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EXTREMAL PROBLEMS OF APPROXIMATION THEORY IN LINEAR SPACES

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We propose an approach that enables one to pose and completely solve main extremal problems in approximation theory in abstract linear spaces. This approach coincides with the traditional one in the case of approximation of sets of functions defined and square integrable with respect to a given σ-additive measure on manifolds in R^m, $m \geq 1$.

1. Spaces S^p_Φ

In the present work, we develop the approach proposed in [1–7], which enables one to pose classical extremal problems of approximation theory in general linear spaces and find their exact solutions. The spaces S^p_φ considered in [1–7] are constructed as follows:

Let $\mathfrak{X}$ be an arbitrary linear complex space and let $\varphi = \{\varphi_k\}_{k=1}^\infty$ be a fixed countable system in it. Assume that, for any pair $x, y \in \mathfrak{X}$ in which at least one of the vectors belongs to φ, the scalar product (x, y) is defined and satisfies the following conditions:

(i) $(x, y) = (\overline{y, x})$, where $\bar{z}$ is the complex conjugate of z;

(ii) $(\lambda x_1 + \mu x_2, y) = \lambda(x_1, y) + \mu(x_2, y)$, where λ and μ are arbitrary complex numbers;

(iii) $(\varphi_k, \varphi_l) = \begin{cases} 0, & k \neq l, \\ 1, & k = l. \end{cases}$

Every element $f \in \mathfrak{X}$ is associated with a system of numbers $\hat{f}(k)$ by the equalities

$$\hat{f}(k) = \hat{f}_\varphi(k) = (f, \varphi_k), \quad k = 1, 2, \ldots \quad (k \in N), \tag{1}$$

and, for fixed $p \in (0, \infty)$, we set

$$S^p_\varphi = S^p_\varphi(\mathfrak{X}) = \left\{ f \in \mathfrak{X}: \sum_{k=1}^\infty \left| \hat{f}_\varphi(k) \right|^p < \infty \right\}. \tag{2}$$

Elements $x, y \in S^p_\varphi$ are assumed to be identical if $\hat{x}_\varphi(k) = \hat{y}_\varphi(k)$ for all $k \in N$.

For vectors $x, y \in \mathfrak{X}$, we define the φ-distance between them by the equality

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$$\rho_\varphi(x, y)_p = \left(\sum_{k=1}^{\infty} \left| \hat{x}_\varphi(k) - \hat{y}_\varphi(k) \right|^p \right)^{1/p}.$$

The vector θ such that $\hat{\theta}_\varphi(k) = 0$ for all $k \in N$ is called the zero element of the space S_φ^p. The distance $\rho_\varphi(\theta, x)$, $x \in S_\varphi^p$, is called the φ-norm of an element x and is denoted by $\|x\|_{\varphi,p}$. Thus,

$$\|x\|_{\varphi,p} = \rho_\varphi(\theta, x) = \left(\sum_{k=1}^{\infty} \left| \hat{x}_\varphi(k) \right|^p \right)^{1/p}. \tag{3}$$

The set S_φ^p is a linear space. For $p \geq 1$, the φ-norm satisfies all necessary axioms of a norm. In this case, S_φ^p is a linear normed space containing the orthonormal system φ. For $p = 2$, the space S_φ^2 is a Hilbert space if it is complete. For other $p \in (0, \infty)$, the spaces S_φ^p inherit the most important properties of Hilbert spaces such as the Parseval equality in the form (3), the minimum property of partial Fourier sums, etc.

Let $\psi = \{\psi_k\}_{k=1}^{\infty}$ be a given system of complex numbers. If, for a given element $f \in \mathfrak{X}$, there exists an element $F \in \mathfrak{X}$ for which

$$\hat{F}_\varphi(k) = \psi_k \hat{f}(k), \qquad k \in N,$$

then the vector F is called the ψ-integral of f and is denoted by $F = \mathcal{J}^\psi f$. Let U_φ^p be a unit ball in the space S_φ^p, namely,

$$U_\varphi^p = \{ f \in S_\varphi^p, \ \|f\|_{\varphi,p} \leq 1 \}.$$

By ψU_φ^p, we denote the set of all ψ-integrals of all elements of U_φ^p, i.e.,

$$\psi U_\varphi^p = \{ F \in \mathfrak{X} : \ F = \mathcal{J}^\psi f, \ f \in U_\varphi^p \}.$$

Note that if the space $\mathfrak{X}$ is complete and, furthermore,

$$\psi_k \neq 0, \qquad k \in N,$$

then

$$\psi U_\varphi^p = \left\{ f \in \mathfrak{X} : \sum_{k=1}^{\infty} \left| \frac{\hat{f}_\varphi(k)}{\psi_k} \right|^p \leq 1 \right\},$$

i.e., the set ψU_φ^p is a p-ellipsoid with semiaxes equal to $|\psi_k|$.

In [1–7], the least upper bounds were determined for the best approximations of elements $f \in \psi U_\varphi^p$ by polynomials of the form

$$P_{\gamma_n} = \sum_{k\in\gamma_n} \alpha_k \varphi_k,$$

where γ_n are fixed collections of n natural numbers and α_k are certain coefficients, and the exact values were found for the Kolmogorov widths $d_n(\psi U_\varphi^p; S_\varphi^p)$ and the quantities

$$e_n(\psi U_\varphi^q)_p = \sup_{f\in\psi U_\varphi^n} \inf_{\alpha_k,\gamma_n} \left\| f - P_{\gamma_n} \right\|_{\varphi,p}, \qquad 0 < p, q < \infty,$$

which are called the best n-term approximations of the q-ellipsoids ψU_φ^q in the space S_φ^p.

It should be noted that, eventually, all these problems are reduced to the corresponding extremal problems for numerical series with positive terms whose solutions can be obtained in explicit form.

The system of numbers (1) can be regarded as the set of values of a certain function $\hat{f}(t)$ defined on the integer-valued lattice Z^m in the Euclidean space R^m of dimension m, $m \geq 1$, with properly enumerated points $t \in Z^m$. Moreover, the operator of scalar multiplication can be interpreted as an operator acting from $\mathfrak{X}$ onto the corresponding set of functions. Furthermore, the functional defined by the series in (2) can be regarded as an integral constructed with respect to the measure $d\mu$ whose support is the set Z^m.

This approach enables us to propose the following construction:

Let $(R^m, d\mu)$, $m \geq 1$, be the m-dimensional Euclidean space of points $t = (t_1, \ldots, t_m)$ equipped with a σ-additive measure $d\mu$, let A be a μ-measurable subset of $(R^m, d\mu)$ whose μ-measure is equal to a, where either a is finite or $a = \infty$, i.e.,

$$\text{mes}_\mu A = |A|_\mu = a, \qquad a \in (0, \infty],$$

and let $Y = Y(A, d\mu)$ be the set of functions $y = y(t)$ defined on A and measurable with respect to the measure $d\mu$.

Further, let $\mathfrak{X}$ be an arbitrary linear space of vectors x and let Φ be a linear operator acting from $\mathfrak{X}$ into Y, namely,

$$\Phi : \mathfrak{X} \to Y(A, d\mu), \qquad \Phi(x) \overset{\text{df}}{=} \hat{x}, \qquad x \in \mathfrak{X}, \qquad \hat{x} \in Y(A, d\mu).$$

For a given $p \in (0, \infty]$, we denote by $L_p(A, d\mu)$ a subset of functions from $Y(A, d\mu)$ that have the finite norm

$$\| y \|_{L_p(A,d\mu)} = \begin{cases} \left(\int_A |y(t)|^p \, dt \right)^{1/p}, & p \in (0, \infty), \\ \operatorname*{ess\,sup}_{t\in A} |y(t)|, & p = \infty, \end{cases} \tag{4}$$

Let $S_\Phi^p = S_\Phi^p(\mathfrak{X}; Y)$ denote the preimage of the set $L_p(A, d\mu)$ in $\mathfrak{X}$ under the mapping Φ. Thus,

$$S_\Phi^p = S_\Phi^p(\mathfrak{X}; Y) = \{ x \in \mathfrak{X}, \ \| \hat{x} \|_{L_p(A,d\mu)} < \infty \}. \tag{5}$$

Elements $x_1, x_2 \in S_\Phi^p$ are assumed to be identical if $\hat{x}_1(t) = \hat{x}_2(t)$ almost everywhere with respect to the measure μ.

For elements $x_1, x_2 \in S_\Phi^p$, $p \in (0, \infty)$, we define the Φ-distance between them by the equality

$$\rho_\Phi(x_1, x_2) = \rho(x_1, x_2)_{S_\Phi^p} = \| \Phi(x_1 - x_2) \|_{L_p(A, d\mu)} = \left(\int_A |\hat{x}_1(t) - \hat{x}_2(t)|^p \, d\mu \right)^{1/p}.$$

An element θ for which $\hat{\theta}(t) = 0$ almost everywhere on A is called the zero element of the set S_Φ^p.

The distance $\rho_\Phi(\theta; x)$, $x \in S_\Phi^p$, is called the Φ-norm of an element x and is denoted by $\|x\|_p = \|x\|_{p,\Phi}$. Thus, by definition,

$$\|x\|_p = \|x\|_{p,\Phi} = \rho(\theta; x)_{S_\Phi^p} = \|\hat{x}\|_{L_p(A, d\mu)}. \tag{6}$$

In this case, S_Φ^p is a linear metric space; indeed, the operations of addition of elements and their multiplication by numbers defined in $\mathfrak{X}$ remain applicable for any pair $x_1, x_2 \in S_\Phi^p$. Furthermore, for any numbers λ_1 and λ_2, the element $x_3 = \lambda_1 x_1 + \lambda_2 x_2$ belongs to S_Φ^p. Indeed, since $x_3 \in \mathfrak{X}$, we have $\hat{x}_3(t) = \lambda_1 \hat{x}_1(t) + \lambda_2 \hat{x}_2(t)$. If $p \geq 1$, then

$$\|x_3\|_p = \|\hat{x}_3\|_{L_p(A, d\mu)} \leq |\lambda_1| \|\hat{x}_1\|_{L_p(A, d\mu)} + |\lambda_2| \|\hat{x}_2\|_{L_p(A, d\mu)} = |\lambda_1| \|x_1\|_p + |\lambda_2| \|x_2\|_p$$

by virtue of the Minkowski inequality. If $p \in (0, 1)$, then, using the inequality

$$|a + b|^p \leq |a|^p + |b|^p, \quad 0 \leq p < 1,$$

we get

$$\|x_3\|_p = \left(\int_A |\lambda_1 \hat{x}_1(t) + \lambda_2 \hat{x}_2(t)|^p \, d\mu \right)^{1/p}$$

$$\leq \left(|\lambda_1|^p \int_A |\hat{x}_1(t)|^p \, d\mu + |\lambda_2|^p \int_A |\hat{x}_2(t)|^p \, d\mu \right)^{1/p} \leq 2^{1/p} (|\lambda_1| \|x_1\|_p + |\lambda_2| \|x_2\|_p),$$

i.e., $x_3 \in S_\varphi^p$ in all cases.

It is clear that, for $p \geq 1$, the functional $\|\cdot\|_p$ satisfies all necessary axioms of a norm. Therefore, S_Φ^p is a linear normed space for $p \geq 1$.

In the terminology accepted in the theory of integral transformations, the element $\hat{x} = \Phi(x)$ is the image (Φ-image) of an element x, and the set $E(\Phi)$ of values of the operator Φ is the set of images. Thus, the Φ-distance and Φ-norm are the distance and norm in the space of images.

2. Multipliers. Approximating Aggregates and Objects of Approximation

As approximating aggregates for elements $x \in S^p_\Phi$, we use elements from S^p_Φ whose images have supports γ_σ of given measure σ. It is clear that exactly this principle is used in the classical case in the construction of, e.g., trigonometric polynomials for the approximation of a given periodic function if the operator Φ is understood as the mapping of functions into the set of their Fourier coefficients. In the general case, there arise certain problems related to the fact that the spaces S^p_Φ can be incomplete. In this connection, we introduce the following definitions:

Let $\omega = \omega(t)$ be a certain function from $Y(A, d\mu)$. Then we denote by M^ω_Φ the operator acting from $\mathfrak{X}$ into $\mathfrak{X}$ that associates $x \in \mathfrak{X}$ with an element $x_\omega \in \mathfrak{X}$ such that if $\Phi(x) = \hat{x}(t)$, then $\hat{x}_\omega(t) = \Phi(x_\omega) = \omega(t)\hat{x}(t)$ almost everywhere. The operator M^ω_Φ is called the multiplier of the operator Φ generated by the function ω. Let $\Omega_\Phi(\mathfrak{X}) = \Omega_\Phi(\mathfrak{X}, \mathfrak{X})$ denote the subset of functions ω from $Y(A, d\mu)$ for which the multipliers M^ω_Φ exist.

If $\mathfrak{N}$ and $\mathfrak{N}'$ are some subsets of $\mathfrak{X}$, $\omega \in \Omega_\Phi(\mathfrak{X})$, and the operator M^ω_Φ maps $\mathfrak{N}$ into $\mathfrak{N}'$, then we say that M^ω_Φ has the type $(\mathfrak{N}, \mathfrak{N}')$. In particular, if M^ω_Φ maps S^p_Φ into S^p_Φ, then the operator M^ω_Φ has the type (S^p_Φ, S^p_Φ) or, briefly, the type (p, p). The set of functions ω generating operators of the type (p, p) is denoted by Ω^p_Φ.

