$, nonzero only in the interval (m_i', m_i''), characterizes the "smearing" (indeterminacy) of the mass of a resonance. If in (39) we replace $f^{s_i}(m)$ by a delta function, then R, as before, will be a linear sum of spaces R_i.

The operator $\left(P_\alpha^{(i)}\right)^2$ in R_i is determined in the following manner:

$$\left(P_\alpha^{(i)}\right)^2 R_i = \int \oplus \left(P_\alpha^{(i)}\right)^2 R^{s_i}(m) f^{s_i}(m) dm = \int \oplus m^2 R^{s_i}(m) f^{s_i}(m) dm. \tag{1'}$$

The operators $P_\alpha^{(i)}$, $M_{\mu\nu}^{(i)}$, M^2, P^2 can be determined by an analogous method. A more detailed presentation of results obtained by taking account of "smearing" of the resonances will be given in another paper.

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