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PROBLEMS OF APPROXIMATION THEORY IN LINEAR SPACES

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We present a survey of results related to the approximation characteristics of the spaces S^p_φ and their generalizations. The proposed approach enables one to obtain solutions of problems of classical approximation theory in abstract linear spaces in explicit form. The results obtained yield statements that are new even in the case of approximations in the functional Hilbert spaces L_2.

In the present paper, we give a survey of results related to the approximation characteristics of the spaces S^p_φ and their generalizations. This work is a result of investigations aimed at finding new approaches to problems of the theory of approximation of functions of many variables and, in particular, periodic functions. There are many problems in this theory, and the most important of them are, probably, the following: the choice of approximating aggregates and the choice of classes of functions and approximation characteristics. In the one-dimensional case, the form of the simplest aggregate is determined by the natural order of a natural series, whereas in the multidimensional case, i.e., in the case where one deals with a set $\mathfrak{X}$ (a Banach space of functions $f(t) = f(t_n, \ldots, t_m)$, $t \in R^m$, of m variables), the choice of the simplest aggregates becomes problematic. The first difficulties here begin with the problem of the choice of an analog of a partial sum for the multiple series

$$\sum_{k \in Z^m} c_k, \quad k = (k_1, \ldots, k_m), \tag{0.1}$$

where Z^m is the integer lattice in R^m.

It seems quite natural to introduce "rectangular" sums; the corresponding approximating aggregates in the periodic case are trigonometric polynomials of the form

$$\sum_{k_1=-n_1}^{n_1} \ldots \sum_{k_m=-n_m}^{n_m} c_{k_1,\ldots,k_m} e^{i(k_1 t_1 + \ldots + k_m t_m)}. \tag{0.2}$$

However, partial sums of a multiple series can be introduced in many ways, e.g., as follows:

Let $\{G_\alpha\}$ be a family of bounded domains in R^m that depend on the numerical parameter α and are such that any vector $n \in Z^m$ belongs to all domains G_α for sufficiently large values of α. Then the expression

$$\sum_{k \in G_\alpha} c_k$$

is called a partial sum of series (0.1) corresponding to the domain G_α. By analogy, one introduces the corresponding partial sums of the trigonometric series:

$$\sum_{k \in G_\alpha} c_k e^{ikx} = \sum_{k \in G_\alpha} c_{k_1,\ldots,k_m} e^{i(k_1 x_1 + \ldots + k_m x_m)}. \tag{0.3}$$

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It turned out quite soon that, in the case of approximation of functions from the Sobolev classes $W_p^r(R^m)$, instead of the rectangular sums (2) it is "more productive" to use sums of the form (3) constructed on the basis of domains formed by certain hyperbolas. These domains, first introduced by Babenko in [1, 2], were called hyperbolic crosses.

The appearance of the notion of hyperbolic cross had a considerable impact on the development of the theory of approximation of functions of many variables. Many important and interesting results were obtained in this direction (for details, see, e.g., [3–13]).

However, hyperbolic crosses turned out to be of little use in the approximation of functions of classes different from Sobolev classes. In this connection, there naturally arise assumptions that, for every individual class $\mathfrak{N}$ (or any family of these classes), it is necessary to select "its own" family of domains G_α defined by the parameters of this class.

It should also be noted that the quality of many results obtained for Sobolev classes with the use of hyperbolic crosses cannot be regarded as perfect. As a rule, results of approximation in the spaces $L_p(R^m)$ are of order character, whereas exact results are obtained only in Hilbert spaces for $p = 2$. Time will show whether this situation is caused by inadequate analysis or by a disharmony between original data and problems posed. At least, one may assume that, parallel with a successful choice of approximating aggregates, another factor that complicates finding exact results on the basis of approximations in the multidimensional case (and in the one-dimensional case as well) is the long-established practice of considering problems exactly in the spaces $L_p(R^m)$. In the periodic case, the norm in these spaces

$$\|f\|_{L_p(R^m)} = \left(\int\limits_{Q_m} |f(t)|^p dt\right)^{\frac{1}{p}}, \quad Q_m = \{t \in R^m,\ 0 \le t_i \le 2\pi,\ \ i = \overline{1,m}\},$$

characterizes only the mean value of the pth power of the modulus of the function considered, and, possibly, this information is insufficient for the derivation of the required results in the general case.

For $p = 2$, the following equality is well known:

$$\|f\|_{L_2(R^m)} = \left(\sum_{|k|\ge 0} |c_k|^2\right)^{\frac{1}{2}},$$

where $c_k = c_{k_1 \ldots k_m}$ are the Fourier coefficients of the function f. Hence, in this case, the norm of the function f completely characterizes the entire set $\{c_k\}_{k \in Z^m}$ (for other values of p, similar equalities are possible only in trivial cases). Therefore, it seems reasonable to introduce norms of functions using quantities related to their Fourier coefficients. This approach was considered in a series of works of the author and his followers (see [14–32]). In particular, this approach enables one to generalize ideas and methods of approximation theory to abstract linear spaces, which, in turn, enables one to consider functions from general positions of analysis and to obtain fairly informative results, some of which are presented in this paper.

1. Spaces S^p_Φ

Let us define the spaces in which problems of approximation theory will be posed and solved.

Let $\mathfrak{X}$ and Y be linear spaces of vectors x and y, respectively. Assume that a linear operator Φ acting in Y is defined on $\mathfrak{X}$, and a functional f is defined on a certain subset $Y' \subset Y$. Further, let $E(\Phi)$ be the range of values of the operator Φ and let $\mathfrak{X}'$ be the preimage of the set $Y'E(\Phi)$ under the mapping Φ. In this case, we can define a functional f' on $\mathfrak{X}'$ by setting

$$f'(x) = f(\Phi(x)), \quad x \in \mathfrak{X}'. \tag{1.1}$$

Choosing a functional that defines a norm (or a quasinorm) on Y' as f, we establish that equality (1.1) defines an analogous quantity on $\mathfrak{X}'$. These arguments form a basis for further constructions.

Let $(R^m, d\mu)$, $m \geq 1$, be an m-dimensional Euclidean space of points $t = (t_1, \ldots, t_m)$ equipped with a certain σ-finite measure $d\mu$ and let A be a μ-measurable subset of $(R^m, d\mu)$ whose μ-measure is equal to a, where either a is finite or $a = \infty$:

$$\operatorname{mes}_\mu A = |A|_\mu = a, \quad a \in (0, \infty].$$

By $Y = Y(A, d\mu)$ we denote the set of all functions $y = y(t)$ defined on A and measurable with respect to the measure $d\mu$. For given $p \in (0, \infty]$, let $L_p(A, d\mu)$ denote subsets of functions from $Y(A, d\mu)$ for which the following quantity is finite:

$$\|y\|_{L_p(A,d\mu)} = \begin{cases} \left(\int\limits_A |y(t)|^p \, dt \right)^{1/p}, & p \in (0, \infty), \\ \operatorname*{ess\,sup}\limits_{t \in A} |\, y(t)|, & p = \infty. \end{cases} \tag{1.2}$$

It is known that, on $L_p(A, d\mu)$, this functional defines a norm for $p \geq 1$ and a quasinorm for $p \in (0, 1)$.

Now let $\mathfrak{X}$ be a certain linear space of vectors x and let Φ be a linear operator acting from $\mathfrak{X}$ into Y:

$$\Phi \colon \mathfrak{X} \to Y(A, d\mu), \quad \Phi(x) \overset{\text{df}}{=} \hat{x}, \quad x \in \mathfrak{X}, \quad \hat{x} = Y(A, d\mu).$$

We set

$$S^p_\Phi = S^p_\Phi(\mathfrak{X}; Y) = \left\{ x \in \mathfrak{X} \colon \|\hat{x}\|_{L_p(A,d\mu)} < \infty \right\}, \quad p \in (0, \infty]. \tag{1.3}$$

Thus, the set S^p_Φ is the preimage of the set $L_p(A, d\mu)$ in $\mathfrak{X}$ under the mapping Φ.

Elements $x_1, x_2 \in S^p_\Phi$ are assumed to be identical if $\hat{x}_1(t) = \hat{x}_2(t)$ almost everywhere with respect to the measure $d\mu$.

For elements $x_1, x_2 \in S^p_\Phi$, $p \in (0, \infty]$, we define the Φ-distance between them by the equality

$$\rho_\Phi(x_1; x_2)_p = \|\Phi(x_1 - x_2)\|_{L_p(A,d\mu)}.$$

A zero element of the set S^p_Φ is an element θ for which $\hat{\theta}(t) = 0$ almost everywhere on A.

The distance $\rho_\Phi(\theta; x)_p$, $x \in S^p_\Phi$, is called the Φ-norm of the element and is denoted by $\|x\|_p = \|x\|_{p,\Phi}$. Thus, by definition,

$$\|x\|_p = \|x\|_{p,\Phi} = \rho_\Phi(\theta; x)_p = \|\hat{x}\|_{L_p(A,d\mu)}. \tag{1.4}$$

In this case, S^p_Φ is a linear space: the operations of addition of elements and their multiplication by numbers defined in $\mathfrak{X}$ remain true for any pair $x_1, x_2 \in S^p_\Phi$. In addition, for any numbers λ_1 and λ_2, the element

$x_3 = \lambda_1 x_1 + \lambda_1 x_2$ belongs to S^p_Φ. Indeed, since $x_3 \in \mathfrak{X}$, we have $\hat{x}_3(t) = \lambda_1 \hat{x}_1(t) + \lambda_2 \hat{x}_2(t)$. If $p \geq 1$, then, by virtue of the Minkowski inequality, we get

$$\|x_3\|_p = \|\hat{x}_3(t)\|_{L_p(A,d\mu)} \leq |\lambda_1| \, \|\hat{x}_1\|_{L_p(A,d\mu)} + |\lambda_2| \, \|\hat{x}_2\|_{L_p(A,d\mu)} = |\lambda_1\| \, \|x_1\|_p + |\lambda_2| \, \|x_2\|_p.$$

For $p \in (0,1)$, using the inequality

$$|a+b|^p \leq |a|^p + |b|^p, \quad 0 \leq p < 1,$$

we obtain

$$\|x_3\|_p = \left(\int\limits_A |\lambda_1 \hat{x}_1(t) + \lambda_2 \hat{x}_2(t)|^p \, d\mu \right)^{\frac{1}{p}} \leq 2^{1/p}(|\lambda_1| \, \|x_1\|_p + |\lambda_2| \, \|x_2\|_p),$$

i.e., we always have $x_3 \in S^p_\Phi$.

It is clear that the functional $\|\cdot\|_p$ satisfies all axioms of a norm for $p \geq 1$ and all axioms of a quasinorm for $p \in (0,1)$. Therefore, S^p_Φ is a linear normed space for $p \geq 1$ and a space with quasinorm for $p \in (0,1)$.

Consider several simplest realizations of the constructions described. We say that a certain space $\mathfrak{N}$ is a special case of the space S^p_Φ if it can be obtained by the proper choice of the space $\mathfrak{X}$, measure $d\mu$, and operator Φ.

1.1. Space S^p_φ. Let $\mathfrak{X}$ be a linear complex space and let $\varphi = \{\varphi_k\}_{k=1}^\infty$ be a fixed countable system in it. Assume that, for any pair $x, y \in \mathfrak{X}$ in which at least one vector belongs to φ, a certain number (the "scalar product" (x,y)) that satisfies the following conditions is defined:

(i) $(x,y) = \overline{(y,x)}$, where $\bar{z}$ is the complex conjugate of the number z;

(ii) $(\lambda x_1 + \mu x_2, y) = \lambda(x_1, y) + \mu(x_2, y)$, where λ and μ are arbitrary numbers;

(iii) $(\varphi_k, \varphi_l) = \begin{cases} 0, & k \neq l, \\ 1, & k = l. \end{cases}$

We associate every element $x \in \mathfrak{X}$ with a system of numbers $\hat{x}(k)$ by the equalities

$$\hat{x}(k) = \hat{x}_\varphi(k) = (x, \varphi_k), \quad k = 1, 2, \ldots \quad (k \in N), \tag{1.5}$$

and, for fixed $p \in (0, \infty)$, we set

$$S^p_\varphi = S^p_\varphi(\mathfrak{X}) = \left\{ x \in \mathfrak{X} \colon \sum_{k=1}^\infty \left| \hat{f}_\varphi(k) \right|^p < \infty \right\}. \tag{1.6}$$

Elements $x, y \in S^p_\varphi$ are assumed to be identical if $\hat{x}_\varphi(k) = \hat{y}_\varphi(k)$ for all $k \in N$.

For vectors $x, y \in \mathfrak{X}$, the φ-distance between them is defined as follows:

$$\rho_\varphi(x,y)_p = \left(\sum_{k=1}^\infty |\hat{x}_\varphi(k) - \hat{y}_\varphi(k)|^p \right)^{\frac{1}{p}}.$$

A vector θ such that $\hat{\theta}_\varphi(k) = 0$ for all $k \in N$ is called a zero element of the space S^p_φ. The distance $\rho_\varphi(\theta, x)_p$, $x \in S^p_\varphi$, is called the φ-norm of the element x and is denoted by $\|x\|_{p,\varphi}$. Thus,

$$\|x\|_{p,\varphi} = \rho_\varphi(\theta, x)_p = \left(\sum_{k=1}^{\infty} |\hat{x}_\varphi(k)|^p\right)^{\frac{1}{p}}. \tag{1.7}$$

The spaces S^p_φ are special cases of the spaces S^p_Φ. Indeed, in the space $\mathfrak{X}$ we define an operator Φ that associates every $x \in \mathfrak{X}$ with a sequence $y = \{\hat{x}_k\}_{k=1}^{\infty}$. As the set $(R^m, d\mu)$, we take the space R^1 with a measure $d\mu$ whose support is the set Z^1 of integer-valued points k at which $\mu(k) \equiv 1$. We set $A = \{k \in Z^1, k \geq 1\}$. In this case, $Y(A, d\mu)$ is the set of all sequences y, and functional (1.2) has the form

$$\|y\|_{L_p(A,d\mu)} = \left(\sum_{k=1}^{\infty} |y_k|^p\right)^{\frac{1}{p}}, \quad p \in (0, \infty).$$

Let S be the set of all sequences of complex numbers

$$S = \{x = (x_1, x_2, \ldots),\ \ x_k \in C\},$$

in which the operations of addition and multiplication are defined in the standard way:

$$(x_1, x_2, \ldots) + (y_1, y_2, \ldots) = (x_1 + y_1, x_2 + y_2, \ldots),$$

$$\lambda(x_1, x_2, \ldots) = (\lambda x_1, \lambda x_2, \ldots), \quad \lambda \in C.$$

In this case, S is a linear space. As $\mathfrak{X}$, we choose the set S. As φ, we choose the system $e = \{e_k\}_{k=1}^{\infty}$, where $e_k = (\varepsilon_1, \varepsilon_2, \ldots)$ and

$$\varepsilon_i = \begin{cases} 1, & i = k, \\ 0, & i \neq k. \end{cases}$$

We define a "scalar product" by setting

$$(x, e_k) = \hat{x}_e(k) = x_k, \quad (e_k, x) = \bar{x}_k \quad (x = (x_1, \ldots, x_k, \ldots)).$$

For this operation, conditions (ii) and (iii) are automatically satisfied. We associate every element $x \in \mathfrak{X}$ with a system of numbers $\hat{x}(k)$,

$$\hat{x}(k) = x_k, \quad k = 1, 2, \ldots,$$

and, for fixed $p \in (0, \infty)$, according to (1.6), we define spaces S^p_e:

$$S^p_e = S^p_e(\mathfrak{X}) = \left\{x \in \mathfrak{X}\colon \sum_{k=1}^{\infty} |x_k|^p < \infty\right\}.$$

In this case, by virtue of (1.7), the φ-norm of an element $x \in S^p_e$ has the form

$$\|x\|_{p,e} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}}.$$

We see that S^p_e coincide with the known spaces l_p.

As above, we take the space S as $\mathfrak{X}$. As φ, we take the system e' obtained from e by removing some elements e_{i_j}, $j = 1, 2, \ldots,$ from it. Using the scheme considered above, we construct spaces $S^p_{e'}$.

It is clear that the φ-norm in $S^p_{e'}$ constructed according to (1.7) satisfies the inequality

$$\|x\|_{p,e'} \le \|x\|_{p,e},$$

and, hence, $S^p_e \subset S^p_{e'}$. It is also clear that the set $S^p_{e'} \setminus S^p_e$ can be nonempty, i.e., the set $S^p_{e'}$ can be broader than the set l_p.

1.1′. Spaces $S^{p,\mu}_{\varphi}$. These spaces are introduced by analogy with the spaces S^p_{φ}, but the functionals

$$\left(\sum_{k=1}^{\infty} |\cdot|^p\right)^{\frac{1}{p}}$$

in the equalities corresponding to (1.5)–(1.7) are replaced by the functionals

$$\left(\sum_{k=1}^{\infty} |\cdot|^p \mu_k^p\right)^{\frac{1}{p}},$$

where $\mu = \{\mu_k\}_{k=1}^{\infty}$ is a given system of nonnegative numbers, $\mu_k \ge 0$, $k \in N$; in particular, if $\mu_k \equiv 1$, then $S^{p,\mu}_{\varphi} = S^p_{\varphi}$.

It is clear that these spaces are special cases of the spaces S^p_{Φ}. Here, as in the case of the spaces S^p_{φ}, as the set $(R^m, d\mu)$ we take the space R^1 with a measure $d\mu$ concentrated on the set Z^1 of integer-valued points k at which $\mu(k) = \mu_k$. We also set $A = \{k \in Z^1,\ k \ge 1\}$.

For more information on these spaces, see Sec. 4 of the present paper.

1.2. Space $S^p_{\mathfrak{F}}(L)$. As before, let R^m, $m \ge 1$, be the m-dimensional Euclidean space, let $\mathfrak{X} = (\mathfrak{X}_1, \ldots, \mathfrak{X}_m)$ be its elements, let Z^m be the integer lattice in R^m, and let $xy = x_1y_1 + \ldots + x_my_m$, $x, y \in R^m$. Further, let $L = L(R^m, 2\pi)$ denote the set of all functions $f(x) = f(x_1, \ldots, x_m)$ 2π-periodic in each variable and summable with respect to the ordinary Lebesgue measure on the cube of periods Q^m, where

$$Q^m = \{x\colon x \in R^m,\ -\pi \le x_k \le \pi,\ \ k = 1, 2, \ldots, m\}.$$

As $\mathfrak{X}$, we take the space $L(R^m, 2\pi)$. On this space, we define an operator Φ (denoted below by $\mathfrak{F}$) by setting

$$\mathfrak{F}(f) = (2\pi)^{-m/2} \int\limits_{Q^m} f(x)\, e^{-ikx} dx = \hat{f}(k), \quad k \in Z^m.$$

This operator maps the space $L(R^m, 2\pi)$ into the set Y of functions $y(t)$ defined on the integer lattice Z^m. Let $d\mu$ be a measure in the space R^m whose support is the set Z^m, where it is equal to 1. In this case, functional (1.2) takes the form

$$\|y\|_{L_p(R^m,d\mu)}L_p(R^m, d\mu) = \left(\int\limits_{R_m} |y(t)|^p d\mu\right)^{\frac{1}{p}} = \left(\sum_{k\in Z^m} \left|\hat{f}(k)\right|^p\right)^{\frac{1}{p}}, \quad p\in(0,\infty),$$

and the space S^p_Φ $\big($denote it by $S^p_{\mathfrak{F}}(L)\big)$ is defined by the relation

$$S^p_{\mathfrak{F}}(L) = \left\{ f\in L\colon \left(\sum_{k\in Z^m} \left|\hat{f}(k)\right|^p\right)^{\frac{1}{p}} \le \infty \right\}.$$

Note that the spaces $S^p_{\mathfrak{F}}(L)$ coincide with the above-considered spaces $S^p_\varphi(L)$ generated by the set L and the system $\varphi = \{\tau_s\}_{s=1}^\infty$, where

$$\tau_s = (2\pi)^{-m/2} e^{ik_s x}, \quad k_s \in Z^m, \quad s = 1, 2, \ldots,$$

which is obtained from the system

$$(2\pi)^{-m/2} e^{ikx}, \quad k\in Z^m,$$

by an arbitrary fixed enumeration of its terms.