Thus, if $\omega \in \Omega^p_\Phi$ and the operator M^ω_Φ acts from S^p_Φ, then it acts into S^p_Φ. In this case, every $x \in S^p_\Phi$ is associated with an element $x_\omega = M^\omega_\Phi(x)$ for which the following equality holds almost everywhere on A:

$$\hat{x}_\omega(t) = \Phi(x_\omega) = \omega(t)\hat{x}(t), \qquad \hat{x}_\omega \in L_p(A, d\mu). \tag{7}$$

Given $\sigma > 0$, assume that γ_σ is a μ-measurable set in A,

$$\operatorname{mes}_\mu \gamma_\sigma \overset{\text{df}}{=} |\gamma_\sigma| = \sigma, \qquad \sigma \le a, \tag{8}$$

and $\lambda = \lambda(t)$ is a measurable function with support γ_σ. Also assume that, for a given $p \in (0, \infty)$, we have $\lambda \in \Omega^p_\Phi$ and $U_{\gamma_\sigma}(x; \lambda) \overset{\text{df}}{=} x_\lambda = M^\lambda_\Phi(x)$, and, therefore, according to (7), we get

$$\hat{U}_{\gamma_\sigma}(x; \lambda) = \Phi(U_{\gamma_\sigma}(x; \lambda)) = \begin{cases} \lambda(t)\hat{x}(t), & t \in \gamma_\sigma, \\ 0, & t \,\overline{\in}\, \gamma_\sigma, \quad x \in S^p_\Phi. \end{cases} \tag{9}$$

The elements $U_{\gamma_\sigma}(x; \lambda)$ are considered as approximating aggregates for $x \in S^p_\Phi$. In this case, if $\lambda(t) \equiv 1$ on γ_σ, i.e., if $\lambda(t)$ coincides with the characteristic function $\chi_{\gamma_\sigma}(t)$ of the set γ_σ, then we set $U_{\gamma_\sigma}(x; \chi_{\gamma_\sigma}) = U_{\gamma_\sigma}(x)$.

Let $\Gamma_\sigma = \Gamma_\sigma(A)$ be the set of all measurable subsets of A whose measures are equal to σ. We say that, for a given $p > 0$, an operator Φ satisfies condition (A_p) if the functions $\chi_{\gamma_\sigma}(t)$ of all sets $\gamma_\sigma \in \Gamma_\sigma$ belong to Ω^p_Φ for all $\sigma \in [0, a)$. Thus, if Φ satisfies condition (A_p), then all elements $U_{\gamma_\sigma}(x)$ are defined for any $x \in S^p_\Phi$ and are contained in S^p_Φ. The element $U_{\gamma_\sigma}(x)$ is called the restriction of an element x of rank σ, and the element $U_{\gamma_\sigma}(x; \lambda)$ is called the λ-restriction of x of rank σ.

Let p be an arbitrary positive number and let $x \in S^p_\Phi$. Then, by virtue of (6) and (9), we get

$$\left\| x - U_{\gamma_\sigma}(x; \lambda) \right\|_p^p = \left\| \hat{x}(t) - \hat{U}_{\gamma_\sigma}(x; \lambda; t) \right\|^p_{L_p(A, d\mu)} = \int\limits_{\gamma_\sigma} |1 - \lambda(t)|^p |\hat{x}(t)|^p \, d\mu + \int\limits_{A \setminus \gamma_\sigma} |\hat{x}(t)|^p \, d\mu.$$

Hence, we arrive at the following statement:

Assertion 1. *Suppose that* $p \in (0, \infty)$, $x \in S^p_\Phi = S^p_\Phi(\mathfrak{X}; Y)$, $\gamma_\sigma \in \Gamma_\sigma$, *and the operator* Φ *satisfies condition* (A_p). *Then*

$$\mathscr{E}_{\gamma_\sigma}(x)_p \overset{\text{df}}{=} \inf_{\lambda \in \Omega^p_\Phi} \| x - U_{\gamma_\sigma}(x; \lambda) \|_p = \| x - U_{\gamma_\sigma}(x) \|_p.$$

Furthermore, the following equality is true:

$$\mathscr{E}_{\gamma_\sigma}(x)_p = \| x \|_p^p - \int\limits_{\gamma_\sigma} |\hat{x}(t)|^p \, d\mu. \tag{10}$$

Thus, if $\chi_{\gamma_\sigma} \in \Omega^p_\Phi$, then, among all elements $U_{\gamma_\sigma}(x; \lambda)$ generated by the multipliers M^λ_Φ and satisfying condition (9), the element $U_{\gamma_\sigma}(x)$ has the least deviation from an element x in Φ-norm in the space S^p_Φ, i.e., among all λ-restrictions of x of given rank σ, its restriction for $\lambda(t) \equiv 1$ is the closest one to x. It is clear that this property is an analog of the minimum property of Fourier sums in the Hilbert spaces L_2.

Let $\Gamma = \{\gamma_\sigma\}_{\sigma > 0}$, $|\gamma_\sigma| = \sigma$, be a family of measurable subsets of A that exhausts the entire set A for $\sigma \to \infty$, i.e., it possesses the property that any point $t \in A$ is contained in all sets γ_σ for all sufficiently large values of σ and, therefore,

$$\lim_{\sigma \to \infty} \int\limits_{\gamma_\sigma \in \Gamma} |\hat{x}(t)|^p \, d\mu = \int\limits_A |\hat{x}(t)|^p \, d\mu \qquad \forall x \in S^p_\Phi. \tag{11}$$

Combining relations (10) and (11), we get

$$\lim_{\substack{\sigma \to \infty \\ \gamma_\sigma \in \Gamma}} \mathscr{E}_{\gamma_\sigma}(x)_p = 0 \qquad \forall x \in S^p_\Phi.$$

We now define the objects of approximation, namely the unions of elements $x \in \mathfrak{X}$ corresponding to the notion of a class of functions in approximation theory. Such objects, along with approximating aggregates, are introduced with the use of multipliers. However, in this case, it is more convenient to use a somewhat different terminology closer to the traditional one. Let $\psi = \psi(t)$ be an arbitrary function from $\Omega_\Phi(\mathfrak{X})$ and let M^ω_Φ be the multiplier of an operator Φ generated by this function. In this case, the image x_ψ of an element x under the mapping M^ψ_Φ is called the ψ-integral of an element x and is denoted by $M^\psi_\Phi(x) = x_\psi = \mathcal{J}^\psi x$. In certain cases, it is convenient to call x the ψ-derivative of x_ψ and write $x = D^\psi x_\psi$.

Thus, if x_ψ is the ψ-integral of x, then

$$\hat{x}_\psi = \Phi(\mathcal{J}^\Psi x) = \psi(t)\hat{x}(t) \tag{12}$$

almost everywhere.

Let $\mathfrak{N}$ be a certain subset of $\mathfrak{X}$. By $\psi\mathfrak{N}$, we denote the set of ψ-integrals of all $x \in \mathfrak{N}$ for which they exist. In particular, if U_Φ^p is a unit ball in a certain space S_Φ^p, namely,

$$U_\Phi^p = \{x : x \in S_\Phi^p, \ \|x\|_{p,\Phi} \le 1\},$$

then ψU_Φ^p is the set of ψ-integrals of all $x \in U_\Phi^p$ for which these integrals exist.

Comparing relations (12) and (7), we conclude that, as functions ψ for which the definition of ψ-integral is correct, one can choose any function from $\Omega_\Phi(S_\Phi^p)$. In this case, the inclusion $\psi S_\Phi^p \subset S_\Phi^p$ is valid.

3. Approximation Characteristics

In the present work, we consider the following approximation characteristics of the sets ψU_Φ^p. For any $\gamma_\sigma \in \Gamma_\sigma$, we set

$$\mathscr{E}_{\gamma_\sigma}(x)_q = \inf_{\lambda \in \Omega_\Phi^p} \left\| x - U_{\gamma_\sigma}(x; \lambda) \right\|_{q,\Phi}, \qquad x \in S_\Phi^p, \tag{13}$$

$$\mathscr{E}_{\gamma_\sigma}(\psi U_\Phi^p)_q = \sup_{x \in \psi U_\Phi^p} \mathscr{E}_{\gamma_\sigma}(x)_q, \tag{14}$$

and

$$\mathscr{D}_\sigma(\psi U_\Phi^p)_q = \inf_{\gamma_\sigma \in \Gamma_\sigma} \mathscr{E}_{\gamma_\sigma}(\psi U_\Phi^p)_q, \tag{15}$$

In the case of the approximation of periodic functions by trigonometric polynomials, the quantity $\mathscr{E}_{\gamma_\sigma}(x)_q$ corresponds to the best approximation of a function x by polynomials of degree σ, the quantity $\mathscr{E}_{\gamma_\sigma}(\psi U_\Phi^p)_q$ corresponds to the upper bound of these best approximations on a given set of functions, and the quantity $\mathscr{D}_\sigma(\psi U_\Phi^p)_q$ resembles the trigonometric width of order σ of the set ψU_Φ^p.

We also consider the following characteristics, which, in the periodic case, correspond to quantities related to the best σ-term approximation:

$$e_\sigma(x)_q = \inf_{\gamma_\sigma \in \Gamma_\sigma} \mathscr{E}_{\gamma_\sigma}(x)_q = \inf_{\gamma_\sigma \in \Gamma_\sigma} \inf_{\lambda \in \Omega_\Phi^p} \left\| x - U_{\gamma_\sigma}(x; \lambda) \right\|_{q,\Phi} \tag{16}$$

and

$$e_\sigma(\psi U_\Phi^p)_q = \sup_{x \in \psi U_\Phi^p} e_\sigma(x)_q. \tag{17}$$

In what follows, we restrict ourselves to the case $p = q$. Moreover, we assume that the corresponding characteristic functions $\chi_{\gamma_\sigma}(\cdot)$ belong to Ω_Φ^p, i.e., the operator Φ satisfies condition (A_p). In this case, according to Assertion 1, of major interest are quantities (13)–(17), where $\lambda(t) = \chi_{\gamma_\sigma}(t)$. In this connection, we set

$$\mathscr{E}_{\gamma_\sigma}(x)_p = \left\| x - U_{\gamma_\sigma}(x) \right\|_{p,\Phi}, \qquad x \in S_\Phi^p, \tag{18}$$

$$\mathscr{E}_{\gamma_\sigma}(\psi U_\Phi^p)_p = \sup_{x \in \psi U_\Phi^p} \mathscr{E}_{\gamma_\sigma}(x)_p, \tag{19}$$

and

$$\mathscr{D}_\sigma(\psi U_\Phi^p)_p = \inf_{\gamma_\sigma \in \Gamma_\sigma} \mathscr{E}_{\gamma_\sigma}(\psi U^p)_{p,\Phi}. \tag{20}$$

Similarly,

$$e_\sigma(x)_p = \inf_{\gamma_\sigma \in \Gamma_\sigma} \left\| x - U_{\gamma_\sigma}(x) \right\|_{p,\Phi} \tag{21}$$

and

$$e_\sigma(\psi U_\Phi^p)_p = \sup_{x \in \psi U_\Phi^p} e_\sigma(x)_p.$$

4. Quantities $\mathscr{E}_{\gamma_\sigma}(U_\Phi^p)_p$ and $\mathscr{D}_\sigma(\psi U_\Phi^p)_p$

Below, we use the notion of the rearrangement of a function in decreasing order. Apparently, this notion first appeared in works of Hardy and Littlewood (see [8], Chap. X), and then it was successfully used by many authors. We present necessary definitions following ([9], Chap. 6). Note that, in [9], rearrangements of functions of one variable are considered, but the main definitions can also be used in the general case.

Assume that, on a μ-measurable set $A \subset R^m$, $m \geq 1$, $\text{mes}_\mu A = a$, where a is either finite or infinite, a nonnegative μ-measurable function $f(x)$ is defined for which the distribution function

$$m_f(y) = \text{mes}_\mu E_y, \qquad E_y = \{x : x \in A, f(x) \geq y\}, \qquad y \geq 0,$$

takes only finite values for $y > 0$.

The function $t = m_f(y)$ does not increase for all $y \geq 0$ and, moreover, $m_f(0) = a$. If $m_f(y)$ is continuous and strictly decreasing, then, on the interval $t \in (0, a)$, there exists its strictly decreasing inverse $y = \overline{\varphi}(t)$, which is called the rearrangement of the function $f(x)$ in decreasing order. In the general case, depending on $f(\cdot)$, the function $m_f(y)$ can possess intervals of constancy and discontinuities of the first kind on a finite or countable set of points. In order to uniquely determine the inverse function, we improve the graph of the function $m_f(y)$ as follows: At every discontinuity point y_j of the function $m_f(y)$, we supplement its graph with the segment $y = y_j$, $m_f(y_j + 0) \leq t \leq m(y_j + 0)$, and, on every interval $[\alpha, \beta]$ where $m_f(y)$ is constant, we leave only one point in its graph with the coordinates, say, $y = (\alpha + \beta)/2$ and $t = m_f((\alpha + \beta)/2)$. In this case, every $t \in (0, a)$ corresponds to a single point with coordinates $(t, m_f^{-1}(t))$. This mapping defines the function $y = \overline{\varphi}(t)$, which is the rearrangement of the function $\varphi(x)$ in the case under consideration.

For any $y \geq 0$, the Lebesgue measure of the set of points $t \in (0, a)$ on which $\overline{\varphi}(t) \geq y$ is $m_f(y)$. Thus,

$$\text{mes}\{t:\ t \in (0, a),\ \overline{\varphi}(t) \geq y\} = \text{mes}_\mu\{x:\ x \in A,\ f(x) \geq y\} = m_f(y). \tag{22}$$

In particular, this implies that

$$\int_0^a F(\overline{\varphi}(t))\,dt = \int_A F(f(x))\,d\mu \tag{23}$$

for any function F for which these integrals exist (see [8], Chap. X).