1.3. Consider an example where the spaces S^p_Φ can be nonseparable.

We choose the space of functions $L_2(R^m)$ as $\mathfrak{X}$ and the space $L^2(R^m)$ with ordinary Lebesgue norm as A and define the operator Φ by the Fourier transformation

$$\Phi(f) = \hat{f}(t) = \mathfrak{F}(f;t) = (2\pi)^{-m/2} \int\limits_{R^m} f(x)\, e^{-itx} dx, \quad f\in L_2(R^m).$$

It is known (see, e.g., [33], Chap. I) that the operator $\mathfrak{F}$ is unitary with respect to $L^2(R^m)$. Therefore, the Φ-norm $\|f\|_{2,\Phi}$ of an element f coincides with its norm in the space $L^2(R^m)$:

$$\|f\|_{2,F} = \|f\|_{L_2(R^m)}. \tag{1.8}$$

In this case, by virtue of relation (1.3), the space $S^2_\Phi(L_2(R^m), R^m, dx)$ has the form $S^2_\Phi = \{f\colon f\in L_2(R^m)\}$, i.e., $S^2_\Phi = \mathfrak{X} = L_2(R^m)$.

1.4. Following the scheme presented in the previous example, one constructs spaces S^2_Φ by taking any operator unitary on the set $L_2(A, d\mu)$, where A is a certain manifold in R^m and $d\mu$ is a certain σ-finite measure in R^m, instead of the Fourier transformation. For example, let $L_2(A, d\mu)$ be the set $L_2(R^1_+)$ of functions $f(t)$ square summable in the Lebesgue sense on the semiaxis $(0,\infty)$ and let Φ be the Hankel transformation

$$H_v f = H_v(f;x) = \hat{f}(x) = \hat{f}_v(x) = x^{-(v+1/2)} \frac{d}{dx} \int\limits_0^\infty x^{v+1} J_{v+1}(xt) \frac{f(t)}{\sqrt{t}}\, dt, \tag{1.9}$$

where v is a certain number, $v \geq -1$, and $J_\alpha(z)$ is the Bessel function of the first kind of order α.

It is known that the Hankel transformation is generated by the operator H_v, which is unitary on $L_2(R^1_+)$ and coincides with its inverse (see, e.g., [34], Chap. III). Therefore, the following analog of equality (1.15) is true:

$$\|f\|_{2,H_v} = \|f\|_{L_2(R^1_+)}.$$

Hence, $S^2_{H_v} = \{f : L_2(R^1_+)\}$, i.e., in this case, we have

$$S^2_{H_v} = \mathfrak{X} = L_2(R^1_+).$$

1.5. Consider the special case of the spaces S^p_Φ generated by the identity operator, i.e, the case where $\Phi \equiv I$. It is clear then that $\mathfrak{X} = Y(A, d\mu)$, $\hat{x} = x$, and, by virtue of (1.3),

$$S^p_L = \{x \in \mathfrak{X} \colon \|x\|_{L_p(A,d\mu)} < \infty\} = L_p(A, d\mu), \quad p \in (0, \infty).$$

2. Multiplicators. Approximating aggregates and Objects of Approximation

As approximating aggregates for elements $x \in S^p_\Phi$, one uses elements from S^p_Φ whose images have supports γ_σ of a given measure σ. It is clear that, in the classical case, this principle is used in the construction of, e.g., trigonometric polynomials for the approximation of a given periodic function if the operator Φ is understood as a mapping of functions into the set of their Fourier coefficients. In the general case, there arise certain difficulties caused by the fact that the spaces S^p_Φ may be incomplete. In this connection, we introduce the following definitions:

Let $\omega = \omega(t)$ be a certain function from $Y(A, d\mu)$. By M^ω_Φ we denote an operator that acts from $\mathfrak{X}$ into $\mathfrak{X}$ and associates $x \in \mathfrak{X}$ with an element $x_\omega \in \mathfrak{X}$ such that if $\Phi(x) = \hat{x}(t)$, then $\hat{x}_\omega(t) = \Phi(x_\omega) = \omega(t)\hat{x}(t)$ almost everywhere. The operator M^ω_Φ is called the multiplicator of the operator Φ generated by the function ω. Let $\Omega_\Phi(\mathfrak{X}) = \Omega_\Phi(\mathfrak{X}, Y)$ denote the subset of functions ω from $Y(A, d\mu)$ for which the multiplicators M^ω_Φ exist.

If $\mathfrak{N}$ and $\mathfrak{N}'$ are certain subsets of $\mathfrak{X}$, $\omega \in \Omega_\Phi(\mathfrak{X})$, and the operator M^ω_Φ maps $\mathfrak{N}$ into $\mathfrak{N}'$, then we say that M^ω_Φ has the type $(\mathfrak{N}, \mathfrak{N}')$. In particular, if M^ω_Φ maps S^p_Φ into S^p_Φ, then the operator M^ω_Φ has the type (S^p_Φ, S^p_Φ) or, briefly, the type (p, p). Let Ω^p_Φ denote the set of functions ω generating operators of the type (p, p).

Thus, if $\omega \in \Omega^p_\Phi$ and the operator M^ω_Φ acts from S^p_Φ, then it also acts in S^p_Φ; moreover, every $x \in S^p_\Phi$ is associated with an element $x_\omega = M^\omega_\Phi(x)$ for which the following equality holds almost everywhere on A:

$$\hat{x}_\omega(t) = \Phi(x_\omega) = \omega(t)\hat{x}(t), \quad \hat{x}_\omega \in L_p(A, d\mu). \tag{2.1}$$

Assume that, for a given $\sigma > 0$, γ_σ is a μ-measurable set in A,

$$\mathrm{mes}_\mu \gamma_\sigma \stackrel{\mathrm{df}}{=} |\gamma_\sigma| = \sigma, \quad \sigma \leq a,$$

and $\lambda = \lambda(t)$ is a measurable function with support γ_σ. Also assume that, for a given $p \in (0, \infty)$, we have $\lambda \in \Omega^p_\Phi$ and $U_{\gamma_\sigma}(x; \lambda) \stackrel{\mathrm{df}}{=} x_\lambda = M^\lambda_\Phi(x)$, so that, by virtue of (2.1),

$$\hat{U}_{\gamma_\sigma}(x; \lambda) = \Phi(U_{\gamma_\sigma}(x; \lambda)) = \begin{cases} \lambda(t)\hat{x}(t), & t \in \gamma_\sigma, \\ 0, & t \bar{\in} \gamma_\sigma, \quad x \in S^p_\Phi. \end{cases} \tag{2.2}$$

The elements $U_{\gamma_\sigma}(x;\lambda)$ are exactly the elements considered as approximating aggregates for $x \in S^p_\Phi$. If $\lambda(t) \equiv 1$ on γ_σ, i.e., $\lambda(t)$ coincides with the characteristic function $\chi_{\gamma_\sigma}(t)$ of γ_σ, then we set $U_{\gamma_\sigma}(x;\chi_{\gamma_\sigma}) = U_{\gamma_\sigma}(x)$.

Let $\Gamma_\sigma = \Gamma_{\sigma(A)}$ be the set of all measurable subsets from A whose measures are equal to σ. We say that, for a given $p > 0$, the operator Φ satisfies condition (A_p) if, for all sets $\gamma_\sigma \in \Gamma_\sigma$, the functions $\chi_{\gamma_\sigma}(t)$ belong to Ω^p_Φ for any $\sigma \in [0, a)$. Thus, if Φ satisfies condition (A_p), then all elements $U_{\gamma_\sigma}(x)$ are defined for any $x \in S^p_\Phi$ and belong to S^p_Φ. The element $U_{\gamma_\sigma}(x)$ is called the restriction of an element x of rank σ. The element $U_{\gamma_\sigma}(x;\lambda)$ is called the λ-restriction of x of rank σ.

Let p be an arbitrary positive number and let $x \in S^p_\Phi$. Then, by virtue of (1.4) and (2.2), we get

$$\|x - U_{\gamma_\sigma}(x;\lambda)\|^p_p = \|\hat{x}(t) - \hat{U}_{\gamma_\sigma}(x;\lambda;t)\|^p_{L_p(A,d\mu)} = \int\limits_{\gamma_\sigma} |1-\lambda(t)|^p |\hat{x}(t)|^p \, d\mu + \int\limits_{A/\gamma_\sigma} |\hat{x}(t)|^p d\mu.$$

This yields the following statement:

Proposition 2.1. *Suppose that* $p \in (0,\infty)$, $x \in S^p_\Phi = S^p_\Phi(\mathfrak{X}; Y)$, $\gamma_\sigma \in \Gamma_\sigma$, *and the operator* Φ *satisfies condition* (A_p). *Then*

$$\mathcal{E}_{\gamma_\sigma}(x)_p \overset{\text{df}}{=} \inf_{\lambda \in \Omega^p_\Phi} \|x - U_{\gamma_\sigma}(x;\lambda)\|_p \; = \; \|x - U_{\gamma_\sigma}(x)\|_p.$$

Moreover, the following equality is true:

$$\mathcal{E}_{\gamma_\sigma}(x)_p \; = \; \|x\|^p_p - \int\limits_{\gamma_\sigma} |\hat{x}(t)|^p d\mu. \tag{2.3}$$

Thus, if $\chi_{\gamma_\sigma} \in \Omega^p_\Phi$, then, among all elements $U_{\gamma_\sigma}(x;\lambda)$ generated by the multiplicators M^λ_Φ and satisfying condition (2.2), the element $U_{\gamma_\sigma}(x)$ deviates least from the element x with respect to the Φ-norm in the space S^p_Φ, i.e., among all λ-restrictions of x of given rank σ, the restriction of exactly this element for $\lambda(t) \equiv 1$ is closest to x. It is clear that this property is an analog of the minimal property of Fourier sums in the Hilbert spaces L_2.

Let $\Gamma = \{\gamma_\sigma\}_{\sigma>0}$, $|\gamma_\sigma| = \sigma$, be a family of measurable subsets of A that exhausts the entire set A as $\sigma \to \infty$, i.e., it possesses the property according to which any point $t \in A$ belongs to all sets γ_σ for all sufficiently large values of σ, so that

$$\lim_{\sigma\to\infty} \int\limits_{\gamma_\sigma \in \Gamma} |\hat{x}(t)|^p \, d\mu = \int\limits_A |\hat{x}(t)|^p \, d\mu \quad \forall x \in S^p_\Phi. \tag{2.4}$$

Combining relations (2.3) and (2.4), we obtain

$$\lim_{\substack{\sigma\to\infty \\ \gamma_\sigma \in \Gamma}} \mathcal{E}_{\gamma_\sigma}(x)_p = 0 \quad \forall x \in S^p_\Phi.$$

We now define objects of approximation—the unions of elements $x \in \mathfrak{X}$ corresponding to the notion of a class of functions in approximation theory. These objects, as well as approximating aggregates, are introduced with the use of multiplicators. However, in this case, it is more convenient to use a different terminology that is closer to the traditional one. Let $\Psi = \Psi(t)$ be an arbitrary function from $\Omega_\Phi(\mathfrak{X})$ and let M^Ψ_Φ be the multiplicator of the operator Φ generated by this function. In this case, the image x_Ψ of an element x under the mapping M_Φ^Ψ is called the Ψ-integral of the element $x,$ and we write $M_\Phi^\Psi(x) = x_\Psi = \jmath^\Psi x.$ It is sometimes convenient to call x the Ψ-derivative of x_Ψ and write $x = D^\Psi x_\Psi.$

Thus, if x_Ψ is the Ψ-integral of $x,$ then

$$\hat{x}_\Psi = \Phi(\jmath^\Psi x) = \Psi(t)\hat{x}(t) \tag{2.5}$$

almost everywhere.

If $\mathfrak{N}$ is a certain subset of $\mathfrak{X},$ then $\Psi\mathfrak{N}$ denotes the set of Ψ-integrals of all $x \in \mathfrak{N}$ for which they exist. In particular, if U_Φ^p is the unit ball in a certain space $S_\Phi^p,$ i.e.,

$$U_\Phi^p = \{x\colon x \in S_\Phi^p,\ \|x\|_{p,\Phi} \le 1\},$$

then ΨU_Φ^p is the set of Ψ-integrals of all $x \in U_\Phi^p$ for which these integrals exist.

Comparing relations (2.5) and (2.1), we conclude that, as functions Ψ for which the Ψ-integral is well defined, we can take any function from $\Omega_\Phi(S_\Phi^p).$ In this case, we have $\Psi S_\Phi^p \subset S_\Phi^p.$

The sets ΨU_Φ^p are exactly the objects for which traditional problems of approximation theory are considered in the present paper.

3. Approximation Characteristics of the Sets ΨU_Φ^p

Consider the following quantities: For any $\gamma_\sigma \in \Gamma,$ we set

$$\mathcal{E}_{\gamma_\sigma}(x)_q = \inf_{\lambda\in\Omega_\Phi^p} \|x - U_{\gamma_\sigma}(x;\lambda)\|_{q,\Phi} \quad x \in S_\Phi^p,$$

$$\mathcal{E}_{\gamma_\sigma}(\Psi U_\Phi^p)_q = \sup_{x\in\Psi U_\Phi^p} \mathcal{E}_{\gamma_\sigma}(x)_q,$$

and

$$D_\sigma(\Psi U_\Phi^p)_q = \inf_{\gamma_\sigma\in\Gamma_\sigma} \mathcal{E}_{\gamma_\sigma}(\Psi U_\Phi^p)_q.$$

In the case of approximation of periodic functions by trigonometric polynomials, the quantity $\mathcal{E}_{\gamma_\sigma}(x)_q$ corresponds to the best approximation of the function x by polynomials of degree $\sigma,$ the quantity $\mathcal{E}_{\gamma_\sigma}(\Psi U_\Phi^p)_q$ corresponds to an upper bound on a given set of functions of these best approximations, and the quantity $D_\sigma(\Psi U_\Phi^p)_q$ resembles the trigonometric width of order σ of the set $\Psi U_\Phi^p.$

We also consider the following characteristics, which, in the periodic case, correspond to quantities related to the best σ-term approximation:

$$e_\sigma(x)_q = \inf_{\gamma_\sigma\in\Gamma_\sigma} \mathcal{E}_{\gamma_\sigma}(x)_q = \inf_{\gamma_\sigma\in\Gamma_\sigma} \inf_{\lambda\in\Omega_\Phi^p} \|x - U_{\gamma_\sigma}(x;\lambda)\|_{q,\Phi}\, x \in S_\Phi^p \tag{3.1}$$

and

$$e_\sigma(\Psi U_\Phi^p)_q = \sup_{x\in\Psi U_\Phi^p} e_\sigma(x)_q. \tag{3.2}$$

Below, we restrict ourselves to the case $p = q$. We also assume that the corresponding characteristic functions $\chi_{\gamma_\sigma}(\cdot)$ belong to Ω^p_Φ, i.e., the operator Φ satisfies condition (A_p). In this case, according to Proposition 2.1, quantities (3.1) and (3.2) with $\lambda(t) = \chi_{\gamma_\sigma}(t)$ are of prime interest. In this connection, we set

$$\mathcal{E}_{\gamma_\sigma}(x)_p = \|x - U_{\gamma_\sigma}(x)\|_{p,\Phi}, \quad x \in S^p_\Phi, \tag{3.3}$$

$$\mathcal{E}_{\gamma_\sigma}(\Psi U^p_\Phi)_p = \sup_{x \in \Psi U^p_\Phi} \mathcal{E}_{\gamma_\sigma}(x)_p, \tag{3.4}$$

and

$$D_\sigma(\Psi U^p_\Phi)_p = \inf_{\gamma_\sigma \in \Gamma_\sigma} \mathcal{E}_{\gamma_\sigma}(\Psi U^p_\Phi)_p. \tag{3.5}$$

Analogously,

$$e_\sigma(x)_p = \inf_{\gamma_\sigma \in \Gamma_\sigma} \|x - U_{\gamma_\sigma}(x)\|_{p,\Phi} \tag{3.6}$$

and

$$e_\sigma(\Psi U^p_\Phi)_p = \sup_{x \in \Psi U^p_\Phi} e_\sigma(x)_p.$$

3.1. Quantities $\mathcal{E}_{\gamma_\sigma}(\Psi U^p_\Phi)_p$ and $D_\sigma(\Psi U^p_\Phi)_p$. In what follows, we use the notion of a rearrangement of a function in decreasing order. This notion was introduced, apparently for the first time, by Hardy and Littlewood (see [35], Chap. X); later, it was successfully used by many authors. We present here necessary definitions from Korneichuk's monograph [36] (Chap. 6). In the monograph indicated, rearrangements of functions of one variable are considered, but main definitions remain true in the general case.

Assume that a nonnegative μ-measurable function $f(x)$ is defined on a μ-measurable set $A \subset R^m$, $m \geq 1$, $\mathrm{mes}_\mu A = a$, where a is finite or infinite, and its distribution function

$$m_f(y) = \mathrm{mes}_\mu E_y, \quad E_y = \{x\colon x \in A,\ f(x) \geq y\}, \quad y \geq 0,$$

takes only finite values for $y > 0$.

The function $t = m_f(y)$ does not increase for all $y \geq 0$, and, moreover, $m_f(0) = a$. If the function $m_f(y)$ is continuous and strictly decreasing, then a strictly decreasing function $y = \bar{\varphi}(t)$ inverse to it exists on the interval $t \in (0, a)$; this function is called the rearrangement of the function $f(x)$ in decreasing order. In the general case, depending on the function $f(\cdot)$, $m_f(y)$ may have intervals of constancy as well as discontinuities of the first kind at finitely many (or countably many) points. To uniquely determine the function inverse to it, we correct the graph of the function $m_f(y)$ as follows: At each discontinuity point y_j of the function $m_f(y)$, we supplement its graph by the segment $y = y_j$, $m_f(y_j + 0) \leq t \leq m(y_j + 0)$. On each segment $[\alpha, \beta]$ where $m_f(y)$ is constant, we preserve a single point in its graph, say, the point with coordinates $y = (\alpha+\beta)/2$ and $t = m_f((\alpha+\beta)/2)$. In this case, every $t \in (0, a)$ is associated with the unique point with coordinates $(t, m_f^{-1}(t))$. This mapping determines the function $y = \bar{\varphi}(t)$, which is the rearrangement of the function $\varphi(x)$ in the case considered.