In the notation accepted, the following statement is true:

Theorem 1. *Let* $\psi = \psi(t)$ *be an arbitrary function from* $Y(A, d\mu)$ *essentially bounded on* A, *i.e.,*

$$\operatorname*{ess\,sup}_{t \in A} |\psi(t)| = \|\psi\|_M < \infty, \tag{24}$$

and, in the case where the set A *is unbounded, let*

$$\lim_{|t| \to \infty} \psi(t) = 0. \tag{25}$$

Then, for any $\mathfrak{X}$, $A \subset R^m$, $m \geq 1$, $\gamma_\sigma \in \Gamma_\sigma$, $\sigma < a$, *and* $p \in (0, \infty)$ *and any operator* Φ *satisfying condition* (A_p), *the following estimates are true:*

$$\mathcal{E}^p_{\gamma_\sigma}(\psi U^p_\Phi)_p \leq \overline{\varphi}_{\gamma_\sigma}(0+0), \tag{26}$$

where $\overline{\varphi}_{\gamma_\sigma}(v)$ *is the rearrangement of the function*

$$\varphi_\sigma(t) = \varphi_{\gamma_\sigma}(t) = \begin{cases} |\psi(t)|^p, & t \in A \setminus \gamma_\sigma, \\ 0, & t \in \gamma_\sigma, \end{cases} \tag{27}$$

in decreasing order,

$$\mathcal{D}_\sigma(\psi U^p_\Phi)_p \leq \overline{\psi}(\sigma + 0), \tag{28}$$

and $\overline{\psi}(v)$ *is the rearrangement of the function* $|\psi(t)|$ *in decreasing order.*

If, in addition, the functions $\chi_{\gamma_\sigma}(t)$ *belong to the set* $E(\Phi)$ *for any* $\gamma_\sigma \in \Gamma_\sigma$ *and* $\sigma \in (0, a)$, *and their preimages* U_{γ_σ} *have* ψ*-integrals, then relations (26) and (28) are the equalities. In this case,* Γ_σ *contains the set* γ^*_σ *for which the following equalities are true:*

$$\mathscr{E}_{\gamma_\sigma^*}(\psi U_\Phi^p)_p = \mathscr{D}_\sigma(\psi U_\Phi^p)_p = \overline{\psi}(\sigma + 0). \tag{29}$$

This set is defined by the relation

$$\gamma_\sigma^* = \{ t \in A : |\psi(t)| \geq \overline{\psi}(\sigma + 0) \}, \qquad \text{mes}\, \gamma_\sigma^* = \sigma.$$

Proof. Conditions (24) and (25) guarantee that, for the function $|\psi(t)|$, its distribution function

$$m_{|\psi|}(y) = \text{mes}_\mu E_y, \qquad E_y = \{ t \in A : |\psi(t)| \geq y \}, \qquad y \geq 0, \tag{30}$$

takes only finite values from the interval $[0, a]$ for any $y > 0$. Therefore, the quantities $\overline{\varphi}_\sigma(0 + 0)$ and $\overline{\psi}(\sigma + 0)$ are always defined.

We also note that, in the case where $E(\Phi) = L_p(A, d\mu)$, the operator Φ satisfies condition (A_p). Moreover, by virtue of conditions (24) and (25), the requirements that guarantee the equality in relations (26) and (28) are also satisfied.

First, we prove relation (26). If $x \in \psi U_\Phi^p$, then, according to (18), (6), and (12), we get

$$\mathscr{E}_{\gamma_\sigma}^p(x)_p = \left\| \Phi((x) - U_{\gamma_\sigma}(x)) \right\|_{L_p}^p = \left\| \hat{x}(t) - \chi_{\gamma_\sigma}(t)\hat{x}(t) \right\|_{L_p}^p$$

$$= \left\| \psi(t)y(t) - \chi_{\gamma_\sigma}(t)\psi(t)y(t) \right\|_{L_p}^p = \int_{A \setminus \gamma_\sigma} |\psi(t)y(t)|^p \, d\mu, \tag{31}$$

where $L_p = L_p(A, d\mu)$ and y is a certain function from a unit ball U_p in the space $L_p(A, d\mu)$, namely,

$$U_p = \{ y \in L_p(A, d\mu), \ \|y\|_{L_p(A, d\mu)} \leq 1 \}.$$

Therefore, according to (19),

$$\mathscr{E}_{\gamma_\sigma}^p(\psi U_\Phi^p)_p \leq \sup_{y \in U_p} \int_{A_\sigma} |\psi(t)|^p |y(t)|^p \, d\mu, \qquad A_\sigma = A \setminus \gamma_\sigma.$$

If $y \in U_p$, then the function $h = |y(t)|^p$ belongs to the subset U_1^+ of nonnegative functions from U_1. Hence,

$$\mathscr{E}_{\gamma_\sigma}^p(\psi U_\Phi^p)_p \leq \sup_{h \in U_1^+} \int_{A_\sigma} |\psi(t)|^p h(t) \, d\mu = \sup_{h \in U_1^+} \int_{A_\sigma} \varphi_\sigma(t) h(t) \, d\mu.$$

In order to prove (26), it remains to show that

$$\sup_{h \in U_1^+} \int_{A_\sigma} \varphi_\sigma(t) h(t) \, d\mu = \overline{\varphi}_\sigma(0 + 0). \tag{32}$$

The function $\varphi_\sigma(t)$ is essentially bounded on A_σ by virtue of (24). Therefore, its rearrangement in decreasing order is bounded. Hence, the limit

$$\bar{\varphi}_\sigma(0+0) = \lim_{v \to 0+0} \bar{\varphi}_\sigma \overset{\text{df}}{=} y_\sigma \tag{33}$$

exists. Let $e_y = E(t:\ \varphi_\sigma(t) \geq y)$. It is clear that the point y_σ is such that $\text{mes}_\mu\, e_y > 0$ for $0 < y < y_\sigma$ and $\text{mes}_\mu\, e_y = 0$ for $y > y_\sigma$. In particular, this yields

$$y_\sigma = \operatorname*{ess\,sup}_{t \in A_\sigma} \varphi_\sigma(t). \tag{34}$$

If $h \in U_1^+$ and $y \in (0, y_\sigma)$, then

$$\int_{A_\sigma} \varphi_\sigma(t) h(t)\, d\mu \leq \int_{e_y} \varphi_\sigma(t) h(t)\, d\mu + y \int_{A_\sigma \setminus e_y} h(t)\, d\mu \leq \int_{e_y} \varphi_\sigma(t) f(t)\, d\mu, \tag{35}$$

where

$$f(t) = h(t) + (\text{mes}_\mu\, e_y)^{-1} \int_{A_\sigma \setminus e_y} h(t)\, d\mu.$$

Since

$$\int_{e_y} f(t)\, d\mu \leq 1,$$

relations (34) and (35) imply that, for any $h \in U_1^+$,

$$\int_{A_\sigma} \varphi_\sigma(t) h(t)\, d\mu \leq y_\sigma.$$

Hence,

$$\sup_{n \in U_1^+} \int_{A_\sigma} \varphi_\sigma(t) h(t)\, d\mu \leq y_\sigma. \tag{36}$$

By virtue of (33), to prove equality (32) we must show that relation (36) cannot be the strict inequality.

Let y be an arbitrary number from the interval $(0, y_\sigma)$ and let e_y be the set corresponding to it. We put

$$h_y(t) = \begin{cases} (\text{mes}_\mu\, e_y)^{-1}, & t \in e_y, \\ 0, & t \in A_\sigma \setminus e_y, \end{cases}$$

and, hence, we always have $h_y \in U_1^+$. At the same time,

$$y \le \int_{A_\sigma} \varphi_\sigma(t) h_y(t)\, d\mu \le y_\sigma.$$

Passing to the limit as y tends to y_σ, we conclude that, indeed, the strict equality in (36) cannot be realized, which completes the proof of equality (32) and estimate (26).

Considering the lower bounds of both parts of (26) over the set Γ_σ, we get

$$\mathcal{D}_\sigma^p(\psi U_\Phi^p)_p \le \inf_{\gamma_\sigma \in \Gamma_\sigma} \overline{\varphi}_{\gamma_\sigma}(0+0). \tag{37}$$

In view of relation (27), we can conclude that the least value of the quantity $\overline{\varphi}_{\gamma_\sigma}(0+0)$ is realized in the case where $\gamma_\sigma = \gamma_\sigma^*$, and this value is equal to $\overline{\psi}^p(\sigma+0)$:

$$\inf_{\gamma_\sigma \in \Gamma_\sigma} \overline{\varphi}_{\gamma_\sigma}(0+0) = \overline{\varphi}_{\gamma_\sigma^*}(0+0) = \overline{\psi}^p(\sigma+0). \tag{38}$$

This proves relation (28).

Now assume that, for any $\gamma_\sigma \in \Gamma_\sigma$ and $\sigma \in (0, a)$, the function $\chi_{\gamma_\sigma}(t)$ belongs to the set $E(\Phi)$, and its preimage U_{γ_σ} has the ψ-integral, which belongs to S_Φ^p by virtue of (23). For a given $\gamma_\sigma \in \Gamma$, we also assume that $y \in (0, y_\sigma)$, $e_y = E(\varphi_\sigma(t) \ge y)$, $\chi_{e_y}(t)$ is the characteristic function of the set e_y,

$$h_y^*(t) = (\text{mes}_\mu e_y)^{-1/p} \chi_{e_y}(t),$$

U_y is the preimage of the function $h_y^*(t)$ under the mapping of Φ, $\Phi(U_y) = h_y^*(t)$, and $x_\psi = \mathcal{J}^\psi U_y$ is the ψ-integral of the element U_y. By virtue of the above assumptions, all elements constructed exist, and, since

$$\int_A \left| h_y^*(t) \right|^p d\mu = \int_{e_y} \left| h_y^*(t) \right|^p d\mu = 1,$$

we get $x_\psi \in \psi U_\Psi^p$.

For the element x_ψ, relation (31) yields

$$\mathcal{E}_{\gamma_\sigma}^p(x_\psi)_p = \left\| \Phi(x_\psi) - U_{\gamma_\sigma}(x_\psi) \right\|_{L_p}^p = \left\| \psi(t) h_y^*(t) \right\|_{L_p}^p = \text{mes}_\mu e_y \int_{e_y} |\psi(t)|^p d\mu \ge y.$$

Taking into account the arbitrariness of the choice of y from the interval $(0, y_\sigma)$, we can conclude that the set ψU_Φ^p contains elements x for which the values of $\mathcal{E}_{\gamma_\sigma}^p(x)$ are arbitrarily close to the value of y_σ. With regard for relation (33), this means that, in the case considered, relation (26) is, in fact, the equality. Then, according to (37) and (38), relation (33) is also the equality. If, in this case, the set γ_σ^* is chosen from condition (30), then

$$\overline{\varphi}_{\gamma_\sigma^*}(0+0) = \overline{\psi}^p(\sigma+0).$$

Then, according to relation (26) (which is now the equality), we get

$$\mathcal{E}_{\gamma_\sigma^*}(\psi U_\Phi^p)_p \;=\; \bar{\psi}(\sigma+0).$$

This yields (29). Theorem 1 is proved.

5. Quantities $e_\sigma(\psi U_\Phi^p)_p$

In the notation accepted, the following theorem is true:

Theorem 2. *Let* $\psi = \psi(t)$ *be an arbitrary function from* $Y(A, d\mu)$ *essentially bounded on* A *and let this function satisfy condition (25) if the set* A *is unbounded.*

Then, for any $\mathfrak{X}$, $A \subset R^m$, $m \geq 1$, $\sigma < a$, *and* $p \in (0, \infty)$, *and any operator* Φ *satisfying condition* (A_p), *the following relation is true:*

$$e_\sigma^p(\psi U_\Phi^p) \;\leq\; \sup_{\sigma < q \leq a} \frac{q-\sigma}{\displaystyle\int_0^q \frac{dt}{\bar{\psi}^p(t)}}, \tag{39}$$

where $\bar{\psi}(v)$ *is the rearrangement of the function* $|\psi(t)|$ *in decreasing order. The value of the least upper bound in (39) is realized for a certain finite value* $q = q^*$.

If, in addition, the set $E(\Phi)$ *of values of the operator* Φ *coincides with the entire space* $L_p(A, d\mu)$, *then relation (39) is, in fact, the equality.*

Proof. For any $x \in S_\Phi^p$, relations (6) and (21) yield

$$e_\sigma^p(x)_p \;=\; \inf_{\gamma_\sigma \in \Gamma_\sigma} \left\| \Phi(x - U_{\gamma_\sigma}(x)) \right\|_{L_p}^p \;=\; \inf_{\gamma_\sigma \in \Gamma_\sigma} \left\| \hat{x}(t)(1 - \chi_{\gamma_\sigma}(t)) \right\|_{L_p}^p$$

$$= \inf_{\gamma_\sigma \in \Gamma_\sigma} \left(\int_A |\hat{x}(t)|^p \, d\mu - \int_{\gamma_\sigma} |\hat{x}(t)|^p \, d\mu \right)$$

$$= \int_A |\hat{x}(t)|^p \, d\mu - \sup_{\gamma_\sigma \in \Gamma_\sigma} \int_{\gamma_\sigma} |\hat{x}(t)|^p \, d\mu, \qquad L_p \overset{\text{df}}{=} L_p(A, d\mu).$$

Hence,

$$e_\sigma^p(\psi U_\Phi^p)_p \;=\; \sup_{x \in \psi U_\Phi^p} \left(\int_A |\hat{x}(t)|^p \, d\mu - \sup_{\gamma_\sigma \in \Gamma_\sigma} \int_{\gamma_\sigma} |\hat{x}(t)|^p \, d\mu \right). \tag{40}$$

If $x \in \psi U_\Phi^p$, then $\hat{x}(t) = \psi(t)\hat{y}(t)$, where y is a certain element from U_p. Therefore, the following relation is true:

$$\sup_{x\in\psi U_\Phi^p}\left(\int_A |\hat{x}(t)|^p\,d\mu - \sup_{\gamma_\sigma\in\Gamma_\sigma}\int_{\gamma_\sigma}|\hat{x}(t)|^p\,d\mu\right) \le \sup_{y\in U_p}\left(\int_A |\psi(t)|^p|y(t)|^p\,d\mu - \sup_{\gamma_\sigma\in\Gamma_\sigma}\int_{\gamma_\sigma}|\psi(t)|^p|y(t)|^p\,d\mu\right),$$

$$= \sup_{h\in U_1^+}\left(\int_A |\psi(t)|^p h(t)\,d\mu - \sup_{\gamma_\sigma\in\Gamma_\sigma}\int_{\gamma_\sigma}|\psi(t)|^p h(t)\,d\mu\right), \tag{41}$$

where, as above, U_1^+ is the subset of nonnegative functions from U_1.

To determine the value of the right-hand side of (41), we use the statement presented below,. This statement is, apparently, of independent interest, and, therefore, we formulate it as a theorem.

Theorem 3. *Let A be an arbitrary μ-measurable set from R^m, $m \ge 1$, let $\mathrm{mes}_\mu A = a$, where either a is finite or $a = \infty$, let $\varphi(x)$ be a nonnegative function essentially bounded on A, and let*

$$\lim_{|x|\to\infty}\varphi(x) = 0 \tag{42}$$

if the set A is unbounded. Then, for any $\sigma < a$, the following equality is true:

$$\mathcal{E}_\sigma(\varphi) = \sup_{h\in U_1^+}\inf_{\gamma_\sigma\in\Gamma_\sigma}\left(\int_A \varphi(x)h(x)\,d\mu - \int_{\gamma_\sigma\in\Gamma_\sigma}\varphi(x)h(x)\,d\mu\right) = \sup_{\sigma<q\le a}\frac{q-\sigma}{\int_0^q \frac{dt}{\overline{\varphi}(t)}}, \tag{43}$$

where $\Gamma_\sigma = \Gamma_\sigma(A)$ is the set of all μ-measurable subsets γ_σ of A whose measures are equal to σ, and $\overline{\varphi}(t)$ is the decreasing rearrangement of the function $\varphi(x)$.

The least upper bound on the right-hand side of (43) is realized for a certain finite value $q = q^*$.

Assume that Theorem 3 is proved. Then, setting $\varphi(x) = |\psi(x)|^p$ and combining relations (40), (41), and (43), we obtain relation (39).

Note that the strict inequality in (39) can be realized only in the case where the same is true for (41). The strict inequality in (39) is possible only due to the fact that not every function $y \in U_p$ has its preimage in U_Φ^p that possesses the ψ-integral. However, if $E(\Phi) = L_p(A)$, then this is not the case. Indeed, for any $y \in U_p$, its preimage exists, and, by virtue of the boundedness of ψ, the product $\psi(t)y(t)$ belongs to $L_p(A, d\mu)$ and, hence, has its preimage in S_Φ^p, or, more precisely, in ψU_Φ^p. Thus, in this case, relation (39) is, in fact, the equality. Thus, to prove Theorem 2, it remains to prove Theorem 3.

6. Proof of Theorem 3

First, we give several preliminary remarks. Below, we consider the Lebesgue integrals of nonnegative μ-measurable functions $f(x)$ defined on μ-measurable sets A in R^m, $m \ge 1$, namely,

$$I_A(f) = \int_A f(x)\,d\mu.$$

In the case where this value if finite, we write $f(x) \in L(A)$. We introduce the following definition:

Definition 1. *Let* $f \in L(A)$ *and let* γ_σ *be an arbitrary measurable subset of* A *such that*

$$\operatorname{mes}_\mu \gamma_\sigma = \sigma < a = \operatorname{mes}_\mu A,$$

i.e., $\gamma_\sigma \in \Gamma_\sigma = \Gamma_\sigma(A)$. *Then the quantity*

$$\mathcal{J}_\sigma(f) = \sup_{\gamma_\sigma \in \Gamma_\sigma} \int_{\gamma_\sigma} f(x)\,d\mu = \sup_{\gamma_\sigma \in A} \int_{\gamma_\sigma} f(x)\,d\mu \tag{44}$$

is called the principal value of rank σ *of the integral* $I_A(f)$.

The following statement is true:

Proposition 1. *The quantity* $\mathcal{J}_\sigma(f)$ *exists for any* $f \in L(A)$. *Furthermore, the following equality is true:*

$$\mathcal{J}_\sigma(f) = \int_0^\sigma \bar{f}(t)\,dt, \tag{45}$$

where $\bar{f}(t)$ *is the decreasing rearrangement of the function* $f(x)$.

The least upper bound in (44) is realized on a certain set $\gamma_\sigma^* \subset A$, $\operatorname{mes}_\mu \gamma_\sigma^* = \sigma$, *i.e.,*

$$\mathcal{J}_\sigma(f) = \sup_{\gamma_\sigma \in A} \int_{\gamma_\sigma} f(x)\,d\mu = \int_{\gamma_\sigma^*} f(x)\,d\mu = \int_0^\sigma \bar{f}(t)\,dt. \tag{46}$$

Proof. Let

$$f_{\gamma_\sigma}(x) = \begin{cases} f(x), & x \in \gamma_\sigma, \\ 0, & x \,\overline{\in}\, \gamma_\sigma. \end{cases}$$

Then, for any set γ_σ, relation (23) yields

$$\int_{\gamma_\sigma} f(x)\,d\mu = \int_A f_{\gamma_\sigma}(x)\,d\mu = \int_0^a \bar{f}_{\gamma_\sigma}(t)\,dt = \int_0^\sigma \bar{f}_{\gamma_\sigma}(t)\,dt.$$

It is clear that $\bar{f}_{\gamma_\sigma} \le \bar{f}(t)$ for all $t \in (0, \sigma)$. Therefore,

$$\int_{\gamma_\sigma} f(x)\,d\mu \le \int_0^\sigma \bar{f}(t)\,dt \qquad \forall \gamma_\sigma \subset A.$$

Hence, we always have

$$\mathcal{I}_\sigma(f) \le \int_0^\sigma \bar{f}(t)\,dt,$$

and, to prove equalities (45) and (46), it remain to establish the existence of the sets γ_σ^*.

Let the quantity y_σ be defined by the equality $\bar{f}(\sigma) = y_\sigma$. First, assume that y_σ is a point of continuity of the function $m_f(y)$. Then we set $\gamma_\sigma^* = E(f(x) \ge y_\sigma)$.

By virtue of (22), $\mathrm{mes}_\mu\,\gamma_\sigma^* = \sigma$ and $\bar{f}(t) \ge y_\sigma$ for $t \in (0, \sigma)$. Therefore, according to (23), we get

$$\int_{\gamma_\sigma^*} f(x)\,d\mu = \int_0^\sigma \bar{f}(t)\,dt, \tag{47}$$

i.e., equalities (45) and (46) are proved in this case.

If y_σ is a point of discontinuity of the function $m_f(y)$, then the measure of the set $E(f(x) \ge y)$ may be greater than σ. In this case, we set $\gamma_\sigma^* = e_1 + e_2'$, $e_1 = E(f(x) > y_\sigma)$, where e_2' is any measurable part of the set $e_2 = E(f(x) = y_\sigma)$ for which $\mathrm{mes}_\mu\, e_1 + \mathrm{mes}_\mu\, e_2' = \sigma$. It is clear that relation (47) is true for the set γ_σ^* thus defined, which proves the required statement. Note that, in the last case, the set γ_σ^* is not unique.

Definition 2. *Assume that, on a* μ*-measurable set* $A \subset R^m$, $m \ge 1$, $\mathrm{mes}_\mu A = a$, *where either* a *is finite or* $a = \infty$, *a summable function* $f(x)$ *such that*

$$\int_A |f(x)|\,d\mu < \infty$$

is given and $\gamma_\sigma \in \Gamma_\sigma$. *Then the quantity*

$$e_\sigma(f) = \inf_{\gamma_\sigma \in A} \left| \int_A f(x)\,d\mu - \int_{\gamma_\sigma} f(x)\,d\mu \right|$$

is called the best approximation of the integral of a function f *over the set* A *by integrals of rank* σ.

If $f(x) \ge 0$ for all $x \in A$, then $e_\sigma(f)$ can be represented in the form

$$e_\sigma(f) = \int_A f(x)\,d\mu - \sup_{\gamma_\sigma} \int_{\gamma_\sigma} f(x)\,d\mu.$$

Then Proposition 1 yields the following statement:

Proposition 2. *Let a nonnegative summable function* $f(x)$ *be defined on a measurable set* $A \subset R^m$, *where* $m \ge 1$, $\mathrm{mes}_\mu A = a$, *and* a *is either finite or infinite. Then*

$$e_{\sigma}(f) = \inf_{\gamma_{\sigma}\in A}\left(\int_{A} f(x)\,d\mu - \int_{\gamma_{\sigma}} f(x)\,d\mu\right) = \int_{\sigma}^{a} \bar{f}(t)\,dt. \tag{48}$$

In this case, the lower bound in (48) is realized by the set $\gamma_{\sigma}^{*} \in A$, $\text{mes}_{\mu}\,\gamma_{\sigma}^{*} = \sigma$, *defined in Proposition 1.*

If $\mathfrak{N}$ is a certain subset of functions from $L(A)$, then we set

$$\mathcal{E}_{\sigma}(\mathfrak{N}) = \sup_{f\in\mathfrak{N}} e_{\sigma}(f) = \sup_{f\in\mathfrak{N}} \inf_{\gamma_{\sigma}\in A}\left(\int_{A} f(x)\,d\mu - \int_{\gamma_{\sigma}} f(x)\,d\mu\right).$$

Thus, the quantity $\mathcal{E}_{\sigma}(\mathfrak{N})$ is the upper bound of the best approximations of the integrals of functions from the set $\mathfrak{N}$ by integrals of rank σ.

As $\mathfrak{N}$, we now consider the set H_{φ} that consists of functions $f(x)$, $x \in A$. These functions can be represented by products of a certain fixed nonnegative function $\varphi(x)$ and nonnegative functions $h(x)$ that belong to the unit ball U_1^{+} in $L(A)$:

$$H_{\varphi} = \{f(x) = \varphi(x)h(x) :\ h \in U_1^{+}\}.$$

We see that the quantity $\mathcal{E}_{\sigma}(\varphi)$ in (43) coincides with $\mathcal{E}_{\sigma}(H_{\varphi})$, and, hence, it is the upper bound of the best approximations of the integrals of functions $f \in H_{\varphi}$ over the set A by integrals of rank σ.

We now pass directly to the proof of Theorem 3.

It suffices to prove Theorem 3 only in the case of bounded sets A. Indeed, assume that it is proved for any bounded measurable set A from R^m. Let us show its validity in the general case.

We fix an arbitrarily small number $\varepsilon > 0$ and choose a number N_{ε} such that, for all $N \geq N_{\varepsilon}$ and $h \in U_1^{+}$, the following relation is true:

$$\int_{A} f(x)\,d\mu = \int_{A_N} f(x)\,d\mu + \rho, \qquad f(x) = \varphi(x)h(x), \qquad A_N = A \cap K_N, \tag{49}$$

$$K_N = \{x :\ x \in R^m,\ |x| \leq N\}, \qquad \rho \leq \varepsilon.$$

Note that, since $h \in U_1^{+}$, we can take as N_{ε} a number for which the relation $|x| > N_{\varepsilon}$ holds for $\varphi(x) < \varepsilon$ [see condition (42)].