For any $y \geq 0$, the Lebesgue measure of the set of points $t \in (0, a)$ on which $\bar{\varphi}(t) \geq y$ is equal to $m_f(y)$. Thus,

$$\mathrm{mes}\,\{t\colon t \in (0, a),\ \ \bar{\varphi}(t) \geq y\} = \mathrm{mes}_\mu\{x\colon x \in A,\ \ f(x) \geq y\} = m_f(y).$$

In particular, this yields the equality

$$\int_0^a F(\bar{\varphi}(t))dt = \int_A F(f(x))d\mu$$

for any function F for which these integrals exist (see [35], Chap. X).

In the notation introduced, the following statement is true:

Theorem 3.1. *Suppose that* $\Psi = \Psi(t)$ *is an arbitrary function from* $Y(A, d\mu)$ *that is essentially bounded on* A, *i.e.,*

$$\operatorname*{ess\,sup}_{t\in A} |\Psi(t)| = \|\Psi\|_M < \infty, \tag{3.7}$$

and if the set A *is not bounded, then*

$$\lim_{|t|\to\infty} \Psi(t) = 0. \tag{3.8}$$

Then, for arbitrary $\mathfrak{X}$, $A \subset R^m$, $m \geq 1$, $\gamma_\sigma \in \Gamma_\sigma$, $\sigma < a$, *and* $p \in (0, \infty)$, *the following estimates hold for any operator* Φ *that satisfies condition* (A_p):

$$\mathcal{E}^p_{\gamma_\sigma}(\Psi U^p_\Phi)_p \leq \bar{\varphi}_{\gamma_\sigma}(0+0), \tag{3.9}$$

where $\bar{\varphi}_{\gamma_\sigma}(v)$ *is the rearrangement of the function*

$$\varphi_\sigma(t) = \varphi_{\gamma_\sigma}(t) = \begin{cases} |\Psi(t)|^p, & t \in A \setminus \gamma_\sigma, \\ 0, & t \in \gamma_\sigma, \end{cases}$$

in decreasing order,

$$D_\sigma(\Psi U^p_\Phi)_p \leq \bar{\Psi}(\sigma+0), \tag{3.10}$$

and $\bar{\Psi}(v)$ *is the rearrangement of the function* $|\Psi(t)|$ *in decreasing order.*

Furthermore, if, for any $\gamma_\sigma \in \Gamma_\sigma$ *and* $\sigma \in (0, a)$, *the functions* $\chi_{\gamma_\sigma}(t)$ *belong to the range of values* $E(\Phi)$ *of the operator* Φ *and their preimages* U_{γ_σ} *have* Ψ*-integrals, then relations (3.9) and (3.10) are equalities. Moreover, there is a set* γ^*_σ *in* Γ_σ *for which the following equalities are true:*

$$\mathcal{E}_{\gamma^*_\sigma}(\Psi U^p_\Phi)_p = D_\sigma(\Psi U^p_\Phi)_p = \bar{\Psi}(\sigma+0).$$

This set is determined by the relation

$$\gamma^*_\sigma = \{t \in A\colon |\Psi(t)| \geq \bar{\Psi}(\sigma+0)\}, \quad \operatorname{mes}_\mu \gamma^*_\sigma = \sigma.$$

This theorem was proved in [23]. We only note here that conditions (3.7) and (3.8) guarantee that, for any $y > 0$, the distribution function $m_{|\Psi|}(y)$ of the function $|\Psi(t)|$, i.e.,

$$m_{|\Psi|}(y) = \operatorname{mes}_\mu E_y, \quad E_y = \{t \in A\colon |\Psi(t)| \geq y\}, \quad y \geq 0,$$

takes only finite values from the segment $[0, a]$. Therefore, the quantities $\bar{\varphi}_\sigma(0+0)$ and $\bar{\Psi}(\sigma+0)$ are always defined.

Also note that, in the case where $E(\Phi) = L_p(A, d\mu)$, the operator Φ satisfies condition (A_p), and, by virtue of conditions (3.7) and (3.8), the requirements that guarantee the realization of equalities in relations (3.9) and (3.10) are also satisfied.

3.2. Quantities $e_\sigma(\Psi U_\Phi^p)_p$.

In the notation introduced, the following theorem is true:

Theorem 3.2. *Let $\Psi = \Psi(t)$ be an arbitrary function from $Y(A, d\mu)$ that is essentially bounded on A and let it satisfy condition (3.8) if the set A is not bounded.*

Then, for arbitrary $\mathfrak{X}$, $A \subset R^m$, $m \geq 1$, $\sigma \leq a$, and $p \in (0, \infty)$, the following relation holds for any operator Φ that satisfies condition (A_p):

$$e_\sigma^p(\Psi U_\Phi^p) \leq \sup_{\sigma < q \leq a} \frac{q - \sigma}{\displaystyle\int_0^q \frac{dt}{\bar{\Psi}^p(t)}}, \tag{3.11}$$

where $\bar{\Psi}(v)$ is the rearrangement of the function $|\Psi(t)|$ in decreasing order. The value of the least upper bound in (3.11) is attained for a certain finite value $q = q^*$.

Furthermore, if the range of values $E(\Phi)$ of the operator Φ coincides with the entire space $L_p(A, d\mu)$, then relation (3.11) is, in fact, an equality.

Proof. The proof of this theorem is presented in [23]. Its essential part is Theorem 3.3 from [23]. Here, we only outline the key points of the proof of Theorem 3.2.

By virtue of (3.6) and (1.2), for any $x \in S_\Phi^p$ we have

$$e_\sigma^p(x)_p = \inf_{\gamma_\sigma \in \Gamma_\sigma} \|\Phi(x - U_{\gamma_\sigma}(x))\|_{L_p}^p = \inf_{\gamma_\sigma \in \Gamma_\sigma} \|\hat{x}(t)(1 - \chi_{\gamma_\sigma}(t))\|_{L_p}^p$$

$$= \inf_{\gamma_\sigma \in \Gamma_\sigma} \left(\int\limits_A |\hat{x}(t)|^p \, d\mu - \int\limits_{\gamma_\sigma} |\hat{x}(t)|^p \, d\mu \right)$$

$$= \int\limits_A |\hat{x}(t)|^p \, d\mu - \sup_{\gamma_\sigma \in \Gamma_\sigma} \int\limits_{\gamma_\sigma} |\hat{x}(t)|^p \, d\mu, \quad L_p \overset{\mathrm{df}}{=} L_p(A, d\mu).$$

Therefore,

$$e_\sigma^p(\Psi U_\Phi^p)_p = \sup_{x \in \Psi U_\Phi^p} \left(\int\limits_A |\hat{x}(t)|^p \, d\mu - \sup_{\gamma_\sigma \in \Gamma_\sigma} \int\limits_{\gamma_\sigma} |\hat{x}(t)|^p \, d\mu \right). \tag{3.12}$$

If $x \in \Psi U_\Phi^p$, then $\hat{x}(t) = \Psi(t)\hat{y}(t)$, where y is a certain element of U_p. Hence, the following relation is true:

$$\sup_{x\in\Psi U_\Phi^p}\left(\int\limits_A |\hat{x}(t)|^p\,d\mu - \sup_{\gamma_\sigma\in\Gamma_\sigma}\int\limits_{\gamma_\sigma} |\hat{x}(t)|^p\,d\mu\right)$$

$$\leq \sup_{\nu\in U_p}\left(\int\limits_A |\Psi(t)|^p|y(t)|^p\,d\mu - \sup_{\gamma_\sigma\in\Gamma_\sigma}\int\limits_{\gamma_\sigma} |\Psi(t)|^p|y(t)|^p\,d\mu\right)$$

$$= \sup_{h\in U_1^+}\left(\int\limits_A |\Psi(t)|^p|h(t)|\,d\mu - \sup_{\gamma_\sigma\in\Gamma_\sigma}\int\limits_{\gamma_\sigma} |\Psi(t)|^p|h(t)|\,d\mu\right), \tag{3.13}$$

where U_1^+ is the subset of nonnegative functions from U_1.

To determine the value of the right-hand side of (3.13), we use the statement presented below. Since this statement is of independent interest, we formulate it in the form of a theorem.

Theorem 3.3. *Suppose that A is an arbitrary μ-measurable set from $R^m, m \geq 1,$ $\mathrm{mes}_\mu A = a,$ where a is finite or infinite, $\varphi(x)$ is a nonnegative function essentially bounded on A, and if the set A is not bounded, then*

$$\lim_{|x|\to\infty}\varphi(x) = 0.$$

Then, for any $\sigma < a,$ the following equality is true:

$$\mathcal{E}_\sigma(\varphi) = \sup_{h\in U_1^+}\inf_{\gamma_\sigma\in\Gamma_\sigma}\left(\int\limits_A \varphi(x)\,h(x)\,d\mu - \int\limits_{\gamma_\sigma\in\Gamma_\sigma}\varphi(x)\,h(x)\,d\mu\right) = \sup_{\sigma<q\leq a}\frac{q-\sigma}{\displaystyle\int_0^q \frac{dt}{\bar{\varphi}(t)}}, \tag{3.14}$$

where $\Gamma_\sigma = \Gamma_\sigma(A)$ is the set of all μ-measurable subsets γ_σ of A whose measure is equal to $\sigma,$ and $\bar{\varphi}(t)$ is the decreasing rearrangement of the function $\varphi(x)$.

The least upper bound on the right-hand side of (3.14) is attained for a certain finite value $q = q^*$.

Setting $\varphi(x) = |\Psi(x)|^p$ and combining relations (3.12)–(3.14), we obtain (3.11).

Note that inequality (3.11) is strict, provided that inequality (3.13) is also strict. The strict inequality in (3.11) is possible only due to the fact that not every function $y \in U_p$ has its preimage in U_Φ^p that, moreover, has the Ψ-integral. However, in the case where $E(\Phi) = L_p(A),$ this is impossible: every $y \in U_p$ has its preimage and, by virtue of the boundedness of $\Psi,$ the product $\Psi(t)y(t)$ belongs to $L_p(A, d\mu)$ and, hence, also has its preimage in S_Φ^p $\big($more exactly, in $\Psi U_\Phi^p\big)$. Thus, in this case, relation (3.11) is an equality.

4. Extremal Problems in the Spaces $S_\varphi^{p,\mu}$

By now, the most complete and final results have been obtained for the spaces $S_\varphi^{p,\mu}$. Here, we give a brief survey of the obtained results related to the best approximations and widths for the sets ΨU_Φ^p and establish new statements for these quantities in several cases not considered earlier. The spaces $S_\varphi^{p,\mu}$ have already been mentioned in Sec. 1. However, for the sake of completeness and rigor, we present all definitions necessary for what follows.

4.1. Spaces $S_\varphi^{p,\mu}$. Let $\mathfrak{X}$ be a certain linear complex space and let $\varphi = \{\varphi_k\}_{k=1}^\infty$ be a fixed countable system in it. Assume that any pair of elements $x, y \in \mathfrak{X}$ for which at least one vector belongs to φ is associated with a number (x, y) (a "scalar product") so that the following conditions are satisfied:

(i) $(x, y) = \overline{(y, x)}$, where $\bar{z}$ is the complex conjugate of the number z;

(ii) $(\lambda x_1 + \nu x_2, y) = \lambda(x_1, y) + \nu(x_2, y)$, where λ and ν are arbitrary numbers;

(iii) $(\varphi_k, \varphi_l) = \begin{cases} 0, & k \neq l, \\ 1, & k = l. \end{cases}$

Further, let $\mu = \{\mu_k\}_{k=1}^\infty$ be a certain system of nonnegative numbers, i.e., $\mu_k \geq 0$, $k \in N = \{1, 2, \ldots\}$.

We associate every element $x \in \mathfrak{X}$ with a system of numbers $\hat{x}(k) = \hat{x}_\varphi(k)$ by the equalities

$$\hat{x}(k) = \hat{x}_\varphi(k) = (x, \varphi_k), \quad k \in N,$$

and, for fixed $p \in (0, \infty)$, we set

$$S_\varphi^{p,\mu} = S_\varphi^{p,\mu}(\mathfrak{X}) = \left\{ x \in \mathfrak{X}\colon \sum_{k=1}^\infty |\mu_k \hat{x}_\varphi(k)|^p < \infty \right\}.$$

Elements $x, y \in S_\varphi^{p,\mu}$ are assumed to be identical if $\hat{x}_\varphi(k) = \hat{y}_\varphi(k)$ for all $k \in N$. Thus, the set $S_\varphi^{p,\mu}$ is generated by the space $\mathfrak{X}$, systems φ and μ, operation $(\cdot, \cdot)$, and number p.

For $\mu_k \equiv 1$, $k \in N$, as mentioned above, the sets $S_\varphi^{p,\mu}$ coincide with the sets S_φ^p introduced and studied in [14–23]. In the general case, they were considered for the first time in [20].

For arbitrary vectors $x, y \in \mathfrak{X}$, we define the φ, μ-distance between them by the equality

$$\rho(x, y)_{p,\mu} \overset{\text{df}}{=} \|x - y\|_{p,\mu} = \|x - y\|_{p,\mu,\varphi} = \left(\sum_{k=1}^\infty |\hat{x}_\varphi(k) - \hat{y}_\varphi(k)|^p \, \mu_k^p \right)^{\frac{1}{p}}.$$

A vector θ for which $\hat{\theta}_\varphi(k) = 0$ for all $k \in N$ is called a zero element of the space $S_\varphi^{p,\mu}$. The distance $\rho(\theta, x)_{p,\mu}$, $x \in S_\varphi^{p,\mu}$, is called the φ, μ-norm of the element x and is defined by $\|x\|_{p,\mu}$. Thus, by definition,

$$\|x\|_{p,\mu} = \|x\|_{p,\mu,\varphi} = \rho(\theta, x)_{p,\mu} = \left(\sum_{k=1}^\infty |\mu_k \hat{x}_\varphi(k)|^p \right)^{\frac{1}{p}}. \tag{4.1}$$

It was shown in [20] that the set $S_\varphi^{p,\mu}$ is a linear space with the same operations of addition of vectors and their multiplication by numbers as those defined in the entire space $\mathfrak{X}$.

If all numbers μ_k in the system μ differ from zero, then the equality $\|x\|_{p,\mu} = 0$ is possible only for $x = \theta$. This implies that, for $\mu_k > 0$, $k \in N$, the functional $\|\cdot\|_{p,\mu}$ defined by equality (4.1) satisfies all axioms of a norm for $p \geq 1$ and all axioms of a quasinorm for $p \in (0, 1)$. Therefore, if $\mu_k > 0$, $k \in N$, then $S_\varphi^{p,\mu}$ is a linear normed space for $p \geq 1$ and a space with quasinorm for $p \in (0, 1)$; furthermore, it contains an orthogonal system $\varphi = \{\varphi_k\}_{k=1}^\infty$ and, moreover, $\|\varphi_k\|_{p,\mu} = \mu_k$.

Now let f be an arbitrary element of the space $S^{p,\mu}_\varphi$ and let

$$S[\,f\,]_\varphi = \sum_{k=1}^{\infty} \hat{f}_\varphi(k)\varphi_k \tag{4.2}$$

be its formal series in the system φ.

The spaces $S^{p,\mu}_\varphi$ inherit the most important properties of separable Hilbert spaces, namely the Parseval equality in the form (4.1) and the minimal property of partial Fourier sums formulated as follows:

Proposition 4.1. *Let $\{g_\alpha\}$ be a family of bounded subsets of the set N that depend on the parameter α and are such that any number $n \in N$ belongs to all sets $\{g_\alpha\}$ for sufficiently large α.*

Further, let $f \in S^{p,\mu}_\varphi$, let $p \in (0,\infty)$, and let

$$S_\alpha(f) = S_{g_\alpha}(f) = \sum_{k\in g_\alpha} \hat{f}(k)\varphi_k$$

be the partial sum of the Fourier series $S[f]_\varphi$ of the element f that corresponds to the set $\{g_\alpha\}$. Then, among all sums of the form

$$\Phi_\alpha = \sum_{k\in g_\alpha} c_k\varphi_k,$$

the partial sum $S_\alpha(f)$ deviates least from f in the space $S^{p,\mu}_\varphi$, i.e.,

$$\inf_{c_k} \|f-\Phi_\alpha\|_{p,\mu} = \|f - S_\alpha(f)\|_{p,\mu}.$$

Moreover,

$$\|f - S_\alpha(f)\|^p_{p,\mu} = \|f\|^p_{p,\mu} - \sum_{k\in g_\alpha} \left|\mu_k \hat{f}(k)\right|^p$$

and

$$\lim_{\alpha\to\infty} \|f - S_\alpha(f)\|_{p,\mu} = 0.$$

It is clear that this statement is a reformulation of Proposition 2.1. This yields the following statement:

Proposition 4.1′. *Suppose that $f \in S^{p,\mu}_\varphi$, $p \in (0,\infty)$, and*

$$S_n(f) = S_n(f)_\varphi = \sum_{k=1}^{n} \hat{f}(k)\,\varphi_k, \quad n \in N,$$

is a partial sum of series (4.2). Then, among all sums of the form

$$\Phi_n = \sum_{k=1}^{n} c_k\,\varphi_k$$

for given $n \in N,$ *the partial sum* $S_n(f)$ *deviates least from* f *in the space* $S^{p,\mu}_\varphi,$ *i.e.,*

$$\inf_{c_k} \|f - \Phi_n\|_{p,\mu} = \|f - S_n(f)\|_{p,\mu}.$$

Moreover,

$$\|f - S_n(f)\|^p_{p,\mu} = \|f\|^p_{p,\mu} - \sum_{k=1}^{n} \left|\mu_k \hat{f}(k)\right|^p$$

and

$$\lim_{n\to\infty} \|f - S_n(f)\|_{p,\mu} = 0. \tag{4.3}$$

It follows from (4.3) that, for any element $f \in S^{p,\mu}_\varphi,$ its Fourier series (4.2) in the system φ converges to f in the norm of the space $S^{p,\mu}_\varphi,$ i.e., the system φ is complete in $S^{p,\mu}_\varphi,$ and $S^{p,\mu}_\varphi$ is separable.

4.2. ψ-Integrals. In the spaces $S^{p,\mu}_\varphi,$ we select objects of approximation—the unions of elements $f \in \mathfrak{X}$ associated with the notion of a class of functions in approximation theory and corresponding to the sets ΨU^p_Φ.

Let $\psi = \{\psi_k\}_{k=1}^\infty$ be an arbitrary system of complex numbers. If, for a given element $f \in \mathfrak{X}$ with Fourier series (4.2), there exists an element $F \in \mathfrak{X}$ for which

$$S[\,f\,]_\varphi = \sum_{k=1}^{\infty} \psi_k\, \hat{f}(k)\varphi_k, \tag{4.4}$$

i.e.,

$$\hat{F}_\varphi(k) = \psi_k\, \hat{f}(k), \quad k \in N, \tag{4.5}$$

then the vector F is called the ψ-integral of the vector f. In this case, we write $F = \mathcal{J}^\psi f$. If $\mathfrak{N}$ is a certain subset of $\mathfrak{X},$ then $\psi\mathfrak{N}$ denotes the set of ψ-integrals of all elements from $\mathfrak{N}$. In particular, $\psi S^{p,\mu}_\varphi$ is the set of ψ-integrals of all vectors belonging to the space $S^{p,\mu}_\varphi$.

If f and F are connected by relation (4.4) or (4.5), then it is sometimes convenient to call f the ψ-derivative of the element F and write $f = D^\psi F = F^\psi$.