At the same time, for any $\gamma_{\sigma} \in A$ and $h \in U_1^{+}$, we get

$$\int_{\gamma_{\sigma}} f(x)\,d\mu = \int_{\gamma_{\sigma}\cap A_N} f(x)\,d\mu + \rho', \qquad \rho' \leq \varepsilon,$$

whence

$$\sup_{\gamma_{\sigma}\in A}\int_{\gamma_{\sigma}} f(x)\,d\mu = \sup_{\gamma_{\sigma}\in A}\int_{\gamma_{\sigma}\cap A_N} f(x)\,d\mu + \rho'', \qquad \rho'' \leq \varepsilon. \tag{50}$$

Let δ_σ be any measurable subset of A_N, $\mathrm{mes}_\mu \delta_\sigma = \sigma$. Then

$$\sup_{\delta_\sigma \in A_N} \int_{\delta_\sigma} f(x)\,d\mu \le \sup_{\gamma_\sigma \in A} \int_{\gamma_\sigma} f(x)\,d\mu \tag{51}$$

and

$$\sup_{\gamma_\sigma \in A} \int_{\gamma_\sigma \cap A_N} f(x)\,d\mu \le \sup_{\delta_\sigma \in A_N} \int_{\delta_\sigma} f(x)\,d\mu. \tag{52}$$

Combining relations (49)–(52), we get

$$\sup_{\gamma_\sigma \in A} \int_{\gamma_\sigma} f(x)\,d\mu = \sup_{\delta_\sigma \in A_N} \int_{\delta_\sigma} f(x)\,d\mu + \rho^{(3)}, \quad \rho^{(3)} \le \varepsilon.$$

Taking into account (43) and (49), we obtain

$$\begin{aligned}
\mathscr{E}_\sigma(H_\varphi) &= \sup_{h \in U_1^+} \left(\int_A \varphi(x)h(x)\,d\mu - \sup_{\gamma_\sigma \in A} \int_{\gamma_\sigma} \varphi(x)h(x)\,d\mu \right) \\
&= \sup_{h \in U_1^+} \left(\int_{A_N} \varphi(x)h(x)\,d\mu - \sup_{\delta_\sigma \in A_N} \int_{\delta_\sigma} \varphi(x)h(x)\,d\mu \right) + \rho^{(4)} \\
&= \sup_{h \in \overline{H}} \left(\int_{A_N} \varphi(x)h(x)\,d\mu - \sup_{\delta_\sigma \in A_N} \int_{\delta_\sigma} \varphi(x)h(x)\,d\mu \right) + \rho^{(5)} \overset{\mathrm{df}}{=} \mathscr{E}_\sigma(\overline{H}_\varphi) + \rho^{(5)},
\end{aligned} \tag{53}$$

where $\rho^{(4)}$ and $\rho^{(5)}$ satisfy the inequalities

$$|\rho^{(4)}| \le \varepsilon, \quad \rho^{(5)} \le \varepsilon,$$

$\overline{H}_\varphi = \{ f(x) = \varphi(x)h(x) : h \in \overline{H} \}$, and $\overline{H}$ is the subset of functions from U_1^+ for which

$$\int_{A_N} h(x)\,d\mu \le 1.$$

For any N, the sets A_N are bounded and their measures are finite (assume that they are equal to a_N). Therefore, according to the assumption made and equality (43), we have

$$\mathscr{E}_\sigma(\overline{H}_\varphi) = \sup_{\sigma < q < a_N} \frac{q - \sigma}{\int_0^q \frac{dt}{\overline{\varphi}_N(t)}}, \tag{54}$$

where $\bar{\varphi}_N(t)$ is the decreasing rearrangement of the restriction of a function $\varphi(x)$ on the set A_N and

$$\varphi_N(x) = \begin{cases} \varphi(x), & x \in A_N, \\ 0, & x \,\overline{\in}\, A_N. \end{cases} \tag{55}$$

Combining relations (53) and (54), we get

$$\mathcal{E}_\sigma(H_\varphi) = \sup_{\sigma<q<a_N} \frac{q-\sigma}{\int_0^q \frac{dt}{\bar{\varphi}_N(t)}} + \rho^{(5)}. \tag{56}$$

Let us show that

$$\lim_{N\to\infty} \mathcal{E}_\sigma(\overline{H}_\varphi) = \sup_{\sigma<q<a} \frac{q-\sigma}{\int_0^q \frac{dt}{\bar{\varphi}(t)}}. \tag{57}$$

First, note that if $a = \infty$, then, for any fixed σ, the function

$$f_\sigma(q) = \frac{q-\sigma}{\int_0^q \frac{dt}{\bar{\varphi}(t)}} \tag{58}$$

tends to zero as $q \to \infty$. Therefore, there exists a point q^* for which

$$\sup_{\sigma<q<a} f_\sigma(q) = f_\sigma(q^*). \tag{59}$$

It is clear that such a point q^* can also be found for $a < \infty$.

Let us prove the following statement:

Proposition 3. *Let $\varphi_N(x)$ be defined by relation (55). Then, on a certain interval $[0, b_N]$, one has*

$$\bar{\varphi}_N(t) = \bar{\varphi}(t), \qquad t \in [0, b_N], \tag{60}$$

and, furthermore,

$$\lim_{N\to\infty} b_N = a. \tag{61}$$

Proof. For a given natural N_0, let

$$y_0 = y_{N_0} = \operatorname*{ess\,sup}_{x\in A\setminus A_{N_0}} \varphi(x), \tag{62}$$

$$E_{y_0} = \{x : x \in A, \ \varphi(x) \ge y_0\}. \tag{63}$$

If $y_0 = 0$, then there exists N_1 such that, for all $N > N_1$, we have $\varphi(x) = 0$ almost everywhere if $x \in A_N$. Consequently, $\varphi_N(x) \equiv \varphi(x)$, $x \in A_N$, for all $N > N_1$, whence $\overline{\varphi}_N(t) = \overline{\varphi}(t)$ for all $t \in [0, a_N]$. Therefore, it suffices to assume that $y_0 > 0$. By virtue of condition (42), if $y_0 > 0$, then the set E_{y_0} is bounded. Therefore, we can find a number N_{y_0} such that $E_{y_0} \subset A_N \quad \forall N > N_{y_0}$. Hence, for $N > N_{y_0}$, we get $\varphi(x) < y_0$, $x \in A_N$. Therefore, for the distribution functions $m_\varphi(y)$ and $m_{\varphi_N}(y)$, we get $m_\varphi(y) = m_{\varphi_N}(y)$, $y \in [y_0, \mathcal{M}_\varphi = \operatorname{ess\,sup}_{x \in A} \varphi(x)]$. Consequently, the functions $\overline{\varphi}(t)$ and $\overline{\varphi}_N(t)$ coincide on the interval $[0, m_\varphi(y_0)]$, i.e., $\overline{\varphi}_N(t) = \overline{\varphi}(t)$, $t \in [0, m_\varphi(y_0)]$. According to (42) and (62), we have $\lim_{N_0 \to \infty} y_{N_0} = 0$. Therefore, by setting $b_{N_0} = m_\varphi(y_{N_0})$, we arrive at relations (60) and (61).

Returning to the proof of equality (57), we conclude that if the number N_{q^*} is such that $b_N \geq q^*$ for $N > N_{q^*}$, then

$$\sup_{\sigma < q < a_N} \frac{q - \sigma}{\int_0^q \frac{dt}{\overline{\varphi}_N(t)}} = \frac{q^* - \sigma}{\int_0^{q^*} \frac{dt}{\overline{\varphi}(t)}},$$

which proves equality (57).

Combining equalities (56) and (57) and taking into account the arbitrariness of the choice of the number ε, we arrive at the required statement. Thus, it remains to prove the theorem only in the case of bounded sets A.

Assume that the theorem is proved for all bounded sets A in the case where the function $\varphi(x)$ takes an arbitrary finite number of values. Namely, assume that the following statement is true:

Proposition 4. *Let A be an arbitrary bounded measurable set from R^m, $\operatorname{mes} A = a$, and let $\varphi(x)$ be a nonnegative function taking finitely many values on A. Then, for any $\sigma < a$, equality (43) is satisfied.*

Let us prove the following statement:

Proposition 5. *Let A be an arbitrary bounded measurable set from R^m, $\operatorname{mes} A = a$, and let $\varphi(x)$ be an arbitrary nonnegative bounded measurable function. Then equality (43) holds for any $\sigma < a$.*

Proof. Let

$$c = \operatorname{ess\,sup}_{x \in A} \varphi(x)$$

and let n be a certain natural number. We divide the segment $[0, c]$ into n equal parts ρ_k, $k = 1, \ldots, n$, by points $y_i^{(n)}$:

$$c = y_1^{(n)} > y_2^{(n)} > \ldots > y_n^{(n)} > y_{n+1}^{(n)} = 0.$$

If E is an arbitrary measurable subset of A, then we put

$$e_k = \{x \in E : y_k^{(n)} < \varphi(x) \leq y_{k+1}^{(n)}\}, \qquad k = 1, 2, \ldots, n.$$

Then, for any function $h \in U_1^+$, we get

$$\sum\nolimits_n^{(1)} (\varphi, h) \overset{\text{df}}{=} \sum_{k=1}^{n} y_{k+1}^{(n)} \int_{e_k} h(x)\, d\mu \leq \int_E \varphi(x) h(x)\, d\mu \leq \sum_{k=1}^{n} y_k^{(n)} \int_{e_k} h(x)\, d\mu \overset{\text{df}}{=} \sum\nolimits_n^{(2)} (\varphi, h).$$

In this case, we have

$$\sum\nolimits_n^{(2)} (\varphi, h) - \sum\nolimits_n^{(1)} (\varphi, h) = \frac{c}{n} \int_E h(x)\, d\mu \leq \frac{c}{n}.$$

Hence,

$$\int_E \varphi(x) h(x)\, d\mu = \sum\nolimits_n^{(1)} (\varphi, h) + \varepsilon_n^{(1)} = \sum\nolimits_n^{(2)} (\varphi, h) - \varepsilon_n^{(2)},$$

$$0 \leq \varepsilon_n^{(1)} \leq \frac{c}{n}, \qquad 0 \leq \varepsilon_n^{(2)} \leq \frac{c}{n}.$$

Therefore, for any $h \in U_1^+$ and $n \in N$, the following equality is true:

$$\int_A \varphi(x) h(x)\, d\mu - \sup_{\gamma_\sigma \in A} \int_{\gamma_\sigma} \varphi(x) h(x)\, d\mu = \int_A \varphi_n(x) h(x)\, d\mu - \sup_{\gamma_\sigma \in A} \int_{\gamma_\sigma} \varphi_n(x) h(x)\, d\mu + \varepsilon_n^{(3)}, \tag{64}$$

where

$$\varphi_n(x) = y_k^{(n)}, \qquad x \in \{ y_k^{(n)} < \varphi(x) \leq y_{k+1}^{(n)} \}$$

and

$$\left| \varepsilon_n^{(3)} \right| \leq \frac{2c}{n}.$$

Considering the upper bounds of both sides of equality (64) on the set U_1^+ and taking into account the uniform boundedness of the quantities $\varepsilon_n^{(3)}$ by the numbers $2c/n$, we get

$$\mathcal{E}_\sigma(H_\varphi) = \sup_{h \in U_1^+} \left(\int_A \varphi_n(x) h(x)\, d\mu - \sup_{\gamma_\sigma \in A} \int_{\gamma_\sigma} \varphi_n(x) h(x)\, d\mu \right) + \varepsilon_n^{(4)},$$

where

$$\left| \varepsilon_n^{(4)} \right| \leq \frac{2c}{n}.$$

The function $\varphi_n(x)$ takes only n values, and, therefore, according to Proposition 4, we obtain

$$\mathscr{E}_\sigma(H_\varphi) = \sup_{\sigma<q\le a} \frac{q-\sigma}{\int_0^q \frac{dt}{\overline{\varphi}_n(t)}} + \varepsilon_n^{(4)},$$

where $\overline{\varphi}_n(t)$ is the decreasing rearrangement of the function $\varphi_n(x)$. To prove Proposition 5, it remains to note that, for any $q < a$, we have

$$\lim_{n\to\infty} \int_0^q \frac{dt}{\overline{\varphi}_n(t)} = \int_0^q \frac{dt}{\overline{\varphi}(t)}.$$

Thus, to complete the proof of the theorem, it remains to prove Proposition 4, i.e., to prove Theorem 3 in the case of bounded sets A for the functions $\varphi(x)$ that take finitely many values φ_j, $j = 1, 2, \dots, n$, $n \in N$, on A.

Assume that $\varphi(x)$ takes n different values φ_k. We enumerate them in decreasing order as follows:

$$\varphi_1 > \varphi_2 > \dots > \varphi_n. \tag{65}$$

We set

$$e_k = \{x \in A : \varphi(x) = \varphi_k\}, \quad k = 1, \dots, n, \quad \operatorname{mes}_\mu e_k = |e_k|.$$

It is clear that the sets e_k and e_j do not intersect for $k \ne j$, and

$$\bigcup_k e_k = A. \tag{66}$$

Let $h(x)$ be a function nonnegative on A, let

$$h_k(x) = \begin{cases} h(x), & x \in e_k, \\ 0, & x \,\overline{\in}\, e_k, \end{cases} \tag{67}$$

and let $\overline{h}_k(x)$ be the decreasing rearrangement of the function $h_k(x)$ (defined on $[0, |e_k|]$). Also let

$$t_i = \sum_{k=1}^{i} |e_k|, \quad i = 1, 2, \dots, n, \quad t_0 \overset{\text{df}}{=} 0. \tag{68}$$

We define a function $h^*(t)$ by setting

$$h^*(t) = \overline{h}(t - t_{k-1}), \quad t \in [t_{k-1}, t_k) \overset{\text{df}}{=} \Delta_k, \quad k = 1, \dots, n, \tag{69}$$

and prove the following statement:

Proposition 6. *Let A be an arbitrary bounded measurable set from R^m, $\operatorname{mes}_\mu A = a$, let $\varphi(x)$ be a nonnegative function taking finitely many values on A, and let $h(x)$ be a nonnegative function for which*

$$\int_A h(x)\,d\mu \;=\; b \;<\; \infty.$$

Further, let γ_σ *be an arbitrary subset of* A, $\operatorname{mes}_\mu \gamma_\sigma = \sigma \le a$. *Then the function* $h^*(t)$ *defined by relation (69) satisfies the equalities*

$$\int_0^a h^*(t)\,dt \;=\; b$$

and

$$\mathscr{E}_\sigma(\varphi; h) \;\stackrel{\text{df}}{=}\; \int_A \varphi(x)h(x)\,d\mu - \sup_{\gamma_\sigma \subset A} \int_{\gamma_\sigma} \varphi(x)h(x)\,d\mu \;=\; \int_0^a \overline{\varphi}(t)h^*(t)\,dt - \sup_{\delta_\sigma \subset (0,a)} \int_{\delta_\sigma} \overline{\varphi}(t)h^*(t)\,dt, \tag{70}$$

where $\overline{\varphi}(t)$ *is the decreasing rearrangement of the function* $\varphi(x)$ *and* δ_σ *are subsets of* $(0, a)$ *for which* $\operatorname{mes}_\mu \delta_\sigma = \sigma$.