In what follows, we assume that the system φ satisfies the condition

$$\lim_{k\to\infty} |\psi_k| = 0. \tag{4.6}$$

It is clear that this condition guarantees the inclusion $\psi S^{p,\mu}_\varphi \subset S^{p,\mu}_\varphi,$ and the condition of the boundedness of the set of numbers $|\psi_k|,$ $k \in N,$ is necessary and sufficient for this inclusion.

Let

$$U^{p,\mu}_\varphi = \left\{f \in S^{p,\mu}_\varphi \colon \|f\|_{p,\mu} \le 1\right\} \tag{4.7}$$

be the unit ball in the space $S^{p,\mu}_\varphi$ and let $\psi\, U^{p,\mu}_\varphi$ be the set of ψ-integrals of all elements from $U^{p,\mu}_\varphi$. The sets $\psi\, U^{p,\mu}_\varphi$ are the main objects whose approximation properties are studied in the present paper. If

$$\psi_k \ne 0 \quad \forall k \in N,$$

then, by virtue of (4.6) and (4.7), we have

$$\psi\, U_\varphi^{p,\mu} = \left\{ f \in S_\varphi^{p,\mu} : \sum_{k=1}^{\infty} \left| \mu_k \, \frac{\hat{f}(k)}{\psi_k} \right|^p \le 1 \right\}, \tag{4.7$'$}$$

i.e., the set $\psi\, U_\varphi^{p,\mu}$ is a p-ellipsoid in the space $S_\varphi^{p,\mu}$ with semiaxes equal to $|\psi_k|$.

4.3. Approximating Aggregates and Approximation Characteristics. The structure of aggregates used for the approximation of elements $f \in S_\varphi^{p,\mu}$ is determined by the characteristic sequences $\varepsilon(\psi)$, $g(\psi)$, and $\delta(\psi)$ of the system ψ. These characteristic sequences are defined as follows:

Let $\psi = \{\psi_k\}_{k=1}^{\infty}$ be an arbitrary system of complex numbers that satisfies condition (4.6). Then $\varepsilon(\psi) = \varepsilon_1, \varepsilon_2, \ldots$ denotes the set of values of the quantities $|\psi_k|$ enumerated in decreasing order, $g(\psi) = g_1, g_2, \ldots$ denotes the system of sets

$$g_n = g_n^{\psi} = \{k \in N\colon |\psi_k| \ge \varepsilon_n\},$$

and $\delta(\psi) = \delta_1, \delta_2, \ldots$ denotes the sequence of numbers $\delta_n = |g_n|$, where $|g_n|$ is the number of numbers $k \in N$ in the set g_n. The sequences $\varepsilon(\psi)$, $g(\psi)$, and $\delta(\psi)$ are called characteristic for the system ψ. Note that, according to this definition, any number $n^* \in N$ belongs to all sets g_n^{ψ} for sufficiently large n and

$$\lim_{k \to \infty} \delta_k = \infty.$$

In what follows, it is convenient to denote the empty set by $g_0 = g_0^{\psi}$ and assume that $\delta_0 = 0$.

Let the set $S_\varphi^{p,\mu}$ be generated by the space $\mathfrak{X}$, systems φ and μ, and number p, $p > 0$, and let $\psi = \{\psi_k\}_{k=1}^{\infty}$ be an arbitrary system of complex numbers that satisfies condition (4.6).

As approximating aggregates for elements $f \in \psi\, S_\varphi^{p,\mu}$, we consider the polynomials

$$S_n(f)_{\varphi,\psi} = S_{g_n^{\psi}}(f) = \sum_{k \in g_n^{\psi}} \hat{f}(k)\, \varphi_k, \quad n = 1, 2, \ldots, \quad S_0(f)_{\varphi,\psi} = \theta, \tag{4.8}$$

where g_n^{ψ} are elements of the sequence $g(\psi)$ and θ is a zero vector of the space $S_\varphi^{p,\mu}$. We set

$$\begin{aligned} \mathcal{E}_n(f)_{\psi,p,\mu} &= \|f - S_{n-1}(f)_{\varphi,\psi}\|_{p,\mu}, \\ \mathcal{E}_n(\psi\, U_\varphi^{q,\mu})_{\psi,p,\mu} &= \sup_{f \in \psi\, U_\varphi^{q,\mu}} \mathcal{E}_n(f)_{\psi,p,\mu}, \quad p, q > 0. \end{aligned} \tag{4.9}$$

The quantity $\mathcal{E}_n(f)_{\psi,p,\mu}$ is called the approximation of an element $f \in S_\varphi^{p,\mu}$ by Fourier sums constructed for the domains g_{n-1}^{ψ}, and $\mathcal{E}_n(\psi\, U_\varphi^{q,\mu})_{\psi,p,\mu}$ is called the approximation of the set $\psi\, U_\varphi^{q,\mu}$ by these sums in the space $S_\varphi^{p,\mu}$. Further, let

$$E_n(f)_{\psi,p,\mu} = \inf_{c_k} \left\| f - \sum_{k \in g_{n-1}^{\psi}} c_k\, \varphi_k \right\|_{p,\mu}$$

be the best approximation of an element $f \in \psi\, S_\varphi^{p,\mu}$ by polynomials constructed for the domains g_{n-1}^{ψ} and let

$$E_n(\psi\, U_\varphi^{q,\mu})_{\psi,p,\mu} = \sup_{f \in \psi\, U_\varphi^{q,\mu}} E_n(f)_{\psi,p,\mu}, \quad p, q > 0, \tag{4.10}$$

be the best approximation of the set $\psi\, U_\varphi^{q,\mu}$ by these polynomials.

As usual,

$$d_n(\mathfrak{M}; Y) = \inf_{F_n \in \mathcal{F}_n} \sup_{x \in \mathfrak{M}} \inf_{u \in F_n} \|x - u\|_Y$$

is the Kolmogorov width of a set $\mathfrak{M}$ in the space Y. Here, $\mathcal{F}_n$ is the set of all subspaces of dimension $n \in N$ of the space Y. According to the Jensen inequality, for any nonnegative sequence $a = \{a_k\}_{k=1}^{\infty}$, $a_k \geq 0$, we have

$$\left(\sum_{k=1}^{\infty} a_k^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{\infty} a_k^q\right)^{\frac{1}{q}}, \quad 0 < q \leq p.$$

Therefore, the following inclusions are true:

$$S_\varphi^{q,\mu} \subset S_\varphi^{p,\mu}, \quad 0 < q \leq p, \tag{4.11}$$

and

$$\psi\, U_\varphi^{q,\mu} \subset \psi\, U_\varphi^{p,\mu}, \quad 0 < q \leq p. \tag{4.12}$$

In particular, this implies that quantities (4.9) and (4.10) are well defined at least for all systems ψ that satisfy condition (4.6) under the assumption that $0 < q \leq p$.

4.4. Best Approximations and Widths of q-Ellipsoids. The following statement is true:

Theorem 4.1. *Let $\psi = \{\psi_k\}_{k=1}^{\infty}$ be a system of numbers that satisfies condition (4.6). Then, for any $n \in N$ and $0 < q \leq p < \infty$, the following relations are true:*

$$E_n(\psi\, U_\varphi^{q,\mu})_{\psi,p,\mu} \leq \mathcal{E}_n(\psi\, U_\varphi^{q,\mu})_{\psi,p,\mu} \leq \varepsilon_n. \tag{4.13}$$

Furthermore, if

$$\mu_k \neq 0 \quad \textit{and} \quad \psi_k \neq 0 \quad \forall k \in N, \tag{4.14}$$

then the inequality signs in (4.13) are replaced by the equality signs. The quantity ε_n is the nth term of the characteristic sequence $\varepsilon(\psi)$.

Partial sums of the form (4.8) constructed for the domains g_n^{ψ} form an optimal aggregate for the approximation of elements from the sets $\psi\, U_\varphi^{q,\mu}$ in the sense of Kolmogorov widths. More exactly, the following theorem is true:

Theorem 4.2. *Suppose that $\psi = \{\psi_k\}_{k=1}^{\infty}$ and $\mu = \{\mu_k\}_{k=1}^{\infty}$ satisfy conditions (4.6) and (4.14) and*

$$d_\nu\left(\psi\, U_\varphi^{p,\mu}\right)_{p,\mu} = d_\nu(\psi\, U_\varphi^{p,\mu}; S_\varphi^{p,\mu}) = \inf_{F_n \in \mathcal{F}_n} \sup_{f \in \psi\, U_\varphi^{p,\mu}} \inf_{u \in F_n} \|f - u\|_{p,\mu}, \quad \nu = 0, 1, \ldots,$$

are the Kolmogorov widths of the sets $\psi U_\varphi^{p,\mu}$ *in the space* $S_\varphi^{p,\mu}$. *Then, for any* $p \in [1,\infty)$ *and* $n \in N$, *the following equalities are true:*

$$d_{\delta_{n-1}}(\psi U_\varphi^{p,\mu})_{p,\mu} = d_{\delta_{n-1}+1}(\psi U_\varphi^{p,\mu}) = \ldots = d_{\delta_n - 1}(\psi U_\varphi^{p,\mu})_{p,\mu} = \varepsilon_n,$$

where δ_s *and* ε_s, $s = 1, 2, \ldots,$ *are elements of the characteristic sequences* $\delta(\psi)$ *and* $\varepsilon(\psi)$ *of the system* ψ *and* $\delta_0 = 0$.

Now assume that, along with the numbers p and q and the sequence $\mu = \{\mu_k\}_{k=1}^\infty$, a sequence $\lambda = \{\lambda_k\}_{k=1}^\infty$ of nonnegative numbers among which at least one number is not equal to zero is given. On the basis of the space $\mathfrak{X}$ and system φ, we construct the sets $S_\varphi^{p,\mu}$ and $S_\varphi^{q,\lambda}$. If $0 < q \le p$ and the sequences λ and μ coincide, then inclusions (4.11) and (4.12) are true. It is also clear that, for any $p \in (0,\infty)$, we have

$$S_\varphi^{p,\lambda} \subset S_\varphi^{p,\mu},$$

provided that

$$\lambda_k \ge C\mu_k \quad \forall k \in N,$$

where C is an arbitrary positive constant. Therefore, the following inclusion is true:

$$S_\varphi^{q,\lambda} \subset S_\varphi^{p,\mu} \quad \forall p, q, \quad 0 < q \le p, \quad \lambda_k \ge C\mu_k \quad \forall k \in N.$$

Let us establish analogs of Theorems 4.1 and 4.2 in the case where the approximated elements belong to the space $S_\varphi^{q,\lambda}$ and the approximation is sought in the space $S_\varphi^{p,\mu}$. In this case, we also construct approximating aggregates according to formulas (4.8), but, as the sequence ψ appearing in the definition of the domains g_n^ψ and the sequence $\varepsilon(\psi)$, we use the sequence

$$\psi' = \{\psi'_k\}_{k=1}^\infty = \left\{\psi_k \frac{\mu_k}{\lambda_k}\right\}_{k=1}^\infty, \tag{4.15}$$

where the numbers ψ_k, $k \in N$, are the same as in the definition of the approximated sets $\psi U_\varphi^{q,\lambda}$.

Theorem 4.1′. *Suppose that* $\psi = \{\psi_k\}_{k=1}^\infty$, $\mu = \{\mu_k\}_{k=1}^\infty$, *and* $\lambda = \{\lambda_k\}_{k=1}^\infty$ *are fixed sequences that satisfy the condition*

$$\lim_{k\to\infty} \psi_k \frac{\mu_k}{\lambda_k} = 0. \tag{4.16}$$

Then, for any $n \in N$ *and* $0 < q \le p$, *the following relations are true:*

$$E_n(\psi U_\varphi^{q,\lambda})_{\psi',p,\mu} \le \mathcal{E}_n(\psi U_\varphi^{q,\lambda})_{\psi',p,\mu} \le \varepsilon'_n, \tag{4.17}$$

where

$$E_n(\psi U_\varphi^{q,\lambda})_{\psi',p,\mu} = \sup_{f \in \psi U_\varphi^{q,\lambda}} E_n(f)_{\psi',p,\mu},$$

$$E_n(f)_{\psi',p,\mu} = \inf_{c_k} \Bigg\| f - \sum_{k \in g_{n-1}^{\psi'}} c_k\, \varphi_k \Bigg\|_{p,\mu},$$

$$\mathcal{E}_n(\psi\, U_\varphi^{q,\lambda})_{\psi',p,\mu} = \sup_{f \in \psi\, U_\varphi^{q,\lambda}} \mathcal{E}_n(f)_{\psi',p,\mu}, \tag{4.18}$$

$$\mathcal{E}_n(f)_{\psi',p,\mu} = \| f - S_{n-1}(f)_{\varphi,\psi'} \|_{p,\mu},$$

and ε_n' *and* $g_{n-1}^{\psi'}$ *are terms of the characteristic sequence of system (4.15). If, in addition, conditions (4.14) are satisfied, then the inequality signs in (4.17) are replaced by the equality signs.*

In this case, the following analog of Theorem 4.2 holds for the Kolmogorov widths:

Theorem 4.2′. *Suppose that the systems* $\psi = \{\psi_k\}_{k=1}^\infty$ *and* $\mu = \{\mu_k\}_{k=1}^\infty$ *satisfy conditions (4.6) and (4.14). Then, for any sequence* $\lambda = \{\lambda_k\}_{k=1}^\infty$ *that satisfies condition (4.16), the following equalities hold for any* $p \in [1, \infty)$ *and* $n \in N$:

$$d_{\delta_{n-1}'}(\psi\, U_\varphi^{p,\lambda})_{p,\mu} = d_{\delta_{n-1}'+1}(\psi\, U_\varphi^{p,\lambda})_{p,\mu} = \ldots = d_{\delta_n'-1}(\psi\, U_\varphi^{p,\lambda})_{p,\mu} = \varepsilon_n', \tag{4.19}$$

where δ_s' *and* ε_s', $s = 1, 2, \ldots,$ *are elements of the characteristic sequences* $\delta(\psi')$ *and* $\varepsilon(\psi')$ *of system (4.15) and* $\delta_0' = 0$.

4.5. Quantities $\boldsymbol{D_n(\psi\, U_\varphi^{q,\lambda})_{p,\mu}}$**.** Let $x = \{x_k\}_{k=1}^\infty$ be an arbitrary sequence of complex numbers and let $\varepsilon(x) = \{\varepsilon_k(x)\}_{k=1}^\infty$, $g(x) = \{g_k(x)\}_{k=1}^\infty$, and $\delta(x) = \{\delta_k(x)\}_{k=1}^\infty$ be its characteristic sequences. By $\bar{x} = \{\bar{x}_k\}_{k=1}^\infty$ we denote the rearrangement of the sequence $\{|x_k|\}_{k=1}^\infty$ in decreasing order. It is clear that the values $\bar{x}_k$, $k \in N$, can be determined by the relations

$$\bar{x}_k = \varepsilon_n(x), \quad k \in (\delta_{n-1}(x), \delta_n(x)], \quad n \in N, \quad \delta_0 = 0.$$

In this notation, equality (4.19) takes the form

$$d_n(\psi\, U_\varphi^{p,\lambda})_{p,\mu} = \bar{\psi}_{n+1}', \quad n \in N, \tag{4.19′}$$

where $\bar{\psi}_n'$ is the nth term of the sequence $\bar{\psi}'$. Further, let γ_n be an arbitrary collection of n natural numbers and let $\mathcal{F}_n$ be the set of all polynomials of the form

$$P_{\gamma_n} = \sum_{k \in \gamma_n} c_k\, \varphi_k, \tag{4.20}$$

where c_k are certain complex numbers. By virtue of the definition of width and relation (4.19′), we always have

$$\inf_{\gamma_n} \sup_{f \in \psi\, U_\varphi^{p,\lambda}} \inf_{P_{\gamma_n} \in F_n} \| f - P_{\gamma_n} \|_{p,\mu} \geq \bar{\psi}_{n+1}'. \tag{4.21}$$

It follows from Theorem 4.1$'$ that if conditions (4.14) are satisfied, then relation (4.21) is, in fact, an equality. In this connection, in the notation introduced, for any subset $\mathfrak{N} \subset \mathfrak{X}$ we put

$$D_n(\mathfrak{N})_{p,\mu} = \inf_{\gamma_n} \sup_{f\in\mathfrak{N}} \inf_{P_{\gamma_n}\in F_n} \|f - P_{\gamma_n}\|_{p,\mu}. \tag{4.22}$$

Then the relation proved above can be rewritten in the form

$$D_n(\psi\, U_{\varphi}^{q,\lambda})_{p,\mu} = \bar{\psi}'_{n+1}, \quad p = q.$$

[It is clear that the quantity $D_n(\psi\, U_{\varphi}^{q,\lambda})$ corresponds to the quantity defined by (3.5).]

It turns out that this relation remains true for $0 < q < p$. More exactly, the following theorem is true:

Theorem 4.3. *Let $\psi = \{\psi_k\}_{k=1}^{\infty}$, $\mu = \{\mu_k\}_{k=1}^{\infty}$, and $\lambda = \{\lambda_k\}_{k=1}^{\infty}$ be arbitrary systems of numbers that satisfy the condition*

$$\lim_{k\to\infty} \psi_k \frac{\mu_k}{\lambda_k} = 0, \tag{4.23}$$

let γ_n be an arbitrary collection of n natural numbers, and let q and p be arbitrary positive numbers such that $0 < q \le p$. Then the following relations holds for any $n \in N$:

$$D_n(\psi\, U_{\varphi}^{q,\lambda})_{p,\mu} = \inf_{\gamma_n} \sup_{f\in\psi\, U_{\varphi}^{q,\lambda}} \inf_{P_{\gamma_n}\in F_n} \|f - P_{\gamma_n}\|_{p,\mu} = \bar{\psi}'_{n+1}, \tag{4.24}$$

where $\bar{\psi}'_{n+1}$ is the $(n+1)$th term of the sequence $\bar{\psi}' = \{\bar{\psi}_k\}_{k=1}^{\infty}$ that is the rearrangement of the sequence

$$|\psi'_k| = \left|\frac{\psi_k\, \mu_k}{\lambda_k}\right|, \quad k = 1, 2, \dots,$$

in decreasing order.

Theorems 4.1–4.3 were proved in [20]. For $\mu_k = \lambda_k \equiv 1$, these statements were established in [14–16]. It was shown in [20] that the inner lower bound in (4.24) is realized by polynomials of the form (4.20) for $c_k = \widehat{f}(k)$, and the outer lower bound is realized by the set $\gamma_n^* = \{i_k\}_{k=1}^{n}$, where the values i_k, $k = \overline{1, n}$, are such that $|\psi'_{i_k}| = \psi'_k$. Therefore, the following statement is true:

Theorem 4.3$'$. *Suppose that all conditions of Theorem 4.3 are satisfied. Then the following relations hold for any $n \in N$:*

$$D_n(\psi\, U_{\varphi}^{q,\lambda})_{p,\mu} = \sup_{f\in\psi\, U_{\varphi}^{q,\lambda}} \left\| f - \sum_{k\in\gamma_n^*} \widehat{f}(k)\, \varphi_k, \right\|_{p,\mu} = \bar{\psi}'_{n+1}, \tag{4.25}$$

where $\gamma_n^* = \{i_k\}_{k=1}^{n}$ and the values i_k, $k = \overline{1, n}$, are such that

$$\left|\frac{\psi_{i_k}\, \mu_{i_k}}{\lambda_{i_k}}\right| = \bar{\psi}'_k.$$

Combining (4.25) and (4.19′), we conclude that the values of the widths $d_n(\psi\, U_\varphi^{p,\lambda})_{p,\mu}$ in the case of approximation of elements $f \in \psi\, U_\varphi^{p,\lambda}$ in the space $S_\varphi^{p,\mu}$ are realized by Fourier sums constructed exactly for the domains γ_n^*.