Proof. Taking (23) into account, we get

$$b \;=\; \int_A h(x)\,d\mu \;=\; \sum_{k=1}^n \int_{e_k} h(x)\,d\mu \;=\; \sum_{k=1}^n \int_{e_k} h_k(x)\,d\mu$$

$$=\; \sum_{k=1}^n \int_0^{|e_k|} \overline{h}_k(t)\,dt \;=\; \sum_{k=1}^n \int_{\Delta_k} \overline{h}_k(t - t_{k-1})\,dt \;=\; \int_0^a h^*(t)\,dt.$$

Let us show that the function also satisfies equality (70). First, note that, by virtue of (23), the following equalities are true:

$$\int_A \varphi(x)h(x)\,d\mu \;=\; \sum_{k=1}^n \varphi_k \int_{e_k} h_k(x)\,d\mu \;=\; \sum_{k=1}^n \varphi_k \int_0^{|e_k|} \overline{h}_k(t)\,dt$$

$$=\; \sum_{k=1}^n \varphi_k \int_{\Delta_k} \overline{h}_k(t - t_{k-1})\,dt \;=\; \sum_{k=1}^n \varphi_k \int_{\Delta_k} h^*(t)\,dt \;=\; \int_0^a \overline{\varphi}(t)h^*(t)\,dt.$$

Therefore, it remains to verify the relation

$$\mathcal{J}(\varphi h) \;\stackrel{\text{df}}{=}\; \sup_{\gamma_\sigma \subset A} \int_{\gamma_\sigma} \varphi(x)h(x)\,d\mu \;=\; \sup_{\delta_\sigma \subset (0,a)} \int_{\delta_\sigma} \overline{\varphi}(t)h^*(t)\,dt \;=\; \mathcal{J}_\sigma(\overline{\varphi}h^*). \tag{71}$$

Let γ_σ be an arbitrary set from A, $\mathrm{mes}_\mu \gamma_\sigma = |\gamma_\sigma| = \sigma$, and let

$$\eta_k = \gamma_\sigma \cap e_k, \quad k = 1, 2, \ldots, n, \quad \mathrm{mes}_\mu \eta_k = |\eta_k|. \tag{72}$$

Then

$$\int_{\gamma_\sigma} \varphi(x)h(x)\,d\mu = \sum_{k=1}^{n} \varphi_k \int_{\eta_k} h(x)\,d\mu = \sum_{k=1}^{n} \varphi_k \int_0^{|\eta_k|} \bar{h}_k(t)\,dt$$

$$= \sum_{k=1}^{n} \varphi_k \int_{\beta_k} \bar{h}_k(t - t_{k-1})\,dt = \sum_{k=1}^{n} \varphi_k \int_{\beta_k} h^*(t)\,dt = \int_{\bigcup_{k=1}^{n}\beta_k} \bar{\varphi}(t)h^*(t)\,dt, \tag{73}$$

where $\beta_k = \{ t: t \in \Delta_k, \ t - t_{k-1} \in (0, |\eta_k|) \}$. Since

$$\mathrm{mes}_\mu \bigcup_{k=1}^{n} \beta_k = \sum_{k=1}^{n} |\eta_k| = \sigma,$$

we conclude, by virtue of (73), that

$$\mathcal{J}(\varphi h) = \sup_{\gamma_\sigma \subset A} \int_{\gamma_\sigma} \varphi(x)h(x)\,d\mu \le \sup_{\delta_\sigma \in (0,a)} \int_{\delta_\sigma} \bar{\varphi}(t)h^*(t)\,dt = \mathcal{J}_\sigma(\bar{\varphi}h^*). \tag{74}$$

To prove (70), it remains to show that the last relation cannot be the strict equality. For this purpose, taking Proposition 1 into account, we consider a set δ_σ^* with measure σ on $[0, a)$ for which

$$\mathcal{J}_\sigma(\bar{\varphi}h^*) = \int_{\delta_\sigma^*} \bar{\varphi}(t)h^*(t)\,dt \tag{75}$$

and put

$$\nu_k = \delta_\sigma^* \cap \Delta_k, \quad \mathrm{mes}_\mu \nu_k = |\nu_k|, \quad k = 1, \ldots, n.$$

According to (69) and (75), we get

$$\mathcal{J}_\sigma(\bar{\varphi}h^*) = \sum_{k=1}^{n} \varphi_k \int_{\nu_k} h^*(t)\,dt = \sum_{k=1}^{n} \varphi_k \int_{\nu_k} \bar{h}_k(t - t_{k-1})\,dt = \sum_{k=1}^{n} \varphi_k \int_{\alpha_k} \bar{h}_k(t)\,dt, \tag{76}$$

where $\alpha_k = \{ t: t \in [0, |\nu_k|), \ t + t_{k-1} \in \nu_k \}$, $k = 1, 2, \ldots, n$. On the intervals $[0, |\nu_k|)$, the functions $\bar{h}_k(t)$ do not decrease. Hence, the functions $\bar{h}_k(t - t_{k-1})$ do not decrease for $t \in \Delta_k$. Therefore, relations (75) and (76) yield

$$\nu_k = [t_{k-1}, t_{k-1} + |\nu_k|], \quad k = 1, 2, \dots, n,$$

or

$$\nu_k = \{t : t \in \Delta_k, \ \bar{h}_k(t - t_{k-1}) \ge \bar{h}_k(|\nu_k|)\}. \tag{77}$$

Consequently, according to (76), we obtain

$$\mathcal{I}_\sigma(\bar{\varphi}, h^*) = \sum_{k=1}^{n} \varphi_k \int_0^{|\nu_k|} \bar{h}_k(t)\,dt. \tag{78}$$

We now construct a set $\gamma_\sigma^* \subset A$ corresponding to the set $\delta_\sigma^* \subset [0, a)$. For this purpose, we put

$$\eta_k^* = \{x : x \in e_k, \ h(x) \ge \bar{h}_k(|\nu_k|)\}, \quad k = 1, 2, \dots, n, \tag{79}$$

and

$$\gamma_\sigma^* = \bigcup_{k=1}^{n} \eta_k^*. \tag{80}$$

Since $h(x) = h_k(x)$ for $x \in e_k$, by virtue of (22), (77), and (79) we get $\text{mes}_\mu\, \eta_k^* = \text{mes}_\mu\, \nu_k = |\nu_k|$. Hence, according to (80), we have $\text{mes}_\mu\, \gamma_\sigma^* = \sigma$. For such γ_σ^*, we obtain [see relations (72), (73), and (78)]

$$\int_{\gamma_\sigma^*} \varphi(x) h(x)\, d\mu = \sum_{k=1}^{n} \varphi_k \int_0^{|\eta_k^*|} h_k(t)\,dt = \sum_{k=1}^{n} \varphi_k \int_0^{|\nu_k|} h_k(t)\,dt = \mathcal{I}_\sigma(\bar{\varphi}, h).$$

Thus, relation (74) is indeed the equality, i.e., relation (71) is true. Proposition 6 is proved.

Assume, as above, that

$$U_1^+ = \left\{ h(x) : h(x) \ge 0, \ \int_A h(x)\, d\mu \le 1 \right\},$$

the function $\varphi(x)$ takes finitely many values φ_j, $j = 1, 2, \dots, n$, on A, and $H^*(\varphi)$ is the set of functions defined on $[0, a)$ and constructed for every $h \in U_1^+$ according to formula (69). Then Proposition 6 yields the following statement:

Proposition 7. *Let A be an arbitrary bounded measurable set from R^m, $\text{mes}_\mu A = a$, and let $\varphi(x)$ be a nonnegative function taking a finite number n of values φ_j, $j = 1, 2, \dots, n$, on A. Then, for any $\sigma \le a$, the following equality is true:*

$$\mathcal{E}_\sigma(H_\varphi) = \sup_{h \in U_1^+} \mathcal{E}_\sigma(\varphi; h) = \sup_{h^* \in H^*(\varphi)} \left(\int_0^a \overline{\varphi}(t) h^*(t)\, dt - \sup_{\delta_\sigma \in (0,a)} \int_{\delta_\sigma} \overline{\varphi}(t) h^*(t)\, dt \right), \tag{81}$$

where $\overline{\varphi}(t)$ *is the decreasing rearrangement of the function* $\varphi(x)$ *and* δ_σ *are the subsets of* $(0, a)$ *for which* $\text{mes}_\mu \delta_\sigma = \sigma$.

Finally, we establish one more auxiliary statement.

Lemma 1. *Assume that, on an interval* $(0, a)$ *of the real axis* R^1, *where* a *is either finite or infinite, a bounded nonincreasing function* $\alpha(t)$ *satisfying the condition*

$$\lim_{t \to \infty} \alpha(t) = 0$$

in the case $a = \infty$ (*in this case, we write* $\alpha \in \mathcal{A}$) *and the set* $\mathcal{M}$ *of nonnegative functions* $m(t)$ *for which*

$$\int_0^a m(t)\, dt \le 1$$

are given. Then

$$\mathcal{E}_\sigma(\alpha, \mathcal{M}) = \sup_{m \in \mathcal{M}} \left(\int_0^a \alpha(t) m(t)\, dt - \sup_{\delta_\sigma \in (0,a)} \int_{\delta_\sigma} \alpha(t) m(t)\, dt \right) = \sup_{q \in (\sigma, a)} \frac{q - \sigma}{\int_0^a \frac{dt}{\alpha(t)}}, \tag{82}$$

where δ_σ *is an arbitrary measurable set from* $(0, a)$, $\text{mes}\, \delta_\sigma = |\mu| = \sigma$. *The upper bound on the right-hand side of (82) is always realized at a certain point* $q^* \in (\sigma, a)$. *The upper bound on the left-hand side of (82) is realized by a function* $m^* \in \mathcal{M}$, *namely*

$$m^*(t) = \begin{cases} \dfrac{1}{\alpha(t)} \displaystyle\int_0^{q^*} \frac{dx}{\alpha(x)}, & t \in [0, q^*], \\ 0, & t \in (q^*, a). \end{cases}$$

Proof. For given $\alpha \in \mathcal{A}$ and $m \in \mathcal{M}$, let

$$F_\sigma(\alpha, m) = \int_0^a \alpha(t) m(t)\, dt - \sup_{\delta_\sigma} \int_{\delta_\sigma} \alpha(t) m(t)\, dt \tag{83}$$

and let $\delta_\sigma^* = \delta_\sigma^*(m)$ be the set from $(0, a)$ for which

$$F_\sigma(\alpha, m) = \int_0^a \alpha(t)m(t)\,dt - \int_{\delta_\sigma^*} \alpha(t)m(t)\,dt.$$

Note that the existence of such a set δ_σ^* is guaranteed by Proposition 1 and the summability of the product $\alpha(t)m(t)$. Assume, in addition, that

$$y_\sigma = y_\sigma(m) = \inf_{t\in\delta_\sigma^*} \alpha(t)m(t).$$

It is clear that, in this case, we have

$$\delta_\sigma^* = \{t: \alpha(t)m(t) \geq y_\sigma\}, \tag{84}$$

and the inequality $t \in e_\sigma = (0, a)\setminus\delta_\sigma^*$ holds for $\alpha(t)m(t) \leq y_\sigma$. Therefore, for the function

$$\overline{m}(t) = \begin{cases} m(t), & t \in e_\sigma, \\ \dfrac{y_\sigma}{\alpha(t)}, & t \in \delta_\sigma^*, \end{cases} \tag{85}$$

the following equalities are true:

$$\int_{\delta_\sigma^*} \alpha(t)\overline{m}(t)\,dt = \sigma y_\sigma$$

and

$$F_\sigma(\alpha, \overline{m}) = F_\sigma(\alpha, m). \tag{86}$$

We choose a point $c > \sigma$ from the condition

$$\int_0^c \left(\frac{y_\sigma}{\alpha(t)} - \overline{m}(t)\right)dt = \int_c^a \overline{m}(t)\,dt. \tag{87}$$

By virtue of the fact that $\alpha(t)$ is a monotone nonincreasing function, such a point always exists and is unique. We set

$$m'(t) = \begin{cases} \dfrac{y_\sigma}{\alpha(t)}, & t \in (0, c), \\ 0, & t \in [c, a). \end{cases} \tag{88}$$

Taking (87) into account, we obtain

$$\int_0^a \alpha(t)m'(t)\,dt - \int_0^a \alpha(t)\overline{m}(t)\,dt = \int_0^c \alpha(t)\left(\frac{y_\sigma}{\alpha(t)} - \overline{m}(t)\right)dt - \int_c^a \alpha(t)\overline{m}(t)\,dt$$

$$\geq \alpha(c+0)\left(\int_0^c \left(\frac{y_\sigma}{\alpha(t)} - \overline{m}(t)\right)dt - \int_c^a \overline{m}(t)\,dt\right) = 0.$$

By virtue of (86), this yields

$$F_\sigma(\alpha, m') \geq F_\sigma(\alpha, m). \tag{89}$$

It follows from relations (85) and (84) that

$$\int_0^a \overline{m}(t)\,dt \leq \int_0^a m(t)\,dt,$$

and relations (88) and (87) yield

$$\int_0^a m'(t)\,dt = \int_0^c \frac{y_\sigma}{\alpha(t)}\,dt = \int_0^a \overline{m}(t)\,dt.$$

Hence, $m' \in \mathcal{M}$.

Let $\mathcal{M}'$ denote the subset of functions m from $\mathcal{M}$ for which there exists a number $q = q(m)$, $\sigma < q \leq a$, such that $m(t) = 0$ for all $t \in (q, a)$, and the product $\alpha(t)m(t)$ is constant on the interval $[0, q)$:

$$m(t) = \begin{cases} \lambda, & t \in [0, q], \\ 0, & t \in (q, a), \end{cases} \tag{90}$$

where λ is a positive number. The function m from (88) belongs to $\mathcal{M}'$ and satisfies relation (89). Therefore,

$$\mathcal{E}_\sigma(\alpha, \mathcal{M}) = \sup_{m \in \mathcal{M}} F_\sigma(\alpha, m) = \sup_{m \in \mathcal{M}'} F_\sigma(\alpha, m) = \mathcal{E}_\sigma(\alpha, \mathcal{M}').$$

Thus, the problem of the determination of the quantity $\mathcal{E}_\sigma(\alpha, \mathcal{M})$ is reduced to the determination of the quantity $\mathcal{E}_\sigma(\alpha, \mathcal{M}')$.

If $m \in \mathcal{M}'$, then, according to (90), we get

$$F_\sigma(\alpha, m) = \lambda(q - \sigma)$$

and

$$\|m\|_1 \stackrel{\mathrm{df}}{=} \int_0^q m(t)\,dt = \lambda \int_0^q \frac{dt}{\alpha(t)}.$$

Consequently, since $\|m\|_1 \le 1$, we obtain

$$F_\sigma(\alpha, m) = \frac{(q-\sigma)\|m\|_1}{\int_0^q \frac{dt}{\alpha(t)}} \le \frac{q-\sigma}{\int_0^q \frac{dt}{\alpha(t)}}.$$

Therefore,

$$\sup_{m \in \mathcal{M}'} F_\sigma(\alpha, m) \le \sup_{\sigma < q \le a} \frac{q-\sigma}{\int_0^q \frac{dt}{\alpha(t)}}. \tag{91}$$

The existence of the number q^* indicated in Lemma 1 has, in fact, been established above [see relations (58) and (59)]. Therefore, according to (91), we get

$$\sup_{m \in \mathcal{M}'} F_\sigma(\alpha, m) \le \frac{q^*-\sigma}{\int_0^{q^*} \frac{dt}{\alpha(t)}}. \tag{92}$$

The function

$$m^*(t) = \begin{cases} \dfrac{1}{\alpha(t)} \displaystyle\int_0^{q^*} \frac{dx}{\alpha(x)}, & t \in [0, q^*], \\ 0, & t \in (q^*, a), \end{cases}$$

belongs to $\mathcal{M}'$, and, hence, relation (92) is indeed the equality, which completes the proof of Lemma 1.

We continue the proof of Proposition 4. The set $H^*(\varphi)$ from equality (81) is contained in $\mathcal{M}$. Therefore, denoting the right-hand side of this equality by $\mathcal{E}_\sigma(\overline{\varphi}; H^*)$, we get

$$\mathcal{E}_\sigma(\overline{\varphi}; H^*) = \sup_{h \in H^*(\varphi)} F_\sigma(\overline{\varphi}; h^*) \le \sup_{m \in \mathcal{M}} F_\sigma(\overline{\varphi}; m). \tag{93}$$

On the interval $(0, a)$, the function $\overline{\varphi}(t)$ satisfies the conditions imposed on the function $\alpha(t)$ in Lemma 1 (for finite values of a). Therefore, according to Lemma 1, we obtain

$$\sup_{m \in \mathcal{M}} F_\sigma(\overline{\varphi}; m) = \sup_{q \in (\sigma, a)} \frac{q-\sigma}{\int_0^q \frac{dt}{\overline{\varphi}(t)}} = \frac{q^*-\sigma}{\int_0^{q^*} \frac{dt}{\overline{\varphi}(t)}}, \tag{94}$$

where q^* is a certain point from (σ, a) and the upper bound on the left-hand side is realized by the function

$$m^*(t) = \begin{cases} \dfrac{1}{\overline{\varphi}(t)} \displaystyle\int_0^{q^*} \frac{dx}{\overline{\varphi}(x)}, & t \in [0, q^*], \\ 0, & t \in (q^*, a), \end{cases} \tag{95}$$

i.e.,

$$F_\sigma(\overline{\varphi}; m^*) = \frac{q^* - \sigma}{\int_0^{q^*} \frac{dt}{\overline{\varphi}(t)}}.$$

In Proposition 4, the function $\varphi(x)$ takes only a finite number of values φ_k, $k = 1, 2, \ldots, n$. If these values are enumerated in decreasing order, then its rearrangement is constant on the intervals Δ_k [see relations (65)–(69)]:

$$\overline{\varphi}(t) = \varphi_k, \qquad t \in \Delta_k, \qquad k = 1, 2, \ldots, n.$$

Therefore, in this case, the function $m^*(t)$ from (95) is piecewise constant on $[0, q^*]$:

$$m^*(t) = \begin{cases} \dfrac{1}{\varphi_k} \displaystyle\int_0^{q^*} \frac{dx}{\overline{\varphi}(x)} \overset{\text{df}}{=} m_k, & t \in [t_{k-1}, t_k], \quad k = 1, 2, \ldots, t \le q^*, \\ 0, & t \in (q^*, a). \end{cases}$$

This implies that the set U_1^+ contains a function h for which the function h^* constructed according to (69) coincides with m^*. This means that $m^* \in H^*(\varphi)$. Thus, relation (93) is, in fact, the equality. Combining equalities (93) and (94), we complete the proof of Proposition 4 and Theorem 3.

7. Examples

Consider several simplest realizations of the constructions considered above.

1. We say that a certain space $\mathfrak{N}$ is a partial case of the space S^p_Φ if it can be obtained by the proper choice of the space $\mathfrak{X}$, measure $d\mu$, and operator Φ.

In this sense, the spaces considered in [1–5], as well as the spaces S^p_φ introduced in [6], are partial cases of the space S^p_Φ. We show this in a simple but important case.

Let R^m be the m-dimensional, $m \ge 1$, Euclidean space, let $x = (x_1, \ldots, x_m)$ be its elements, let Z^m be the integer-valued lattice in R^m, i.e., the set of all vectors $k = (k_1, \ldots, k_m)$ with integer-valued coordinates, and let $xy = (x_1 y_1 + \ldots + x_m y_m)$, $|x| = \sqrt{x, x}$, and, in particular, $kx = k_1 x_1 + \ldots + k_m x_m)$, $|k| = \sqrt{k_1^2 + \ldots + k_m^2}$.

Further, we denote by $L = L(R^m, 2\pi)$ the set of all functions $f(x) = f(x_1, \ldots, x_m)$ 2π-periodic in each variable and summable with respect to the ordinary Lebesgue measure in the cube of periods

$$Q^m = \{x : x \in R^m, -\pi \le x_k \le \pi, k = 1, 2, \ldots, m\}.$$

In the space $L(R^m, 2\pi)$, we define an operator $\mathcal{F}$ by setting

$$\mathcal{F}(f; k) = (2\pi)^{-m/2} \int_{Q^m} f(x) e^{-ikx}\, dx = \hat{f}(k). \tag{96}$$

The operator $\mathcal{F}$ maps the space $L(R^m, 2\pi)$ into the set Y of functions $y(t)$ given on the integer-valued lattice Z^m in R^m. Now let $d\mu$ be a measure in the space R^m whose support is the set Z^m, where it is equal to 1, i.e., $\mu(k) \equiv 1$, $k \in Z^m$. In this case, the functional defined by equality (4) has the form

$$\|y\|_{L_p(R^m, d\mu)} = \left(\int_{R^m} |y(t)|^p \, d\mu \right)^{1/p} = \left(\sum_{k \in Z^m} \left| \hat{f}(k) \right|^p \right)^{1/p}, \quad p \in (0, \infty).$$

Choosing the space L as $\mathfrak{X}$ and the operator $\mathcal{F}$ as Φ, we obtain the space $S^p_{\mathcal{F}}(L; Y)$:

$$S^p_{\mathcal{F}}(L; Y) = \left\{ f \in L \colon \sum_{k \in Z^m} \left| \hat{f}(x) \right|^p < \infty \right\}.$$

Note that these spaces coincide with the considered spaces S^p_φ generated by the space L, the system $\varphi = \{\varphi_k\}_{k=1}^{\infty}$, where

$$\varphi_k = (2\pi)^{-m/2} e^{ikx}, \quad k = 1, 2, \ldots,$$

and the scalar product $(f, \varphi_k) = \hat{f}(k)$ defined by (96).

If $f = f(x)$ and $g = g(x)$ are arbitrary functions from $L(R^m, 2\pi)$, then their convolution

$$h(x) = (f * g)(x) = (2\pi)^{-m/2} \int_{Q^m} f(t) g(x-t) \, dt \tag{97}$$

also belongs to $L(R^m, 2\pi)$, and

$$\mathcal{F}(h; x) = \hat{h}(k) = (2\pi)^{m/2} \int_{Q^m} h(x) e^{-ikx} \, dx = \hat{f}(k) \hat{g}(k). \tag{98}$$

Therefore, the role of the multiplier M^ω_Φ is played by the convolution operator (97), and the set Ω^p_Φ contains all functions $\omega(t)$ satisfying the equality

$$\omega(k) = \hat{g}(k), \quad g \in L,$$

and such that

$$\sum \left| \hat{f}(k) \right|^p |\omega(k)|^p < \infty$$

for all $f \in S^p_\Phi$. In the case under consideration, the measure of any bounded set γ_σ is a natural number or zero. Therefore, for any function $\lambda(t)$ on γ_σ, the function

$$g_{\gamma_\sigma}(x) = \sum_{k \in \gamma_\sigma} \lambda(k) e^{ikx}$$

is a polynomial of degree $\leq \sigma$, and, hence, it belongs to S^p_Φ for any $p \in (0, \infty)$. Setting

$$U_{\gamma_\sigma}(f; \lambda)(x) = (f * g_{\gamma_\sigma})(x),$$

we get

$$\mathcal{F}(U_{\gamma_\sigma}(f; \lambda)) = \begin{cases} \lambda(k)\hat{f}(k), & k \in \gamma_\sigma, \\ 0, & k \,\overline{\in}\, \gamma_\sigma, \end{cases}$$

by virtue of (98), i.e., the function $U_{\gamma_\sigma}(f; \lambda)$, which is also a polynomial of degree σ, satisfies condition (9). In particular, if $\lambda_k = 1$ for $k \in \gamma_\sigma$, then $U_{\gamma_\sigma}(f)$ is a polynomial with the numbers of harmonics belonging to γ_σ, and its coefficients are the corresponding Fourier coefficients of the function f. Such polynomials are called the Fourier sums of a function f constructed on the sets γ_σ.

In this case, ψ-integrals are defined in the following way: Let $\psi = \{\psi(k)\}_{k \in Z^m}$ be an arbitrary system of complex numbers and let $f \in L$. Then a ψ-integral of a function f is an arbitrary function $u = \mathcal{J}^\psi f$ from L for which

$$\mathcal{F}(u; k) = \psi(k)\hat{f}(k).$$

In particular, if $\psi \in \Omega^p_\Phi$, then $\mathcal{J}^\psi f$ is given by the formula

$$\mathcal{J}^\psi f = f * \Psi,$$

where $\psi = \psi(x)$ is a function summable on R^m and such that its Fourier series has the form

$$S[\psi] = \sum_{k \in Z^m} \psi(k) e^{ikx}.$$

In this case, we have

$$\psi S^p_{\mathcal{F}} \subset S^p_{\mathcal{F}}, \quad p \in (0, \infty).$$

The results corresponding to those obtained in Theorems 1 and 2 for the case under consideration are presented in [1–5].

We now give examples of the spaces S^p_Φ that have nothing common with the scheme of the construction of the spaces S^p_φ in Sec. 1.

2. Consider an example in which the spaces S_Φ^p are definitely nonseparable. Let $L_2(R^m)$ be the space of all functions $f(x) = f(x_1, \ldots, x_m)$ Lebesgue-measurable on R^m, $m \geq 1$, and such that

$$\|f\|_{L_2(R^m)} = \left(\int\limits_{R^m} |f(x)|^2 \, dx \right)^{1/2} < \infty.$$

As $\mathfrak{X}$ and Y, we choose the spaces $L_2(R^m)$ and define the operator Φ by the Fourier transformation

$$\Phi(f) = \hat{f}(t) = \mathcal{F}(f; t) = (2\pi)^{-m/2} \int\limits_{R^m} f(x) e^{-itx} \, dx.$$

As is known (see, e.g., [11, Chap. I], the operator $\mathcal{F}$ is unitary on $L_2(R^m)$. Hence, the Φ-norm $\|f\|_{2,\Phi}$ of an element f coincides with its norm in the space $L_2(R^m)$:

$$\|f\|_{2,\mathcal{F}} = \|f\|_{L_2(R^m)}. \tag{99}$$

In this case, by virtue of formula (5), the space $S_\Phi^2 = S_\Phi^2(L_2(R^m), L_2(R^m), dx)$ has the form $S_\Phi^2 = \{f: f \in L_2(R^m)\}$, i.e., $S_\Phi^2 = \mathfrak{X} = L_2(R^m) = Y$.