Note that, in the case of approximation of periodic functions by trigonometric polynomials, the quantities $D_n(\mathfrak{N})_{p,\mu}$ are associated with so-called trigonometric widths. For this reason, we can call these quantities, e.g., basis widths of order n of the set $\mathfrak{N}$ in the space $S_\varphi^{p,\mu}$. Then it follows from Theorem 4.3′ that the values of the basis widths $D_n(\psi\, U_\varphi^{q,\lambda})_{p,\mu}$ for $0 < q \le p$ are also realized by Fourier sums constructed for the domains γ_n^*.

4.6. Best Approximations of q-Ellipsoids in the Spaces $S_\varphi^{p,\mu}$ for $q > p$. Let us obtain an analog of Theorem 4.1′ (and, hence, of Theorem 4.1) for $0 < p < q < \infty$. In this case, conditions (4.16) are insufficient for the inclusion

$$\psi\, U_\varphi^{q,\lambda} \subset S_\varphi^{p,\mu}.$$

This inclusion is now guaranteed by the conditions

$$\sum_{k=1}^{\infty} |\psi'_k|^{p\,q/(q-p)} < \infty, \quad \psi'_k = \psi_k\,\frac{\mu_k}{\lambda_k}, \quad k \in N, \tag{4.26}$$

which can easily be verified by using the Hölder inequality.

Theorem 4.4. *Let ψ, μ, and λ be sequences and let p and q be nonnegative numbers $(q > p > 0)$ that satisfy condition (4.26).*

Then, for any natural n, the following equality is true:

$$E_n(\psi\, U_\varphi^{q,\lambda})_{\psi',p,\mu} = \mathcal{E}_n(\psi\, U_\varphi^{q,\lambda})_{\psi',p,\mu} = \left(\sum_{k=\delta'_{n-1}+1}^{\infty} (\bar{\psi}'_k)^{\frac{p\,q}{q-p}}\right)^{\frac{q-p}{p\,q}}, \tag{4.27}$$

where $\bar{\psi}' = \{\bar{\psi}_k\}_{k=1}^{\infty}$ is a sequence for which

$$\bar{\psi}'_k = \varepsilon'_k, \quad \delta'_{n-1} < k \le \delta'_n, \quad n \in N,$$

and ε'_n and δ'_n are terms of the characteristic sequences $\varepsilon(\psi')$ and $\delta(\psi')$.

For $\mu_k = \lambda_k \equiv 1$, this theorem was proved in [18], Sec. 11.8. The reasonings presented there, in fact, can be used in the general case.

Proof. Since the first equality in (4.27) follows from Proposition 4.1, it suffices to verify the validity of the second equality.

Let ε'_n, δ'_n, and $g'_n = g_n(\psi') = \{k \in N\colon |\psi'_k| \ge \varepsilon'_n\}$ be the characteristic sequences of the system ψ' and let

$$S_n(f)_{\varphi,\psi'} = S_{g_n^{\psi'}} = \sum_{k \in g_n^{\psi'}} \widehat{f}_\varphi(k)\,\varphi_k$$

be the polynomials constructed according to (4.8) for elements $f \in \psi U_\varphi^{q,\lambda}$. Using relations (4.9), we get

$$\mathcal{E}_n^p(f)_{\psi',p,\mu} = \left\| f - S_{n-1}(f)_{\varphi,\psi'} \right\|_{p,\mu}^p = \sum_{k \notin g_{n-1}^{\psi'}} |\mu_k \, \widehat{f}_\varphi(k)|^p$$

$$= \sum_{k \notin g_{n-1}^{\psi'}} |\psi_k|^p |\mu_k \, \widehat{f}_\varphi^{\,\psi}(k)|^p = \sum_{k \notin g_{n-1}^{\psi'}} |\psi_k'|^p |\widehat{f}_\varphi^{\,\psi}(k)|^p \lambda_k^p. \tag{4.28}$$

Let $i_k, \ \ k = 1, 2, \ldots,$ denote natural numbers chosen from the condition

$$|\psi_{i_k}'| = \bar{\psi}_k', \quad \text{where} \quad \bar{\psi}_k = \varepsilon_k' \quad \text{for} \quad k \in (\delta_{n-1}', \delta_n']. \tag{4.29}$$

Then we can rewrite (4.28) in the form

$$\mathcal{E}_n^p(f)_{\psi',p,\mu} = \sum_{k=\delta_{n-1}'+1} \left| \bar{\psi}_k' \, \widehat{f}_\varphi^{\,\psi}(i_k) \lambda_{i_k} \right|^p. \tag{4.30}$$

We set

$$m_k = |\widehat{f}_\varphi^{\,\psi}(i_k) \lambda_{i_k}|^q. \tag{4.31}$$

In this case,

$$\lambda_{i_k}^p |\widehat{f}_\varphi^{\,\psi}(i_k)|^p = {m_k}^{p/q}$$

and, hence,

$$\mathcal{E}_n^p(f)_{\psi',p,\mu} = \sum_{k=\delta_{n-1}'+1}^{\infty} (\bar{\psi}_k')^p \, m_k^r, \quad r = \frac{p}{q}.$$

If $f \in \psi \, U_\varphi^{q,\lambda},$ then

$$\sum_{k=1}^{\infty} |\lambda_k \, \widehat{f}_\varphi^{\,\psi}(k)|^p \le 1.$$

Taking relations (4.18), (4.31), and (4.30) into account, we get

$$\mathcal{E}_n^p(\psi \, U_\varphi^{q,\lambda})_{\psi',p,\mu} \le \sup \left\{ \sum_{k=\delta_{n-1}'+1}^{\infty} (\bar{\psi}_k')^p \, m_k^r, \ \ r = \frac{p}{q}, \ \ \sum_{k=1}^{\infty} m_k \le 1 \right\}. \tag{4.32}$$

Setting $(\bar{\psi}_k')^p = \alpha_k,$ we rewrite condition (4.26) in the form

$$\sum_{k=1}^{\infty} \alpha_k^{\frac{1}{1-r}} < \infty, \quad \alpha_k > 0 \quad \forall k \in N. \tag{4.33}$$

Consequently, the determination of the right-hand side of (4.32) reduces to the solution of the extremal problem

$$\sum_{k=\delta'_{n-1}+1}^{\infty} \alpha_k\, x_k^r \to \sup \tag{4.34}$$

under the conditions $x_k \geq 0$ and $\sum\limits_{k=\delta_{n-1}+1} x_k = 1,$ and the numbers α_k form a nonincreasing sequence and satisfy condition (4.33).

The solutions $\bar{x}_k$ of this problem were obtained in [18], Sec. 11.8. They have the form

$$\bar{x}_k = \alpha_k^{\frac{1}{1-r}} \left(\sum_{i=\delta'_{n-1}+1}^{\infty} \alpha_i^{\frac{1}{1-r}} \right)^{-1}, \quad k = \delta'_{n-1}+1, \delta'_{n-1}+2, \ldots\,. \tag{4.35}$$

Combining relations (4.32) – (4.35), we get

$$\mathcal{E}_n^p(\psi\, U_\varphi^{q,\lambda})_{\psi',p,\mu} \leq \sum_{k=\delta'_{n-1}+1}^{\infty} \alpha_k \bar{x}_k^r = \left(\sum_{k=\delta'_{n-1}+1}^{\infty} \alpha_k^{\frac{1}{1-r}} \right)^{1-r} = \left(\sum_{k=\delta'_{n-1}+1}^{\infty} (\bar{\psi}'_k)^{\frac{p\,q}{q-p}} \right)^{\frac{q-p}{p\,q}}.$$

To complete the proof of the theorem, it remains to show that this relation is, in fact, an equality. To this end, it suffices to show that, for any $\varepsilon > 0,$ the set $\psi\, U_\varphi^{q,\lambda}$ contains an element f_ε for which

$$\mathcal{E}_n(f_\varepsilon)_{\psi',p,\mu} > \left(\sum_{k=\delta'_{n-1}+1}^{\infty} (\bar{\psi}'_k)^{\frac{p\,q}{q-p}} \right)^{\frac{q-p}{p\,q}} - \varepsilon. \tag{4.36}$$

We construct this element using the procedure proposed in [18], Sec. 11.8.

Let

$$h = \sum_{k=1}^{\infty} c_{i_k}\, \varphi_{i_k},$$

where the numbers i_k are chosen according to (4.29) and the numbers c_{i_k} are such that

$$c_{i_k}^q\, \lambda_{i_k}^q = \begin{cases} 0, & k = 1, 2, \ldots, \delta'_{n-1}, \\ \alpha_k^{\frac{1}{1-r}} \sigma_2^{-1}(\delta'_{n-1}), & k > \delta'_{n-1}, \end{cases}$$

$$\sigma_2(s) = \sum_{i=s}^{\infty} \alpha_i^{\frac{1}{1-r}}.$$

It is clear that

$$\|h\|_{q,\lambda} = \sum_{k=1}^{\infty} c_{i_k}^q\, \lambda_{i_k}^q = 1. \tag{4.37}$$

Now let ε be an arbitrary positive number and let N_ε be so large that, for all $n > N_\varepsilon$, the following inequality is true:

$$\sigma_2^{-r}(\delta'_{n-1}) \sum_{k=N_\varepsilon+1}^{\infty} \alpha_k^{\frac{1}{1-r}} < \varepsilon.$$

By virtue of (4.37), the element

$$h_\varepsilon = \sum_{k=1}^{N_\varepsilon} c_{i_k}\, \varphi_{i_k}$$

belongs to U_φ^q. Therefore, the element $f_\varepsilon = \mathcal{J}^\psi\, h_\varepsilon$ belongs to $\psi\, U_\varphi^q$. Calculating the value of $\mathcal{E}_n(f_\varepsilon)_{\psi',p,\mu}$ according to (4.9), we establish the validity of estimate (4.36). This completes the proof of Theorem 4.4.

4.7. Quantities $D_n(\psi\, U_\varphi^{q,\lambda})_{p,\mu}$ for $q > p > 0$. Let us find analogs of Theorems 4.3 and 4.3′ for $q > p > 0$. As before, let γ_n be an arbitrary collection of n natural numbers, let $\mathcal{F}_n$ be the set of all polynomials of the form (4.20), let

$$E_{\gamma_n}(f)_{p,\mu} = \inf_{P_{\gamma_n} \in F_n} \|f - P_{\gamma_n}\|_{p,\mu}, \quad f \in S_\varphi^{p,\mu}, \tag{4.38}$$

be the best approximation of an element f by polynomials constructed on the basis of the given collection γ_n of n basis vectors, let

$$\mathcal{E}_{\gamma_n}(f)_{p,\mu} = \|f - S_{\gamma_n}(f)\|_{p,\mu}, \quad S_{\gamma_n}(f) = \sum_{k \in \gamma_n} \widehat{f}_\varphi(k)\, \varphi_k, \tag{4.38′}$$

$$E_{\gamma_n}(\mathfrak{N})_{p,\mu} = \sup_{f \in \mathfrak{N}} E_{\gamma_n}(f)_{p,\mu}$$

be an upper bound of the quantities $E_{\gamma_n}(f)_{p,\mu}$ on a certain subset $\mathfrak{N}$ from $S_\varphi^{p,\mu}$, and let

$$\mathcal{E}_{\gamma_n}(\mathfrak{N}) = \sup_{f \in \mathfrak{N}} \mathcal{E}_{\gamma_n}(f)_{p,\mu}.$$

In this notation, the quantity $D_n(\psi\, U_\varphi^{q,\lambda})_{p,\mu}$ defined by (4.22) has the form

$$D_n(\psi\, U_\varphi^{q,\lambda})_{p,\mu} = \inf_{\gamma_n} E_{\gamma_n}(\psi\, U_\varphi^{q,\lambda})_{p,\mu}. \tag{4.39}$$

We introduce additional notation. If $\psi = \{\psi_k\}_{k=1}^\infty = \{\psi(k)\}_{k=1}^\infty$ is a certain system of complex numbers, $\mu = \{\mu_k\}_{k=1}^\infty$ and $\lambda = \{\lambda_k\}_{k=1}^\infty$ are systems of nonnegative numbers, and γ_n is a fixed collection of n natural numbers, then we set

$$\psi_{\gamma_n} = \{\psi_{\gamma_n}(k)\}_{k=1}^\infty,$$

where

$$\psi_{\gamma_n}(k) = \begin{cases} 0, & k \in \gamma_n, \\ \psi_k = \psi(k), & k \notin \gamma_n, \end{cases}$$

$$\psi' = \{\psi'(k)\}_{k=1}^{\infty}, \quad \psi'(k) = \psi'_k = \psi_k \frac{\mu_k}{\lambda_k}, \tag{4.40}$$

$$\psi'_{\gamma_n} = \{\psi'_{\gamma_n}(k)\}_{k=1}^{\infty}, \tag{4.41}$$

and

$$\psi'_{\gamma_n}(k) = \begin{cases} 0, & k \in \gamma_n, \\ \psi'_k, & k \notin \gamma_n. \end{cases}$$

In the notation introduced, the following statement is true:

Theorem 4.5. *Suppose that numbers p and q and sequences ψ, μ, and λ are such that $q > p > 0$ and condition (4.26) is satisfied.*

Then, for any natural n, the following equality is true:

$$E_{\gamma_n}(\psi\, U_\varphi^{q,\lambda})_{p,q} = \mathcal{E}_{\gamma_n}(\psi\, U_\varphi^{q,\lambda})_{p,q} = \left(\sum_{k=1}^{\infty} (\bar{\psi}'_{\gamma_n}(k))^{\frac{p\,q}{q-p}} \right)^{\frac{q-p}{p\,q}}, \tag{4.42}$$

where $\bar{\psi}'_{\gamma_n}(k)$, $k \in N$, is the rearrangement of the sequence $|\psi'_{\gamma_n}(k)|$, $k \in N$, in decreasing order.

Proof. First, note that equality (4.27) is a special case of (4.42). Indeed, let $g_{n-1}^{\psi'}$ be the $(n-1)$th term of the characteristic sequence of the domains $g_k^{\psi'}$ for the system ψ', i.e.,

$$g_{n-1}^{\psi'} = \{k \in N\colon |\psi'(k)| \geq \varepsilon_{n-1}\}, \tag{4.43}$$

and let $n' \overset{\text{df}}{=} \delta'_{n-1} = |g_{n-1}^{\psi'}|$ be the number of natural numbers contained in $g_{n-1}^{\psi'}$. We choose the collection $\gamma^*_{n'}$ from the condition $\gamma^*_{n'} = g_{n-1}^{\psi'}$. According to (4.40) and (4.41), we have

$$\bar{\psi}'_{\gamma^*_{n'}}(k) = \bar{\psi}'(k + n'), \quad k = 1, 2, \dots . \tag{4.44}$$

Therefore, by virtue of (4.43), (4.44), and (4.27), we get

$$E_{\gamma^*_{n'}}(\psi\, U_\varphi^{q,\lambda})_{p,q} = \mathcal{E}_{\gamma^*_{n'}}(\psi\, U_\varphi^{q,\lambda})_{p,q} = \left(\sum_{k=n'+1}^{\infty} (\bar{\psi}'(k))^{\frac{p\,q}{q-p}} \right)^{\frac{q-p}{p\,q}} = E_n(\psi\, U_\varphi^{q,\lambda})_{\psi'\, p,\mu}.$$

Thus,

$$E_{\gamma^*_{\delta_{n-1}}}(\psi\, U_\varphi^{q,\lambda})_{p,q} = \mathcal{E}_{\gamma^*_{\delta_{n-1}}}(\psi\, U_\varphi^{q,\lambda})_{p,q} = E_n(\psi\, U_\varphi^{q,\lambda})_{\psi'\, p,\mu} = \mathcal{E}_n(\psi\, U_\varphi^{q,\lambda})_{\psi'\, p,\mu}.$$

The proof of this theorem is analogous to the proof of Theorem 4.4. Therefore, we only dwell on its key points. By virtue of Proposition 4.1, it suffices to prove only the second equality in (4.42). To this end, we first write the following analog of equality (4.28) by using (4.38′) and (4.41):

$$\mathcal{E}^p_{\gamma_n}(f)_{p,\mu} = \sum_{k\notin\gamma_n} |\mu_k \widehat{f}_\varphi(k)|^p = \sum_{k\notin\gamma_n} |\psi'_k|^p |\widehat{f}^{\,\psi}_\varphi(k)|^p \lambda^p_k = \sum_{k=1}^{\infty} |\psi'_{\gamma_n}(k)|^p |\widehat{f}^{\,\psi}(k)|^p \lambda^p_k \tag{4.45}$$

where $i_k,\ k \in 1, 2, \ldots,$ denote natural numbers chosen from the condition

$$\psi'_{\gamma_n}(i_k) = \bar{\psi}'_{\gamma_n}(k), \quad k = 1, 2, \ldots .$$

Then we rewrite (4.45) in the form

$$\mathcal{E}^p_{\gamma_n}(f)_{p,\mu} = \sum_{k=1}^{\infty} |\bar{\psi}'_{\gamma_n}(k)\, \widehat{f}^{\,\psi}_\varphi(i_k)\, \lambda_{i_k}|^p.$$

Performing substitution (4.31), we obtain the following analog of inequality (4.32):

$$\mathcal{E}^p_{\gamma_n}(\psi\, U^{q,\lambda}_\varphi)_{p,\mu} \le \sup\left\{ \sum_{k=1}^{\infty} (\bar{\psi}'_{\gamma_n}(k))^p\, m^r_k, \quad r = \frac{p}{q}, \quad \sum_{k=1}^{\infty} m_k \le 1 \right\}.$$

To complete the proof of the theorem, it suffices to repeat the corresponding arguments used in the proof of Theorem 4.4.

Considering lower bounds of both sides of equality (4.42) over all possible collections γ_n of n natural numbers, we conclude that the least lower bound of the right-hand of (4.42) is realized by the collection γ^*_n defined by the relation

$$\gamma^*_n = \{ i_k \in N \colon |\psi'_{i_k}| = |\bar{\psi}'_{i_k}|,\ k = 1, 2, \ldots, n \}, \quad n = 1, 2, \ldots .$$

According to (4.40) and (4.41), we have

$$\bar{\psi}'_{\gamma^*_n}(k) = \bar{\psi}'(k+n), \quad k = 1, 2, \ldots .$$

Therefore, by virtue of (4.39), we get

$$D_n(\psi\, U^{q,\lambda}_\varphi)_{p,\mu} = \left(\sum_{k=1}^{\infty} (\bar{\psi}'_{\gamma^*_n}(k))^{\frac{p\,q}{q-p}} \right)^{\frac{q-p}{p\,q}} = \left(\sum_{k=n+1}^{\infty} (\bar{\psi}'(k))^{\frac{p\,q}{q-p}} \right)^{\frac{q-p}{p\,q}}.$$

Thus, the following statement is true:

Theorem 4.6. *Suppose that numbers p and q and sequences ψ, μ, and λ are such that $q > p > 0$ and condition (4.26) is satisfied. Then, for any natural n, the following equality holds:*

$$D_n(\psi\, U^{q,\lambda}_\varphi)_{p,\mu} = \left(\sum_{k=n+1}^{\infty} (\bar{\psi}'_k)^{\frac{p\,q}{q-p}} \right)^{\frac{q-p}{p\,q}}, \tag{4.46}$$

where $\bar{\psi}'_k$, $k \in N$, is the rearrangement of the sequence $\{|\psi'_k|\}_{k=1}^{\infty}$ in decreasing order.