The set $\Omega_\Phi^2 = \Omega_\Phi^2(L_2(R^m))$ coincides with the set of all functions ω for which the product $\omega(t)\hat{f}(t)$ is contained in $L_2(R^m)$ for any $f \in L_2(R^m)$, and the function $f_\omega(t) = M_\Phi^\omega f(t)$ satisfying equality (7) is given by the formula

$$f_\omega(t) = \mathcal{F}^{-1}(\omega \hat{f}; t) = (2\pi)^{m/2} \int\limits_{R^m} \omega(x) \hat{f}(x) e^{itx} \, dx. \tag{100}$$

It is clear that the set $\Omega_\Phi^2(L_2(R^m))$ contains all functions essentially bounded on R^m. In particular, $\Omega_\Phi^2(R^m)$ contains all essentially bounded functions $\lambda_\sigma = \lambda_\sigma(t)$ whose supports are bounded sets γ_σ for which condition (8) is satisfied, and, hence, the operator $\mathcal{F}$ satisfies condition (A_2). In this case, according to (100), the functions $U_{\gamma_\sigma}(f; \lambda; x)$ have the form

$$U_{\gamma_\sigma}(f; \lambda; x) = (2\pi)^{-m/2} \int\limits_{R^m} \lambda_\sigma(t) \hat{f}(t) e^{ixt} \, dt = (2\pi)^{-m/2} \int\limits_{R^m} f(z) \check{\lambda}_\sigma(x - z) \, dz, \tag{101}$$

where

$$\check{\lambda}_\sigma(v) = \mathcal{F}^{-1}(\lambda_\sigma; v) = (2\pi)^{-m/2} \int\limits_{R^m} \lambda_\sigma(t) e^{ivt} \, dt.$$

The proof of equality (101) is based on the well-known Plancherel theory. It is also known that, in this case, the functions $U_{\gamma_\sigma}(f; \lambda; x)$ are entire functions of exponential type (see, e.g., [10, Chap. V]).

In the case under consideration, the ψ-integrals of functions $f \in L_2(R^m)$ are defined as follows: Let $\psi = \psi(t)$ be a certain function from $\Omega^2_{\mathcal{F}}$ and let $f \in L_2(R^m)$. Then the ψ-integral of f is the function $f_\psi = \mathcal{J}^\psi f$ from $L_2(R^m)$ for which

$$\mathcal{J}(f_\psi; t) = \psi(t)\hat{f}(t).$$

Note that if $\psi \in L_2(R^m)$, then the function f_ψ is representable in the form

$$f_\psi(x) = \mathcal{J}^\psi f(x) = (2\pi)^{-m/2} \int_{R^m} f(z)\check{\psi}(x-z)\,dz, \qquad \check{\psi}(v) = (2\pi)^{-m/2} \int_{R^m} \psi(t)e^{ivt}\,dt.$$

In this case, Theorem 1 yields the following statement:

Theorem 1′. *Let* $\psi = \psi(t) = \psi(t_1, \ldots, t_m)$ *be an arbitrary function essentially bounded on* R^m, $m \geq 1$,

$$\operatorname*{ess\,sup}_{t \in R^m} |\psi(t)| < \infty,$$

and such that

$$\lim_{|t| \to \infty} |\psi(t)| = 0,$$

and let ψU_2 *be the set of* ψ*-integrals of all functions from* $U_2(R^m) = \{\varphi : \|\varphi\|_{L_2(R^m)} \leq 1\}$. *Further, let* Γ_σ *be the set of all Lebesgue-measurable subsets* $\gamma_\sigma \subset R^m$ *whose measures are equal to* σ, $\sigma \in (0, \infty)$. *Then, for any* $\gamma_\sigma \in \Gamma_\sigma$, *the following equalities are true:*

$$\mathcal{E}^2_{\gamma_\sigma}(\psi U_2) = \sup_{f \in \psi U_2} \left\| f(\cdot) - U_{\gamma_\sigma}(f; \cdot) \right\|^2_{L_2(R^m)} = \bar{\varphi}_{\gamma_\sigma}(0+0),$$

where $\bar{\varphi}_{\gamma_\sigma}(v)$ *is the decreasing rearrangement of the function*

$$\varphi_{\gamma_\sigma}(t) = \begin{cases} |\psi(t)|^2, & t \in R^m \setminus \gamma_\sigma, \\ 0, & t \in \gamma_\sigma, \end{cases}$$

and

$$D_\sigma(\psi U_2) = \inf_{\gamma_\sigma \in \Gamma_\sigma} \mathcal{E}_{\gamma_\sigma}(\psi U_2) = \bar{\psi}(\sigma + 0),$$

where $\bar{\psi}(v)$ *is the decreasing rearrangement of the function* $|\psi(t)|$.

The set Γ_σ *contains the set* γ^*_σ *for which*

$$\mathcal{E}_{\gamma^*_\sigma}(\psi U_2) = D_\sigma(\psi U_2) = \bar{\psi}(\sigma + 0).$$

This set is defined by the relation

$$\gamma_\sigma^* = \{ t \in R^m : |\psi(t)| \ge \overline{\psi}(\sigma + 0) \}, \qquad \text{mes}\, \gamma_\sigma^* = \sigma.$$

By analogy, using Theorem 2, we obtain the following statement:

Theorem 2′. *Under the conditions of Theorem 1′ and in the notation accepted therein, the following equality is true:*

$$e_\sigma^p(\psi U_2) = \sup_{f \in \psi U_2} \inf_{\gamma_\sigma \in \Gamma_\sigma} \left\| f(\cdot) - U_{\gamma_\sigma}(f; \cdot) \right\|_{L_2(R^m)}^p = \sup_{q > \sigma} \frac{q - \sigma}{\int_0^q \frac{dt}{\overline{\psi}^p(t)}}, \tag{102}$$

where $\overline{\psi}(v)$ is the rearrangement of the function $|\psi(t)|$ in decreasing order. The least upper bound on the right-hand side of (102) is realized for a certain finite value $q = q^*$.

3. Using the scheme presented in the second example, we can obtain analogs of Theorems 1′ and 2′ in the case where, instead of the Fourier transformation, one takes an arbitrary operator Φ unitary on the set $L_2(A, d\mu)$, where A is a certain manifold in R^m and μ is a certain σ-additive measure in R^m. We present the corresponding reasoning for the case where $L_2(A, d\mu)$ is the set $L_2(R_+^1)$ of functions $f(t)$ square summable in the Lebesgue sense on the semiaxis $(0, \infty)$, and Φ is the Hankel transformation, i.e.,

$$H_\nu f = H_\nu(f; x) = \hat{f}(x) = \hat{f}_\nu(x) = x^{-(\nu+1/2)} \frac{d}{dx} \int_0^\infty x^{\nu+1} \dot{\mathcal{J}}_{\nu+1}(xt) \frac{f(t)}{\sqrt{t}}\, dt,$$

where ν is a certain number, $\nu > -1$, and $\dot{\mathcal{J}}_\alpha(z)$ is the Bessel function of the first kind of order α.

As is known, the Hankel transformation generates the operator H_ν, which is unitary on $L_2(R^1)$ and coincides with its inverse (see, e.g., [10, Chap. III]). Therefore, the following analog of equality (99) is true:

$$\|f\|_{2, H_\nu} = \|f\|_{L_2(R_+^1)}.$$

Hence, $S_{H_\nu}^2 = \{ f : L_2(R_+^1) \}$, i.e., $S_{H_\nu}^2 = X = L_2(R_+^1) = Y$ in this case as well.

As in the previous example, the set $\Omega_{H_\nu}^2 = \Omega_{H_\nu}^2(L_2(R_+^1))$ is the set of functions ω for which the product $\omega(t)\hat{f}(t)$ is contained in $L_2(R_+^1)$ if $f \in L_2(R_+^1)$, and the function $f_\omega(t) = M_{H_\nu}^\omega f(t)$ satisfying equality (7) is defined by the equality

$$f_\omega(t) = H_\nu(\omega \hat{f}; t) = t^{-(\nu+1/2)} \frac{d}{dt} \int_0^\infty t^{\nu+1} \dot{\mathcal{J}}_{\nu+1}(xt) \omega(x) \frac{\hat{f}(x)}{\sqrt{x}}\, dx. \tag{103}$$

In particular, if Γ_σ is the set of all Lebesgue-measurable subsets γ_σ of R_+^1 whose measures are equal to σ, $\sigma \in (0, \infty)$, and $\gamma_\sigma \in \Gamma_\sigma$, then

$$U_{\gamma_\sigma}(f; x) \;=\; H_\nu(\chi_{\gamma_\sigma}\hat{f}; t) \;=\; t^{-(\nu+1/2)}\,\frac{d}{dt}\int\limits_{\gamma_\sigma} t^{\nu+1}\mathcal{J}_{\nu+1}(xt)\,\frac{\hat{f}(x)}{\sqrt{x}}\,dx. \tag{104}$$

If $\psi \in \Omega^2_{H_\nu}$ and $f \in L_2(R^1_+)$, then the ψ-integral of a function f is defined by formula (103) for $\omega(t) = \psi(t)$. If we now denote the set of ψ-integrals of all functions from $U_2(R^1_+) = \{\varphi : \|\varphi\|_{L_2(R^1_+)} \le 1\}$ by ψU_2, then the functions $U_{\gamma_\sigma}(f; \cdot)$ defined by (104) satisfy the statement obtained from Theorems 1′ and 2′ by the replacement of R^m by R^1_+.

4. Consider the partial case of the spaces S^p_Φ generated by the identity operator, i.e., the case $\Phi = I$. It is clear that, in this case, $\mathfrak{X} = Y(A, d\mu)$, $\hat{x} = x$, and, according to (5), we get

$$S^p_I \;=\; \{x \in \mathfrak{X} : \|x\|_{L_p(A, d\mu)} < \infty\} \;=\; L_p(A, d\mu), \qquad p \in (0, \infty].$$

The set Ω^Φ_I contains all μ-measurable functions ω for which the product $\omega(t)x(t)$ belongs to $L_p(A, d\mu)$ for all $x \in L_p(A, d\mu)$. In particular, if $\gamma_\sigma \in \Gamma_\sigma(A)$, then, for any $\sigma \in (0, \infty)$, the inclusion $\lambda_{\gamma_\sigma} \in \Omega^p_I$ holds for any essentially bounded function $\lambda = \lambda(t)$ with support γ_σ. The multiplier M^ω_I multiplies the element $x(t)$ by $\omega(t)$. Therefore,

$$U_{\gamma_\sigma}(x, \lambda; t) \;=\; \begin{cases} \lambda(t)x(t), & t \in \gamma_\sigma, \\ 0, & t \in A \setminus \gamma_\sigma, \;\; \lambda \in \Omega^p_I, \end{cases}$$

and, correspondingly,

$$U_{\gamma_\sigma}(x; t) \;=\; \begin{cases} x(t), & t \in \gamma_\sigma, \\ 0, & t \in A \setminus \gamma_\sigma. \end{cases}$$

In this case, the unit ball U^p_I coincides with the unit ball U_p in $U_p(A, d\mu)$, i.e.,

$$U^p_I \;=\; \{x : x \in S^p_I = L_p(A, d\mu), \; \|x\|_{L_p(A, d\mu)} \le 1\},$$

and, for a given function $\psi = \psi(t)$, the set ψU^p_I is the set of products $\psi(t)x(t)$, $x \in U^p_I$.

In the case under consideration, Theorems 1 and 2 yield the following statement:

Theorem 4. *Let* $\psi = \psi(t)$ *be an arbitrary function from* $Y(A, d\mu)$ *essentially bounded on* A *and let condition (25) be satisfied in the case where the set* $A \subset R^m$, $m \ge 1$, *is unbounded. If* $\gamma_\sigma \in \Gamma_\sigma(A)$, $\sigma < a = \mathrm{mes}_\mu A$, *then the following equalities are true:*

$$\mathcal{E}^p_{\gamma_\sigma}(\psi U^p_I)_p \;=\; \sup_{x \in \psi U_p}\; \inf_{\lambda \in \Omega^p_I} \left\| x(t) - U_{\gamma_\sigma}(x; \lambda; t) \right\|^p_{L_p(A,\, d\mu)}$$

$$=\; \sup_{x \in \psi U_p} \left\| x(t) - U_{\gamma_\sigma}(x; t) \right\|^p_{L_p(A,\, d\mu)} \;=\; \overline{\varphi}_\sigma(0+0), \qquad p \in (0, \infty),$$

where $\bar{\varphi}_{\gamma_\sigma}(v)$ *is the decreasing rearrangement of the function*

$$\varphi_\sigma(t) = \begin{cases} |\psi(t)|^p, & t \in A \setminus \gamma_\sigma, \\ 0, & t \in \gamma_\sigma, \end{cases}$$

and

$$D_\sigma(\psi U_I^p) = \inf_{\gamma_\sigma \in \Gamma_\sigma(A)} \sup_{x \in \psi U_p} \left\| x(t) - U_{\gamma_\sigma}(x; t) \right\|_{L_p(A,\, d\mu)} = \bar{\psi}(\sigma + 0),$$

where $\bar{\psi}(v)$ *is the decreasing rearrangement of the function* $|\psi(t)|$.

For the quantities

$$e_\sigma(\psi U_I^p) = \sup_{x \in \psi U_p} \inf_{\gamma_\sigma \in \Gamma_\sigma(A)} \left\| x(t) - U_{\gamma_\sigma}(x; t) \right\|_{L_p(A,\, d\mu)},$$

the following equality is true:

$$e_\sigma^p(\psi U_I^p) = \sup_{\sigma < q \le a} \frac{q - \sigma}{\displaystyle\int_0^q \frac{dt}{\bar{\psi}^p(t)}}. \qquad (105)$$

The least upper bound on the right-hand side of (105) is realized for a certain finite value $q = q^*$.

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