Note that, in the general case, $\bar{\psi}'_k,\ k \in N,$ is a steplike sequence. Therefore, by virtue of equality (4.24), the quantity $D_n(\psi\, U^{q,\lambda}_\varphi)_{p,\mu}$ has the same behavior for $p \geq q > 0.$ If $p < q,$ then, according to (4.46), this quantity strictly decreases as the parameter n increases.

4.8. Best n-Term Approximations. Let $S^{p,\mu}_\varphi = S^{p,\mu}_\varphi(\mathfrak{X})$ be the space defined by the space $\mathfrak{X},$ system $\varphi,$ number $p > 0,$ and sequence $\mu.$ Further, let $f \in S^{p,\mu}_\varphi$ and let $E_{\gamma_n}(f)_{p,\mu}$ be the quantity of the best approximation of the element f by polynomials constructed on the basis of the given collection γ_n of n basis vectors defined by (4.38). The quantity

$$e_n(f)_{p,\mu} = \inf_{\gamma_n} E_{\gamma_n}(f)_{p,\mu} \tag{4.47}$$

is called the best n-term approximation of the element f in the space $S^{p,\mu}_\varphi,$ and the quantity

$$e_n(\mathfrak{N})_{p,\mu} = \sup_{f \in \mathfrak{N}} e_n(f)_{p,\mu} \tag{4.48}$$

is called the best n-term approximation of a subset $\mathfrak{N}$ of $S^{p,\mu}_\varphi$ in the space $S^{p,\mu}_\varphi.$ It is clear that the quantities $e_n(f)_{p,\mu}$ and $e_n(\mathfrak{N})_{p,\mu}$ correspond to quantities (3.1) and (3.2).

Apparently for the first time, quantities analogous to those defined by equality (4.48) were considered by Stechkin in [37]. Later, they were studied in the theory of approximation of periodic functions by many authors (see, e.g., [38–50]).

As sets $\mathfrak{N},$ we consider the sets $\psi\, U^{q,\lambda}_\varphi$ of ψ-integrals of all elements from the unit balls in the spaces $S^{q,\lambda}_\varphi$ under the conditions that guarantee the inclusion

$$S^{q,\lambda}_\varphi \subset S^{p,\mu}_\varphi.$$

Theorem 4.7. *Let p and q be real numbers such that $p \geq q > 0$ and let $\psi,\ \mu,$ and λ be sequences that satisfy condition (4.23). Then, for any $n \in N,$ the following equality is true:*

$$e^p_n(\psi\, U^{q,\lambda}_\varphi)_{p,\mu} = \sup_{l > n} (l - n) \left(\sum_{k=1}^{l} (\bar{\psi}'_k)^{-q} \right)^{-\frac{p}{q}} = (l^* - n) \left(\sum_{k=1}^{l^*} (\bar{\psi}'_k)^{-q} \right)^{-\frac{p}{q}},$$

where $\bar{\psi}' = \{\bar{\psi}'_k\}_{k=1}^\infty$ is the rearrangement of the sequence

$$|\psi'_k| = \left| \frac{\psi_k\, \mu_k}{\lambda_k} \right|, \quad k = 1, 2, \ldots,$$

in decreasing order and l^* is a certain natural number.

This theorem and Theorem 4.8 for $\mu_k = \lambda_k \equiv 1,\ k \in N,$ were proved in [14, 15] (see also [18]). In the general case, they were proved in [20]. An important role in the proof of Theorem 4.7 is played by the following statement proved in [14] (see also [18]):

Lemma 4.1. *Let $\alpha = \{\alpha_k\}_{k=1}^\infty$ be a nonincreasing sequence of positive numbers $\alpha_k \geq 0,\ k \in N,$ for which*

$$\lim_{k\to\infty} \alpha_k = 0$$

and let $m = \{m_k\}_{k=1}^{\infty}$ *be a sequence of nonnegative numbers* $m_k \geq 0,\ \ k \in N,$ *that satisfies the condition*

$$|m| = \sum_{k=1}^{\infty} m_k \leq 1.$$

Suppose that r *is an arbitrary number,* $r \geq 1,$ *and*

$$S^{(r)}(m) = \sum_{k=1}^{\infty} \alpha_k\, m_k^r, \qquad S_{\gamma_n}^{(r)}(m) = \sum_{k \in \gamma_n} \alpha_k\, m_k^r,$$

where γ_n *is an arbitrary collection of* n *natural numbers,*

$$\mathcal{E}_n(m) = \mathcal{E}_n^{\alpha,r}(m) = S^{(r)}(m) - \sup_{\gamma_n} S_{\gamma_n}^{(r)}(m),$$

and

$$\mathcal{E}_n = \mathcal{E}_n^{\alpha,r} = \sup_{|m| \leq 1} \mathcal{E}_n^{\alpha,r}(m).$$

Then, for any natural $n,$ *there exists a number* $l^* > n$ *such that*

$$\mathcal{E}_n = (l^* - n) \left(\sum_{k=1}^{l^*} \alpha_k^{-1/r} \right)^{-r}.$$

The number l^* *is determined by the equality*

$$\sup_{l > n} (l - n) \left(\sum_{k=1}^{l} \alpha_k^{-1/r} \right)^{-r} = (l^* - n) \left(\sum_{k=1}^{l^*} \alpha_k^{-1/r} \right)^{-r}.$$

Moreover, for the sequence $m' = \{m'_k\}_{k=1}^{\infty},$ *where*

$$m'_k = \begin{cases} \alpha_k^{-1/r} \left(\displaystyle\sum_{i=1}^{l^*} \alpha_i^{-1/r} \right)^{-r}, & k = 1, 2, \ldots, l^*, \\ 0, & k > l^*, \end{cases}$$

the following equality is true:

$$\mathcal{E}_n(m') = (l^* - n) \left(\sum_{i=1}^{l^*} \alpha_i^{-1/r} \right)^{-r}.$$

For $q < p,$ the following statement is true:

Theorem 4.8. *Let p and q be arbitrary numbers such that $q > p$ and let ψ, μ, and λ be sequences for which condition (4.26) is satisfied. Then, for any $n \in N$, the following equality is true:*

$$e_n^p(\psi\, U_\varphi^{q,\lambda})_{p,\mu} = \bar{\sigma}_1^{-\frac{p}{q}} \left[(s-n)^{\frac{q}{q-p}} + \bar{\sigma}_1^{\frac{q}{q-p}}\, \bar{\sigma}_2 \right]^{\frac{q-p}{q}},$$

where

$$\bar{\sigma}_1 = \bar{\sigma}_1(s) = \sum_{k=1}^{s} (\bar{\psi}'_k)^{-q}, \qquad \bar{\sigma}_2 = \bar{\sigma}_2(s) = \sum_{k=s+1}^{\infty} (\bar{\psi}'_k)^{-pq/(q-p)},$$

$\bar{\psi}' = \{\bar{\psi}'_k\}_{k=1}^{\infty}$ is the rearrangement of the sequence

$$|\psi'_k| = |\psi_k \frac{\mu_k}{\lambda_k}|, \quad k = 1, 2, \ldots,$$

in decreasing order, and the number s is chosen from the condition

$$(\bar{\psi}'_s)^{-q} \le \frac{1}{s-n} \sum_{k=1}^{s} (\bar{\psi}'_k)^{-q} < (\bar{\psi}'_{s+1})^{-q}.$$

The number s always exists and is unique.

Proof. The proof of this theorem is based on the following analog of Lemma 4.1 proved in [16] (see also [18]):

Lemma 4.2. *Let $\alpha = \{\alpha_k\}_{k=1}^{\infty}$ be a nonincreasing sequence of positive numbers $\alpha_k \ge 0$, $k \in N$, such that, for a given $r \in (0,1)$, one has*

$$\sum_k \alpha_k^{\frac{1}{1-r}} < \infty$$

and let $m = \{m_k\}_{k=1}^{\infty}$ be a sequence of nonnegative numbers $m_k \ge 0$, $n \in N$, that satisfies the condition

$$|m| = \sum_{k=1}^{\infty} m_k \le 1.$$

Let $S^{(r)}(n_1)$, $S_{\gamma_n}^{(r)}(m)$, $\mathcal{E}_n(m)$, and $\mathcal{E}_n$ be the quantities defined in Lemma 4.1.

Then, for any natural n, the following relation is true:

$$\mathcal{E}_n = \mathcal{E}_n(\alpha; r) = \sigma_1^{-r}(s) \left[(s-n)^{\frac{1}{1-r}} + \sigma_1^{\frac{1}{1-r}}\, \sigma_2(s) \right]^{1-r},$$

where

$$\sigma_1(s) = \sum_{k=1}^{s} \alpha_k^{-1/r}, \qquad \sigma_2(s) = \sum_{k=s+1}^{s} \alpha_k^{\frac{1}{1-r}},$$

and the number s is chosen from the condition

$$a_s^{-1/r} \le \frac{\sigma_1(s)}{s-n} \le \alpha_{s+1}^{-1/r}, \quad s > n.$$

The number s always exists and is unique.

The upper bound in the relation

$$\mathcal{E}_n = \sup_{|m| \le 1} \mathcal{E}_n(m)$$

is realized by the sequence $m = \{m_k\}_{k=1}^{\infty}$, where

$$m_k = \begin{cases} \left(\dfrac{t_s}{\alpha_k}\right)^{1/r}, & k = 1, 2, \dots, s, \\ \dfrac{1 - t_s^{1/r}\sigma_1(s)}{\sigma_2(s)}\, \alpha_k^{1/(1-r)}, & k > s, \end{cases}$$

$$t_s = \left(\sigma_1(s) + \left(\frac{\sigma_1(s)}{s-n}\right)^{1/(1-r)} \sigma_2(s)\right)^{-r}.$$

4.9. Best n-Term Approximations with Restrictions. Let Γ_n be the set of all collections γ_n of n natural numbers.

In this case, we can represent the quantity $e_n(f)_{p,\mu}$ defined by (4.47) in the form

$$e_n(f)_{p,\mu} = \inf_{\gamma_n \in \Gamma_n} E_{\gamma_n}(f)_{p,\mu}.$$

Parallel with $e_n(f)_{p,\mu}$, one can also consider the quantities

$$e_n(f; \Gamma_n')_{p,\mu} = \inf_{\gamma_n \in \Gamma_n'} E_{\gamma_n}(f)_{p,\mu},$$

where Γ_n' is a certain proper subset of Γ_n. In this connection, it is convenient to call the quantity $e_n(f)_{p,\mu}$ the absolute best n-term approximation and to call the quantity $e_n(f; \Gamma_n')_{p,\mu}$ the best n-term approximation with restrictions, keeping in mind that the term "restriction" stands for the choice of the subset Γ_n'.

As Γ_n', we consider two subsets $\Gamma_n^{(1)}$ and $\Gamma_n^{(2)}$. Let $\Gamma_n^{(1)}$ denote the set of collections

$$\gamma_n^{(1)} = \{in+1, in+2, \dots, (i+1)n\}, \quad i = 0, 1, \dots,$$

and let $\Gamma_n^{(2)}$ denote the set of collections

$$\gamma_n^{(2)} = \{i+1, i+2, \dots, i+n\}, \quad i = 0, 1, \dots.$$

It is clear that we always have

$$\Gamma_n^{(1)} \subset \Gamma_n^{(2)} \subset \Gamma_n,$$

whence

$$e_n(f)_{p,\mu} \le e_n(f;\Gamma_n^{(2)})_{p,\mu} \le e_n(f;\Gamma_n^{(1)})_{p,\mu}.$$

Therefore, by setting

$$e_n(\mathfrak{N};\Gamma'_n) = \sup_{f\in\mathfrak{N}} e_n(f;\Gamma'_n),$$

where $\mathfrak{N}$ is a certain subset of $S_\varphi^{p,\mu}$, we obtain the estimates

$$e_n(\mathfrak{N})_{p,\mu} \le e_n(\mathfrak{N};\Gamma_n^{(2)})_{p,\mu} \le e_n(\mathfrak{N};\Gamma_n^{(1)})_{p,\mu}.$$

As before, we use the sets $\psi\, U_\varphi^{q,\lambda}$ of ψ-integrals of all elements of the unit ball $U_\varphi^{q,\lambda}$ in the space $S_\varphi^{q,\lambda}$ as the sets $\mathfrak{N}$.

First, we consider the case where $p \ge q > 0$. The following theorem is true:

Theorem 4.9. *Let p and q be real numbers such that $p \ge q > 0$ and let ψ, μ, and λ be sequences for which the quantities*

$$|\psi'_k| = \left|\frac{\psi_k \mu_k}{\lambda_k}\right|, \quad k = 1, 2, \ldots, \tag{4.49}$$

do not increase and tend to zero. Then, for any $n \in N$, the following equalities are true:

$$e_n(\psi\, U_\varphi^{q,\lambda};\Gamma^{(1)})_{p,\mu} = e_n(\psi\, U_\varphi^{q,\lambda};\Gamma^{(2)})_{p,\mu} = (l^*-1)^{1/p}\left(\sum_{k=1}^{l^*}|\psi'_{(k-1)n+1}|^{-q}\right)^{-1/q},$$

where l^* is a natural number for which

$$\sup_{l>1}(l-1)^{1/p}\left(\sum_{k=1}^{l}|\psi'_{(k-1)n+1}|^{-q}\right)^{-1/q} = (l^*-1)\left(\sum_{k=1}^{l^*}|\psi'_{(k-1)n+1}|^{-q}\right)^{-1/q}.$$

The number l^* always exists.

Proof. For $\mu_k \equiv \lambda_k \equiv 1$, $k \in N$, this theorem was, in fact, proved in [20]. In the general case, it was proved in [51]. Its proof is based on the arguments of [20] and the following statement proved therein:

Lemma 4.3. *Let $\alpha = \{\alpha_k\}_{k=1}^\infty$ be a nonincreasing sequence of positive numbers for which*

$$\lim_{k\to\infty}\alpha_k = 0 \tag{4.50}$$

and let $m = \{m_k\}_{k=1}^\infty$ be a sequence of nonnegative numbers such that

$$|m| = \sum_{k=1}^\infty m_k \le 1. \tag{4.51}$$

(*In this case, one has $\alpha \in A$ and, hence, $m \in \mathcal{M}$.*)

Suppose that, for each $n \in N,$ *one has*

$$F_{n,r}(\alpha, m) = \sum_{k=1}^{\infty} \alpha_k \, m_k^r - \sup_{\gamma_n \in \Gamma_n^{(1)}} \sum_{k \in \gamma_n} \alpha_k \, m_k^r, \quad \alpha \in A, \quad m \in \mathcal{M}, \quad r > 0,$$

and

$$\sigma_{n,r}(\alpha) = \sup_{m \in \mathcal{M}} F_{n,r}(\alpha, m).$$

Then, for any $r \geq 1$ *and* $n \in N,$ *the following equality is true:*

$$\sigma_{n,r}(\alpha) = \sup_{q > 1} (q-1) \left(\sum_{i=1}^{q} \alpha_{(i-1)n+1}^{-1/r} \right)^{-r}. \tag{4.52}$$

The upper bound on the right-hand side of (4.52) is always attained for a certain value q^*. *Moreover, for the sequence* $m' = \{m'_k\}_{k=1}^{\infty}$ *from* $\mathcal{M},$ *where*

$$m'_k = \begin{cases} \alpha_{(i-1)n+1}^{-1/r} \left(\sum\limits_{j=1}^{q^*} \alpha_{(j-1)n+1}^{-1/r} \right)^{-1}, & k = (i-1)n+1, \ i = 1, 2, \ldots, q^*, \\ 0 & \textit{for the other values of } k, \end{cases}$$

the following equality holds:

$$F_{n,r}(\alpha, m') = (q^* - 1) \left(\sum_{i=1}^{q^*} \alpha_{(i-1)n+1}^{-1/r} \right)^{-r}.$$

For $q > p > 0,$ the following statement is true:

Theorem 4.10. *Let* p *and* q *be real numbers such that* $q > p > 0$ *and let* $\psi,$ $\mu,$ *and* λ *be sequences for which quantities (4.49) do not increase, tend to zero, and satisfy condition (4.26). Then, for any* $n \in N,$ *the following equality holds:*

$$e_n^p(\psi \, U_\varphi^{q,\lambda}; \Gamma_n^{(1)})_{p,\mu} = \widetilde{\sigma}_1^{\,-\frac{p}{q}}(s) \left[(s-1)^{\frac{q}{q-p}} + \widetilde{\sigma}_1^{\,\frac{p}{q-p}}(s) \, \widetilde{\sigma}_2(s) \right]^{\frac{q-p}{q}},$$

where

$$\widetilde{\sigma}_1(s) = \sum_{k=1}^{s} \left(\sum_{i=(k-1)n+1}^{k\,n} |\psi'_i|^{\frac{p\,q}{q}} \right)^{-\frac{q-p}{q}},$$

$$\widetilde{\sigma}_2(s) = \sum_{k=s\,n+1}^{\infty} |\psi'_k|^{\frac{p\,q}{q-p}}.$$

The number s *is chosen from the condition*

$$\left(\sum_{k=(s-1)n+1}^{s\,n} |\psi'_k|^{\frac{p\,q}{q-p}}\right)^{-\frac{q-p}{p}} \leq \frac{\widetilde{\sigma}_1(s)}{s-1} < \left(\sum_{k=s\,n+1}^{(s+1)n} |\psi'_k|^{\frac{p\,q}{q-p}}\right)^{-\frac{q-p}{p}}.$$

The number s *always exists and is unique.*

This theorem was proved in [51]. The proof is based on the following analog of Lemma 4.3:

Lemma 4.4. *Let* $\alpha = \{\alpha_k\}_{k=1}^{\infty}$ *be a nonincreasing sequence of positive numbers for which condition (4.50) is satisfied and, for a given* $r \in (0,1)$, *one has*

$$\sum_{k=1}^{\infty} \alpha_k^{\frac{1}{1-r}} < \infty$$

and let $m = \{m_k\}_{k=1}^{\infty}$ *be a sequence of nonnegative numbers for which condition (4.51) is satisfied.* (*In this case, one has* $\alpha \in A_r$ *and, as before,* $m \in \mathcal{M}$.)

Suppose that, for each $n \in N$,

$$F_{n,r}(\alpha, m) = \sum_{k=1}^{\infty} \alpha_k m_k^r - \sup_{\gamma_n \in \Gamma_n^{(1)}} \sum_{k \in \gamma_n} \alpha_k m_k^r, \quad \alpha \in A_r, \ \ m \in \mathcal{M}, \ \ r \in (0,1),$$

and

$$\sigma_{n,r}(\alpha) = \sup_{m \in \mathcal{M}} F_{n,r}(\alpha, m). \tag{4.53}$$

Then, for any $r \in (0,1)$ *and* $n \in N$, *the following equality is true:*

$$\sigma_{n,r}(\alpha) = \bar{\sigma}_1^{-r}(s)\Big[(s-1)^{\frac{1}{1-r}} + \bar{\sigma}_1^{\frac{r}{1-r}}(s)\,\bar{\sigma}_2(s)\Big]^{1-r},$$

where

$$\bar{\sigma}_1(s) = \bar{\sigma}_1(\alpha; s) = \sum_{k=1}^{s} \left(\sum_{i=(k-1)n+1}^{k\,n} \alpha_i^{\frac{1}{1-r}}\right)^{-\frac{1-r}{r}},$$

$$\bar{\sigma}_2(s) = \bar{\sigma}_2(\alpha; s) = \sum_{k=s\,n+1}^{\infty} \alpha_i^{\frac{1}{1-r}},$$

and the number s *is chosen from the condition*

$$a_s^{-\frac{1}{r}} \leq \frac{\bar{\sigma}_1(s)}{s-1} < a_{s+1}^{-\frac{1}{r}}, \quad a_j = \left(\sum_{i=(j-1)n+1}^{j\,n} \alpha_i^{\frac{1}{1-r}}\right)^{1-r}, \quad j = 1, 2, \ldots .$$

The number s always exists and is unique.

The upper bound in (4.53) is realized by the sequence m^* for which

$$m_k^* = \mu_i^* \, \alpha_k^{\frac{1}{1-r}} \, a_i^{-\frac{1}{1-r}}, \qquad (i-1)n + 1 \le k \le i\,n, \quad i = 1, 2, \ldots,$$

where

$$\mu_i^* = \begin{cases} \left(\dfrac{t_s}{a_i}\right)^{1/r}, & i = 1, 2, \ldots, s, \\[2ex] \dfrac{1 - t_s^{\frac{1}{r}}\, \bar{\sigma}_1(s)}{\bar{\sigma}_2(s)}\, a_i^{\frac{1}{1-r}}, & i > s, \\[2ex] t_s = \left(\bar{\sigma}_1(s) + \left(\dfrac{\bar{\sigma}_1(s)}{s-1}\right)^{\frac{1}{1-r}} \bar{\sigma}_2(s)\right)^{-r}. \end{cases}$$

Results concerning the quantity $e_n\left(\psi\, U_\varphi^{q,\lambda}; \Gamma_n^{(2)}\right)_{p,\mu}$ for $q > p > 0$ are given in [51].

5. Approximation of Individual Elements in the Spaces S_Φ^p

In Sec. 4 , we have considered extremal problems for various approximation characteristics of the sets $\psi\, U_\varphi^{p,\mu}$. In this section, we present results concerning the best approximations of individual elements from the spaces S_Φ^p.

As in Sec. 4.3, let

$$E_n(f) = E_n(f)_{\psi,p,\mu} = \inf_{c_k} \left\| f - \sum_{k \in g_{n-1}^\psi} c_k\, \varphi_k \right\|_{p,\mu} \tag{5.1}$$

be the best approximation of an element $f \in \psi\, S_\varphi^{p,\mu}$ by polynomials constructed for the domains g_{n-1}^ψ. In the case where the sequence μ is such that $\mu_k \equiv 1$, we omit the index μ in all objects considered. In this notation, the following statements were proved in [16]:

Theorem 5.1. *Suppose that $f \in S_\varphi^p$, $p > 0$, and a sequence $\psi = \{\psi_k\}_{k=1}^\infty$ satisfies condition (4.6). Then the series*

$$\sum_{k=2}^\infty (\varepsilon_k^p - \varepsilon_{k-1}^p) E_k^p(f)_{\psi,p}$$

converges and, for any $n \in N$, the following equality is true:

$$E_n^p(f)_{\psi,p} = \varepsilon_n^p\, E_n^p(f^\psi)_{\psi,p} + \sum_{k=n+1}^\infty (\varepsilon_k^p - \varepsilon_{k-1}^p)\, E_k^p(f^\psi)_{\psi,p}, \tag{5.2}$$

where the quantities $E_n(x)_{\psi,p}$ are determined by equality (5.1) and ε_k, $k = 1, 2, \ldots,$ are elements of the characteristic sequence $\varepsilon(\psi)$.

Theorem 5.2. *Suppose that* $f \in S^p_\varphi,$ $p > 0,$ *and a sequence* $\psi = \{\psi_k\}_{k=1}^\infty$ *satisfies condition (4.6). Also assume that*

$$\lim_{k\to\infty} \varepsilon_k^{-1} E_k(f)_{\psi,p} = 0.$$

In order that

$$f \in S^p_\varphi,$$

it is necessary and sufficient that the series

$$\sum_{k=2}^\infty (\varepsilon_k^{-p} - \varepsilon_{k-1}^{-p}) E_k^p(f)_{\psi,p}$$

be convergent. If this series converges, then the following equality holds for any $n \in N:$

$$E_n^p(f)_{\psi,p} = \varepsilon_n^{-p} E_n^p(f^\psi)_{\psi,p} + \sum_{k=n+1}^\infty (\varepsilon_k^{-p} - \varepsilon_{k-1}^{-p}) E_k^p(f)_{\psi,p},$$

where the quantities $E_n(x)_{\psi,p}$ *and* ε_k *have the same sense as in Theorem 5.1.*

Theorem 5.1 establishes the relationship between the best approximation of an element f and the best approximations of its derivatives. As is known, in approximation theory similar statements are usually called direct theorems. In this sense, Theorem 5.2 is an inverse theorem: it establishes the existence of derivatives of an element f on the basis of properties of the best approximation of this element and gives information about the best approximation of these derivatives.

Taking into account that the quantities ε_n are strictly decreasing and using relation (5.2), we get

$$E_n^p(f)_{\psi,p} \le \varepsilon_n^p E_n^p(f^\psi)_{\psi,p} \quad \forall f \in \psi S^p_\varphi \quad \forall n \in N. \tag{5.3}$$

Note that, by virtue of Proposition 4.1, we always have

$$\lim_{n\to\infty} E_n^p(f^\psi)_{\psi,p} = 0.$$

On important subsets $\mathfrak{N}$ of $\psi S^p_\varphi,$ relation (5.3) gives an exact result. Consider one of these cases.

As $\mathfrak{N},$ we take the set ψU^q_φ for $0 < q < p.$ If $f \in \psi U^q_\varphi,$ then $f^\psi \in U^q_\varphi$ and, moreover, $f^\psi \in U^p_\varphi.$ Therefore, $\|f^\psi\|_p \le 1$ and, hence, $E_n^p(f^\psi)_{\psi,p} \le 1.$ Consequently,

$$E_n^p(f)_{\psi,p} \le \varepsilon_n^p \quad \forall f \in \psi U^q_\varphi, \quad 0 < q \le p. \tag{5.4}$$

On the other hand, let k' be an arbitrary point from the set $g_n^\psi \setminus g_{n-1}^\psi$ and let $f_* = \psi_{k'} \varphi_{k'},$ $(\psi_{k'}) \ne 0.$ Since $f_*^\psi = \varphi_{k'},$ we have $\|f_*^\psi\|_q = 1$ for any $q > 0.$ Therefore, $f_* \in \psi U^q_\varphi$ for any $q > 0.$ It is clear that

$$E_n(f_*)_{\psi,p} = \|f_*\|_{\varphi,p} = \psi_{k'} = \varepsilon_n. \tag{5.5}$$

Thus, combining relations (5.4) and (5.5) and setting

$$E_n(\psi\, U^q_\varphi)_{\psi,p} = \sup_{f\in\psi\, U^q_\varphi} E_n(f)_{\psi,p}, \qquad \mathcal{E}_n(\psi\, U^q_\varphi)_{\psi,p} = \sup_{f\in\psi\, U^q_\varphi} \mathcal{E}_n(f)_{\psi,p},$$

we arrive at the following statement:

Theorem 5.3. *Let* $\psi = \{\psi_k\}_{k=1}^{\infty}$ *be a system of numbers that satisfies conditions (4.6) and (4.14). Then, for any* $n \in N$ *and* $0 < q \le p < \infty,$ *the following equalities are true:*

$$E_n(\psi\, U^q_\varphi)_{\psi,p} = \mathcal{E}_n(\psi\, U^q_\varphi)_{\psi,p} = \varepsilon_n,$$

where ε_n *is the* n*th term of the characteristic sequence* $\varepsilon(\psi)$.

Note that this statement is a special case of Theorem 4.1.

6. Applications of the Results Obtained to Problems of Approximation of Periodic Functions of Many Variables

Consider one possible concretization of the spaces $S^p_\varphi = S^p_\varphi(\mathfrak{X})$ for which one can deduce statements on approximation of periodic functions from the general results obtained in Secs. 4 and 5.

Assume that R^m is an m-dimensional $(m \ge 1)$ Euclidean space, $x = (x_1, \ldots, x_m)$ are its elements, Z^m is the integer lattice in R^m (the set of vectors $k = (k_1, \ldots, k_m)$ with integral coordinates), $xy = x_1y_1 + \ldots + x_my_m$, $|x| = \sqrt{(xx)}$, and, in particular, $kx = k_1x_1 + \ldots + k_mx_m$, $|k| = \sqrt{k_1^2 + \ldots + k_m^2}$.

Further, let $L = L(R^m)$ be the set of all functions $f(x) = f(x_1, \ldots, x_m)$ 2π-periodic in each variable and summable in the cube of periods Q^m, where

$$Q^m = \{x\colon x \in R^m,\ -\pi \le x_k \le \pi,\ k = 1, \ldots, m\}.$$

If $f \in L$, then $S[f]$ denotes the Fourier series of the function f in the trigonometric system

$$(2\pi)^{-m/2}\, e^{ikx}, \quad k \in Z^m, \tag{6.1}$$

i.e.,

$$S[f] = (2\pi)^{-m/2} \sum_{k\in Z^m} \widehat{f}(k)\, e^{ikx}, \qquad \widehat{f}(k) = (2\pi)^{-m/2} \int\limits_{Q^m} f(t)\, e^{ikt} dt. \tag{6.2}$$

If functions equivalent with respect to the Lebesgue measure are assumed to be indistinguishable, then we can take the space $L(R^m)$ as $\mathfrak{X}$; as the system φ we can take the trigonometric system $\tau = \{\tau_s\}$, $s \in N$, where

$$\tau_s = (2\pi)^{-m/2}\, e^{ik_s x}, \qquad k_s \in Z^m, \quad s = 1, 2, \ldots,$$

which is obtained from system (1.6) by an arbitrary fixed enumeration of its elements. In this case, the scalar product is defined in the following way:

$$(f, \tau_s) = (2\pi)^{-m/2} \int\limits_{Q^m} f(t)\, e^{-ik_s t} dt = \widehat{f}(k_s) = \widehat{f}_\tau(k_s).$$

For fixed $p \in (0, \infty)$, according to (1.6) we set

$$S^p_\tau = S^p_\tau\,(L(R^m)) = \left\{ f \in L(R^m))\colon\ \sum_{s=1}^{\infty} |\widehat{f}(k_s)|^p \leq \infty \right\}. \tag{6.3}$$

The "φ-norm" in the space S^p_τ is introduced according to (1.7) as follows:

$$\|f\|_{p,\tau} = \left(\sum_{s=1}^{\infty} \left| \widehat{f}(k_s) \right|^p \right)^{\frac{1}{p}}. \tag{6.4}$$

By virtue of equalities (6.3) and (6.4), the quantities $\|\cdot\|_{p,\tau}$ and the spaces S^p_τ do not depend on the enumeration of system (6.1). For this reason, in what follows, we set $S^p_\tau = S^p$.

Now let $\psi = \{\psi(k)\}_{k \in Z^m}$ be an arbitrary system of complex numbers (a multiple sequence). For functions $f \in L$, parallel with (6.2) we consider the series

$$(2\pi)^{-m/2} \sum_{k \in Z^m} \psi(k)\, \widehat{f}(k)\, e^{ikx}.$$

If this series for a given function f and a system ψ is the Fourier series for a function F from L, then we say that F is the ψ-integral of the function f and write $F(x) = \mathcal{J}^\psi(f; x)$. It is sometimes convenient to call the function f the ψ-derivative of the function F and write $f(x) = D^\psi(F; x) = F^\psi(x)$. The set of ψ-integrals of all functions $f \in L$ is denoted by L^ψ. If $\mathfrak{N}$ is a certain subset of L, then $L^\psi\,\mathfrak{N}$ denotes the set of ψ-integrals of all functions from $\mathfrak{N}$. It is clear that if $f \in L^\psi$, then the Fourier coefficients of the functions f and f^ψ are connected by the relation

$$\widehat{f}(k) = \psi(k)\, \widehat{f}^{\,\psi}(k), \quad k \in Z^m.$$

As $\mathfrak{N}$, we consider the unit ball U^p in the space S^p:

$$U^p = \{f\colon f \in S^p,\ \|f\|_p \leq 1\}.$$

In this case, we set $L^\psi\, U^p = L^\psi_p = L^\psi_p(R^m)$. For the system ψ, we assume that

$$\lim_{|k| \to \infty} \psi(k) = 0. \tag{6.5}$$

Note that if $f \in L^\psi S^p$ and $|\psi(k)| \leq C$, $k \in Z^m$, $C > 0$, then $f \in S^p$, i.e., condition (6.5) always guarantees the inclusion $L^\psi_p \subset S^p$.

We define the characteristic sequences $\varepsilon(\psi)$, $g(\psi)$, and $\delta(\psi)$ as follows:

$\varepsilon(\psi) = \varepsilon_1, \varepsilon_2, \ldots$ is the set of the values of $|\psi(k)|$, $k \in Z^m$, arranged in decreasing order;

$g(\psi) = \{g_n\}_{n=1}^\infty$, where

$$g_n = g^\psi_n = \{k \in Z^m\colon |\psi(k)| \geq \varepsilon_n\};$$

$\delta(\psi) = \{\delta_n\}_{n=1}^\infty$, where $\delta_n = \delta^\psi_n = |g_n|$ is the number of numbers $k \in Z^m$ belonging to the set g_n.

In the case considered, by virtue of condition (6.5), the sequences $\varepsilon(\psi)$ and $g(\psi)$ are determined, as in Sec. 4.3, with regard for the fact that $k \in Z^m$. As before, we assume that $g_0 = g_0^\psi$ is the empty set and $\delta_0 = \delta_0^\psi = 0$.

As approximating aggregates for the functions $f \in L^\psi$, we take the following trigonometric polynomials, which are analogs of sums (4.8):

$$S_n(f;x) = S_{g_n^\psi}(f;x) = (2\pi)^{-m/2} \sum_{k \in g_n^\psi} \widehat{f}(k)\, e^{ikx}, \tag{6.6}$$

$$n \in N, \quad S_0(f;x) = 0,$$

where g_n^ψ are elements of the sequence $g(\psi)$.

Assume that p and q are arbitrary numbers, $p, q > 0$,

$$\mathcal{E}_n(f)_{\psi,p} = \|f(x) - S_{n-1}(f;x)\|_{S^p}, \tag{6.7}$$

$$\mathcal{E}_n(L_q^\psi)_p = \sup_{f \in L_q^\psi} \mathcal{E}(f)_{\psi,p}, \quad n = 1, 2, \ldots, \tag{6.8}$$

$$E_n(f)_{\psi,p} = \inf_{a_k} \|f(x) - (2\pi)^{-m/2} \sum_{k \in g_{n-1}^\psi} a_k\, e^{ikx}\|_{S^p}, \tag{6.9}$$

and

$$E_n(L_q^\psi)_p = \sup_{f \in L_q^\psi} E_n(f)_{\psi,p}, \quad n = 1, 2, \ldots . \tag{6.10}$$

Further, let

$$d_n(L_p^\psi)_p = \inf_{F_n \in G_n} \sup_{f \in L_p^\psi} \inf_{u \in F_n} \|f - u\|_{S^p}, \quad n \in N, \qquad d_0(L_p^\psi)_p \overset{\text{df}}{=} \sup_{f \in L_p^\psi} \|f\|_{S^p},$$

where G_n is the set of all n-dimensional subspaces in S^p, be the Kolmogorov widths of the classes L_p^ψ and let

$$e_n(L_q^\psi)_p = \sup_{f \in L_q^\psi} \inf_{a_k, \gamma_n} \|f(x) - (2\pi)^{-m/2} \sum_{k \in \gamma_n} a_k\, e^{ikx}\|_{S^p},$$

where γ_n is an arbitrary collection of n vectors $k \in Z^m$, be the quantity of the best n-term approximation of the class L_q^ψ in the space S^p.

In the notation introduced, the following statements (analogs and, in fact, special cases of the theorems proved in Secs. 4 and 5) are true:

Theorem 6.1. *Let $\psi = \{\psi_k\}_{k \in Z^m}$ be a system of numbers that satisfy conditions (6.5) and are such that*

$$\psi(k) \neq 0 \quad \forall k \in Z^m. \tag{6.11}$$

Then, for any $n \in N$ *and* $0 < q \le p < \infty$, *the following equalities are true:*

$$E_n(L^\psi_q)_p = \mathcal{E}_n(L^\psi_q)_p = \varepsilon_n, \tag{6.12}$$

$$e_n(L^\psi_q)_p = \sup_{l>n}(l-n)\left(\sum_{k=1}^{l}\bar\psi_k^{-q}\right)^{-\frac{p}{q}} = (l^*-n)\left(\sum_{k=1}^{l^*}\bar\psi_k^{-q}\right)^{-\frac{p}{q}}, \tag{6.13}$$

where $\bar\psi = \{\bar\psi_k\}_{k=1}^\infty$ *is the sequence defined by the relations*

$$\bar\psi_k = \varepsilon_n \quad \textit{for} \quad k \in (\delta_{n-1}, \delta_n], \quad n = 1, 2, \ldots,$$

ε_n *and* δ_n *are terms of the characteristic sequences of the system* ψ, *and* l^* *is a certain natural number, which always exists under the conditions of the theorem.*

The following statements are analogs of Theorems 5.1 and 5.2:

Theorem 6.2. *Suppose that* $f \in L^\psi_p$, $p > 0$, *and* $\psi = \{\psi_k\}_{k\in Z^m}$ *is a system of numbers that satisfies conditions (6.5). Then the series*

$$\sum_{k=1}^{\infty}(\varepsilon_k^p - \varepsilon_{k-1}^p)E_k^p(f^\psi)_{\psi,p}$$

converges and, for any $n \in N$, *the following equality holds:*

$$E_n^p(f)_{\psi,p} = \varepsilon_n^p\, E_n^p(f^\psi)_{\psi,p} + \sum_{k=n+1}^{\infty}(\varepsilon_k^p - \varepsilon_{k-1}^p)E_k^p(f^\psi)_{\psi,p},$$

where the quantities $E_\nu(\cdot)_{\psi,p}$ *are defined by (6.9) and* ε_k *are elements of the characteristic sequence* $\varepsilon(\psi)$ *of the system* ψ.

Theorem 6.3. *Suppose that* $f \in S^p$, $p > 0$, *and the system* $\psi = \{\psi_k\}_{k\in Z^m}$ *satisfies condition (6.9). Also assume that*

$$\lim_{k\to\infty}\varepsilon_k^{-1}\,E_k(f)_{\psi,p} = 0.$$

In order that

$$f \in L^\psi_p,$$

it is necessary and sufficient that the following series be convergent:

$$\sum_{k=1}^{\infty}(\varepsilon_k^{-p} - \varepsilon_{k-1}^{-p})E_k^p(f)_{\psi,p}.$$

If this series converges, then the following equality holds for any $n \in N$:

$$E_n^p(f^{\psi})_{\psi,p} = \varepsilon_n^{-p}\, E_n^p(f)_{\psi,p} + \sum_{k=n+1}^{\infty} (\varepsilon_k^{-p} - \varepsilon_{k-1}^{-p}) E_k^p(f)_{\psi,p},$$

where the quantities $E_n(\cdot)_{\psi,p}$ *and* ε_k *have the same sense as in Theorem 6.2.*

Let $\psi = \{\psi_k\}_{k\in Z^m}$ be an arbitrary system of numbers that satisfies condition (6.5). We enumerate all vectors $k \in Z^m$ in a certain order using a natural index s. We say that the system ψ belongs to a set $A_{p,q}$ for some values of p and q, $q > p > 0$, if

$$\sum_{s=1}^{\infty} |\psi(k_s)|^{\frac{p\,q}{q-p}} < \infty. \tag{6.14}$$

It is clear that the sets $A_{p,q}$ are independent of the method of enumeration of the numbers $k \in Z^m$ and are completely determined by the quantities $|\psi(k)|$ and numbers p and q.

Theorem 6.4. *Suppose that, for given* p *and* q *such that* $q > p > 0$, *a system* $\psi = \{\psi_k\}_{k\in Z^m}$, *belongs to the set* $A_{p,q}$. *Then*

$$E_n(L_q^{\psi})_p = \mathcal{E}_n(L_q^{\psi})_p = \left(\sum_{k=\delta_{n-1}+1}^{\infty} \bar{\psi}^{\frac{p\,q}{q-p}}\right)^{\frac{q-p}{p\,q}}, \quad n = 1, 2, \ldots,$$

and

$$e_n^p(L_q^{\psi})_p = \widetilde{\sigma}_1^{-\frac{p}{q}}(s)\Big[(s-n)^{\frac{q}{q-p}} + \widetilde{\sigma}_1^{\frac{p}{q-p}}(s)\,\widetilde{\sigma}_2(s)\Big]^{\frac{q-p}{q}},$$

where

$$\widetilde{\sigma}_1(s) = \sum_{k=1}^{s} \bar{\psi}_k^{-q}, \qquad \widetilde{\sigma}_2(s) = \sum_{k=s+1}^{\infty} \psi_k^{\frac{p\,q}{q-p}},$$

$\bar{\psi} = \{\psi_k\}_{k=1}^{\infty}$ *is a sequence for which*

$$\bar{\psi}_k = \varepsilon_k \quad \text{for} \quad k \in (\delta_{n-1}, \delta_n], \;\; n = 1, 2, \ldots,$$

ε_n *and* δ_n *are terms of the characteristic sequences* $\varepsilon(\psi)$ *and* δ_{ψ}, *and the number* s *is chosen from the condition*

$$\bar{\psi}_s^{-q} \le \frac{1}{s-n} \sum_{k=1}^{s} \bar{\psi}_k^{-q} < \psi_{s+1}^{-q}. \tag{6.15}$$

The number s *always exists and is unique.*

The proofs of these theorems are based on the corresponding theorems from previous sections.

Using the systems ψ appearing in these statements as a starting point, we enumerate all vectors $k \in Z^m$ so that all vectors k from the sets $g_n^\psi \setminus g_{n-1}^\psi$ are enumerated by numbers $s \in (\delta_{n-1}, \delta_n]$ in a certain fixed order. Then we define a sequence $\psi' = \{\bar{\psi}_s\}_{s=1}^\infty$ by setting

$$\psi'_s = \psi(k_s), \quad s = 1, 2, \ldots . \tag{6.16}$$

Then

$$S[f] = (2\pi)^{-m/2} \sum_{k \in Z^m} \widehat{f}(k)\, e^{ikx} = (2\pi)^{-m/2} \sum_{s=1}^\infty \widehat{f}(k_s)\, e^{i\,k_s\,x},$$

and, according to (6.16),

$$\mathcal{J}^{\psi'}(f; x) = (2\pi)^{-m/2} \sum_{s=1}^\infty \psi(k_s) \widehat{f}(k_s)\, e^{i\,k_s\,x} = \mathcal{J}(f; x) \quad \forall f \in L.$$

Therefore, $L^{\psi'} = L^\psi$. Further, note that $L^{\psi'} = \psi' U^p$, where $\psi' U^p$ is the set defined according to (4.7′), namely

$$\psi' U^p = \{f \in L \colon f^{\psi'} \in U^p\},$$

where $U^p = U^p_\varphi$ and $\varphi = \{2\pi^{-m/2} e^{i\,k_s\,x}\}_{s=1}^\infty$. Furthermore, the sequences $\mathcal{E}(\psi')$ and $\mathcal{E}^\psi$, as well as $\delta(\psi')$ and $\delta(\psi)$, coincide, and the following equalities are true:

$$S_{g_n^{\psi'}}(f) = S_{g_n^\psi}(f; x), \quad \mathcal{E}_n(f)_{\psi', p} = \mathcal{E}_n(f)_{\psi, p}, \quad \mathcal{E}_n(\psi' U^q)_p = \mathcal{E}_n(L_q^\psi)_p,$$

$$E_n(f)_{\psi', p} = E_n(f), \quad E_n(\psi' U^q)_p = E_n(L_q^\psi)_p \,.$$

The left-hand sides of these equalities are determined by (4.8) and (4.9), and their right-hand sides are determined by (6.6) – (6.10). It is clear that $e_n(L_q^\psi)_p = e_n(\psi' U^q)$ and $\bar{\psi}' = \bar{\psi}$. This implies that equality (6.12) follows from (4.13), equality (6.13) follows from Theorem 4.7, Theorems 6.2 and 6.3 follow from Theorems 5.1 and 5.2, and Theorem 6.4 follows from Theorems 4.4 and 4.8.

For the completeness of presentation, we reformulate Theorem 4.2 for the Kolmogorov widths of the sets L_p^ψ.

Theorem 6.5. *Let* $\psi = \{\psi_k\}_{k \in Z^m}$ *be a system of numbers that satisfies conditions (6.5) and (6.11) and let* $p \in [1, \infty)$. *Then the following relation holds for any* $n \in N$:

$$d_{\delta_{n-1}}(L_p^\psi; S^p) = d_{\delta_{n-1+1}}(L_p^\psi; S^p) = \ldots = d_{\delta_{n-1}}(L_p^\psi; S^p) = \mathcal{E}_n(L_p^\psi)_p = \varepsilon_n,$$

where ε_n *and* δ_n *are terms of the characteristic sequences* $\varepsilon(\psi)$ *and* $\delta(\psi)$.

7. Remarks

7.1. On the Sequences $\boldsymbol{\Psi}$**.** In all previous constructions, the sequences ψ play the key role: they determine approximated sets, the approximation apparatus is constructed on their basis, and approximation characteristics are expressed in their terms. Except for conditions of the form (6.9) and (6.21), without which considerations become almost meaningless, no restrictions have been imposed on the sequences ψ in the present paper. For this reason, the systems ψ and their characteristic sequences $\varepsilon(\psi)$, $g(\psi)$, and $\delta(\psi)$ may be fairly complicated in the general case.

In the multidimensional case, apparently the most natural and simplest systems are the systems ψ for which the numbers $\psi(k)$ are represented by the products

$$\psi(k) = \psi(k_1, \ldots, k_m) = \prod_{j=1}^{m} \psi_j(k_j), \quad k_j \in Z^1, \quad j = \overline{1,m}, \tag{7.1}$$

of values of the one-dimensional sequences $\psi_j = \{\psi_j(k_j)\}_{k_j=1}^{\infty}$. Furthermore, if

$$\psi(-k_j) = \overline{\psi_j(k_j)}, \quad j = \overline{1,m},$$

where $\bar{z}$ denotes the complex conjugate of the number z, then the sets g_n^{ψ} are symmetric with respect to all coordinate planes. It is easy to verify that

$$\sum_{k \in Z^m} \psi(k)\, e^{ikt} = \sum_{k \in Z_+^m} 2^{m-q(k)} \prod_{j=1}^{m} |\psi_j(k_j)| \cos\left(k_j t_j - \frac{\beta_{k_j}\pi}{2}\right), \tag{7.2}$$

where $Z_+^m = \{k \in Z^m,\ k_i \geq 0,\ i = \overline{1,m}\}$, $q(k)$ is the number of zero coordinates of the vector k, and the numbers β_{k_j} are defined by the equalities

$$\cos\frac{\beta_{k_j}\pi}{2} = \frac{Re\,\psi_j(k_j)}{|\psi_j(k_j)|}, \qquad \sin\frac{\beta_{k_j}\pi}{2} = \frac{Im\,\psi_j(k_j)}{|\psi_j(k_j)|}.$$

In this case, the set L^{ψ} of ψ-integrals of real functions φ from $L(R^m)$ consists of real functions f, and if, in addition, the series in (7.2) is the Fourier series of a certain summable function $\mathcal{D}_{\psi}(t)$, then a necessary and sufficient condition for the inclusion $f \in L^{\psi}\mathfrak{N}$ is the representability of f by a convolution of the form

$$f(x) = (2\pi)^{-m} \int\limits_{Q^m} \varphi(x-t)\, \mathcal{D}_{\psi}(t) dt,$$

where $\varphi \in \mathfrak{N}$ and, almost everywhere, $\varphi(x) = f^{\psi}(x)$. In particular, this means that the classes $L^{\psi}\mathfrak{N}$ cover the classes of functions represented by convolutions with fixed summable kernels (see, e.g., [52], Sec. 1.9).

7.2. On Relationship between the Spaces S^p and L_p. Let $L_p = L_p(R^m)$, $p \in [1, \infty)$, be the space of functions $f \in L$ with finite norm $\|\cdot\|_{L_p}$, where

$$\|f\|_{L_p} = \left(\int\limits_{Q^m} |f(t)|^p dt\right)^{1/p}. \tag{7.3}$$

The known Hausdorff–Young theorem establishes a relationship between the sets L_p and S^p (see, e.g., [53], Sec. XII.2). This theorem states that the following assertions are true:

I. If $f \in L_p$, $p \in (1, 2]$, and $\widehat{f}(k)$ are the Fourier coefficients of the function f defined by the relation

$$\widehat{f}(k) = (2\pi)^{-m/2} \int\limits_{Q^m} f(t)\, e^{-ikt} dt,$$

then

$$\left(\sum_{k \in Z^m} |\widehat{f}(k)|^{p'} \right)^{1/p'} \leq (2\pi)^{m(\frac{1}{2} - \frac{1}{p})} \, \|f\|_{L_p}, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

II. Let $\{c_k\}_{k \in Z^m}$ be a sequence of complex numbers for which

$$\sum_{k \in Z^m} |c_k|^p < \infty, \quad p \in (1, 2].$$

Then there exists a function $f \in L_{p'}$ for which $\widehat{f}(k) = c_k$ and

$$\|f\|_{L_{p'}} \leq (2\pi)^{m(\frac{1}{2} - \frac{1}{p})} \left(\sum_{k \in Z^m} |c_k|^p \right)^{1/p}, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

It follows from this theorem that if $p \in (1, 2\,]$, then

$$L_p \subset S^{p'} \quad \text{and} \quad \|f\|_{S^{p'}} \leq (2\pi)^{m(\frac{1}{2} - \frac{1}{p})} \, \|f\|_{L_p}, \tag{7.4}$$

$$S^p \subset L_{p'} \quad \text{and} \quad \|f\|_{L_{p'}} \leq (2\pi)^{m(\frac{1}{2} - \frac{1}{p})} \, \|f\|_{S^p}. \tag{7.5}$$

In particular, for $p = p' = 2$, the following equalities are true:

$$L_2 = S^2 \quad \text{and} \quad \|\cdot\|_{L_2} = \|\cdot\|_{S^2}. \tag{7.6}$$

By virtue of relations (7.4) and (7.5), the theorems proved for the spaces S^p also contain information for the spaces L_p, which is more complete by virtue of relation (7.6) in the case $p = 2$.

Since this case is very important, we present the corresponding exact statements.

As before, let $\psi = \{\psi_k\}_{k \in Z^m}$ be an arbitrary system of complex numbers and let $L^\psi \mathfrak{N}$ be the set of ψ-integrals of all functions $f \in \mathfrak{N}$, where $\mathfrak{N}$ is a certain subset of $L = L(R^m)$, $m \geq 1$. As $\mathfrak{N}$, we take the unit ball U_{L_2} in the space L_2:

$$U_{L_2} = \{f \colon f \in L_2, \; \|f\|_{L_2} \leq 1\}. \tag{7.7}$$

Here, the norm $\|\cdot\|_{L_2}$ is defined by equality (7.3) for $p = 2$. In this case, we set $L^\psi U_{L_2} = U^\psi_{L_2}$.

Assuming that condition (6.5) is satisfied, we determine the characteristic sequences $\varepsilon(\psi)$, $g(\psi)$, and $\delta(\psi)$ and the polynomials $S_n(f; x)$ according to (6.6). For $f \in U^\psi_{L_2}$, we set

$$\mathcal{E}^\psi_n(f)_{L_2} = \|f(x) - S_{n-1}(f; x)\|_{L_2}, \quad \mathcal{E}_n(U^\psi_{L_2})_{L_2} = \sup_{f \in U^\psi_{L_2}} = \mathcal{E}^\psi_n(f)_{L_2},$$

$$E_n^{\psi}(f)_{L_2} = \inf_{a_k} \left\| f(x) - (2\pi)^{-m/2} \sum_{k \in g_{n-1}^{\psi}} a_k\, e^{ikx} \right\|_{L_2},$$

and

$$E_n(U_{L_2}^{\psi})_{L_2} = \sup_{f \in U_{L_2}^{\psi}} E_n^{\psi}(f)_{L_2}.$$

Also let

$$d_n(U_{L_2}^{\psi})_{L_2} = \inf_{F_n \in G_n} \sup_{f \in L_p^{\psi}} \inf_{u \in F_n} \|f - u\|_{L_2}, \quad n \in N, \qquad d_0(U_{L_2}^{\psi}) = \sup_{f \in U_{L_2}^{\psi}} \|f\|_{L_2},$$

where G_n is the set of all n-dimensional subsets in L_2, and let

$$e_n(U_{L_2}^{\psi})_{L_2} = \sup_{f \in U_{L_2}^{\psi}} \inf_{a_k, \gamma_n} \left\| f(x) - (2\pi)^{-m/2} \sum_{k \in \gamma_n} a_k\, e^{ikx} \right\|_{L_2},$$

where γ_n is an arbitrary collection of n vectors $k \in Z^m$, be the quantity of the best n-term approximation of the class $U_{L_2}^{\psi}$ in the space L_2. The following statement is true:

Theorem 7.1. *Let $\psi = \{\psi_k\}_{k \in Z^m}$ be a system of numbers satisfying conditions (6.3) and (6.9). Then, for any $n \in N$, the following equalities are true:*

$$E_n(U_{L_2}^{\psi})_{L_2} = \mathcal{E}_n(U_{L_2}^{\psi}) = \varepsilon_n, \tag{7.8}$$

$$d_{\delta_{n-1}}(U_{L_2}^{\psi})_{L_2} = d_{\delta_{n-1}+1}(U_{L_2}^{\psi}) = \ldots = d_{\delta_n - 1}(U_{L_2}^{\psi})_{L_2} = E_n(U_{L_2}^{\psi})_{L_2} = \varepsilon_n, \tag{7.9}$$

$$e_n^2(U_{L_2}^{\psi})_{L_2} = \sup_{l > n} (q - n) / \sum_{s=1}^{l} \bar{\psi}_s^{-2} = (l^* - n) / \sum_{s=1}^{l^*} \bar{\psi}_s^{-2}, \tag{7.10}$$

where ε_s and δ_s are elements of the characteristic sequences $\varepsilon(\psi)$ and $\delta(\psi)$, $\delta_0 = 0$, l^* is a certain natural number, and

$$\bar{\psi}_s = \varepsilon_n, \quad \delta_{n-1} < s < \delta_n, \quad n = 1, 2, \ldots .$$

Proof. By virtue of (7.6) and (7.7), we have $U_{L_2} = U^2$ and, hence, $U_{L_2}^{\psi} = L_2^{\psi}$. Therefore,

$$\mathcal{E}_n(U_{L_2}^{\psi})_{L_2} = \mathcal{E}_n(L_2^{\psi})_2, \quad E_n(U_{L_2}^{\psi})_{L_2} = E_n(L_2^{\psi})_2, \quad d_n(U_{L_2}^{\psi})_{L_2} = d_n(L_2^{\psi})_2,$$

and

$$e_n(U_{L_2}^{\psi})_{L_2} = e_n(L_2^{\psi})_2, \quad n = 1, 2, \ldots .$$

This implies that equalities (7.7) – (7.10) follow from relations (6.12), (6.15), and (6.13).

Note that, in the one-dimensional case, i.e., for $m = 1$, equalities (7.8) and (7.9) (in somewhat different terminology) were obtained in 1936 by Kolmogorov in his well-known paper [54], which gave rise to investigations of various functional classes. In the general case, these equalities can also be obtained by analyzing the results and ideas presented by Tikhomirov in his monograph [55] (Sec. 4.4).

Equality (7.10) is, apparently, new even in the one-dimensional case.

It should also be noted that, as follows from equalities (7.8) and (7.9), in the space L_2 the values of the widths of the sets $U^{\psi}_{L_2}$ are realized by approximations by sums (6.6), i.e., by polynomials that are the best in the sense of widths in the spaces S^p for all $p \in [1, \infty)$ for the classes L^{ψ}_p. This allows us to conjecture that sums (6.6) also form the best approximation apparatus (in the sense of Kolmogorov widths) in the spaces L_p for all $p \geq 1$ for the corresponding sets $U^{\psi}_{L_p}$, namely

$$U^{\psi}_{L_p} = L^{\psi} U_{L_p}, \qquad U_{L_p} = \{f\colon f \in L_p,\ \|f\|_{L_p} \leq 1\},$$

which are direct generalizations of the known Sobolev spaces obtained from $U^{\psi}_{L_p}$ by taking $\psi(k)$ in the form (7.1) in the case where

$$\psi_j(k_j) = \begin{cases} 1, & k_j = 0, \\ (ik_j)^{r_j}, & k_j \neq 0, \quad j = \overline{1, m}, \end{cases} \tag{7.11}$$

where r_j are certain real numbers.

Let $m = 2$ and let the sequences $\psi_1(k_1)$ and $\psi_2(k_2)$ be defined by equalities (7.11) under the condition that $r_1 = r_2 = r > 0$.

For the first time, the classes $U^{\psi}_{L_2}$ defined by these sequences were considered from the viewpoint of determination of their widths by Babenko in [1, 2], where, in fact, relation (7.9) was also obtained in the case indicated.

In the case considered, the characteristic sequence $\varepsilon(\psi)$ consists of elements $\varepsilon_n = n^{-r}$, $n \in N$, of the set g^{ψ}_n, i.e., the set of vectors $k = (k_1, k_2) \in Z^2$ that satisfy the condition

$$k_1' k_2' \leq n,$$

where

$$k_j' = \begin{cases} 1, & k_j = 0, \\ |k_j|, & k_j \neq 0, \quad j = 1, 2. \end{cases}$$

These sets appeared for the first time in Babenko's works indicated above and are now customarily called hyperbolic crosses.

All these comments were presented by the author in [15], where one can also find more detailed results for the periodic case, including specific numerical examples.

The main results of this paper were announced by the author in [56].

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