APPROXIMATION OF THE CLASSES $C_{\beta}^{\psi}H_\omega$
BY GENERALIZED ZYGUND SUMS

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We obtain asymptotic equalities for the least upper bounds of approximations by Zygmund sums in the uniform metric on the classes of continuous $2\pi$-periodic functions whose $(\psi, \beta)$-derivatives belong to the set $H_\omega$ in the case where the sequences $\psi$ that generate the classes tend to zero not faster than a power function.

Let $L$ be the space of $2\pi$-periodic functions $f(t)$ summable in $(0, 2\pi)$ with the norm
\[ \|f\|_L = \frac{\pi}{-\pi} \int |f(t)| dt , \]
let $M$ be the space of measurable, essentially bounded, $2\pi$-periodic functions $f(t)$ with the norm
\[ \|f\|_M = \text{ess sup}_t |f(t)| , \]
and let $C$ be the space of continuous $2\pi$-periodic functions $f(t)$ with the norm
\[ \|f\|_C = \max_t |f(t)| . \]

By $C_{\beta}^{\psi}$ we denote the classes of continuous $2\pi$-periodic functions introduced by Stepanets [1, 2] as follows: Let $f \in C$ and let
\[ S[f] = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \] (1)
be its Fourier series. If a sequence $\psi = \psi(k), \ k \in \mathbb{N},$ of real numbers and a number $\beta \in \mathbb{R}$ are such that the series
\[ \sum_{k=1}^{\infty} \frac{1}{\psi(k)} \left( a_k \cos \left( kx + \frac{\beta \pi}{2} \right) + b_k \sin \left( kx + \frac{\beta \pi}{2} \right) \right) \]

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is the Fourier series of a certain function \( \varphi \in L \), then \( \varphi(\cdot) \) is called the \((\psi, \beta)\)-derivative of the function \( f(\cdot) \) and is denoted by \( f^w_\beta(\cdot) \). In this case, we say that the function \( f(\cdot) \) belongs to the set \( C^w_\beta \). If \( f \in C^w_\beta \) and

\[
\| f^w_\beta \|_M \leq 1,
\]

then we say that \( f \in C^w_{\beta, \infty} \). If \( f \in C^w_\beta \) and \( f^w_\beta \in H_\omega \), where

\[
H_\omega = \{ \varphi \in C : |\varphi(t_1) - \varphi(t_2)| \leq \omega(|t_1 - t_2|) \quad \forall t_1, t_2 \in \mathbb{R} \}
\]

and \( \omega(t) \) is a fixed modulus of continuity, then we write \( f \in C^w_\beta H_\omega \).

For \( \psi(k) = k^{-r}, \ r > 0 \), the classes \( C^w_{\beta, \infty} \) and \( C^w_\beta H_\omega \) coincide with the known Weyl–Nagy classes \( W^r_\beta \) and \( W^r_\beta H_\omega \), respectively (see, e.g., [2, pp. 25–33]).

In what follows, we assume that the sequence \( \psi(k) \) that defines the classes \( C^w_\beta H_\omega \) is the restriction of a certain continuous function \( \psi(t) \) of a continuous argument \( t \) that belongs to the set \( \mathcal{M} \).

\[
\mathcal{M} = \left\{ \psi(t), t \geq 1 : \psi(t) > 0, \psi(t_1) - 2\psi\left(\frac{t_1 + t_2}{2}\right) + \psi(t_2) \geq 0 \ orall t_1, t_2 \in [1; \infty), \lim_{t \to \infty} \psi(t) = 0 \right\}
\]

to the set \( \mathbb{N} \). Following Stepanets (see, e.g., [3, p. 160]), we consider the following subsets \( \mathcal{M}_0, \mathcal{M}_C, \) and \( \mathcal{M}_\infty^+ \) of the set \( \mathcal{M} \):

\[
\mathcal{M}_0 = \{ \psi \in \mathcal{M} : 0 < \mu(\psi; t) \leq K < \infty \ \forall t \geq 1 \},
\]

\[
\mathcal{M}_C = \{ \psi \in \mathcal{M} : 0 < K_1 \leq \mu(\psi; t) \leq K_2 < \infty \ \forall t \geq 1 \},
\]

\[
\mathcal{M}_\infty^+ = \{ \psi \in \mathcal{M} : \mu(\psi; t) \uparrow \infty, t \to \infty \},
\]

where

\[
\mu(\psi; t) = \frac{t}{\eta(\psi; t) - t},
\]

\[
\eta(\psi; t) = \psi^{-1}\left(\frac{\psi(t)}{2}\right),
\]

\( \psi^{-1}(\cdot) \) is the function inverse to \( \psi(\cdot) \), and the constants \( K, K_1, \) and \( K_2 \) may, generally speaking, depend on the function \( \psi \). Natural representatives of the set \( \mathcal{M}_C \) are, e.g., the functions \( t^{-r}, \ r > 0 \), representatives of the set \( \mathcal{M}_0 \setminus \mathcal{M}_C \) are the functions \( \ln(t + e^{-at}), \ a > 0 \), and representatives of the set \( \mathcal{M}_\infty^+ \) are functions of the form \( e^{-at'}, \ a > 0, \ r > 0 \). Let \( \mathcal{M}' \) denote the subset of functions \( \psi(\cdot) \) from \( \mathcal{M} \) for which the following condition is satisfied:
We also set $\mathcal{M}_0 = \mathcal{M}_0 \cap \mathcal{M}'$.

Let $f(x)$ be a summable $2\pi$-periodic function and let series (1) be its Fourier series. Consider polynomials of the form

$$Z^\varphi_n(f; x) = \frac{a_0}{2} + \sum_{k=1}^{n-1} \left(1 - \frac{\varphi(k)}{\varphi(n)}\right)(a_k \cos kx + b_k \sin kx), \quad n \in \mathbb{N}, \quad (2)$$

where $\varphi(k)$ are the values of a certain function $\varphi \in F$ at integer points, and $F$ is the set of all continuous functions $\varphi(u)$ monotonically increasing to infinity on $[1, \infty)$. The polynomials $Z^\varphi_n(f; x)$ were introduced in [4, 5] and are called the generalized Zygmund sums. It is clear that if $\varphi(t) = t^s$, $s > 0$, then $Z^\varphi_n(f; x)$ coincide with the classical Zygmund sums $Z_n^s(f; x)$, i.e., with polynomials of the form

$$Z_n^s(f; x) = \frac{a_0}{2} + \sum_{k=1}^{n-1} \left(1 - \frac{k^s}{n^s}\right)(a_k \cos kx + b_k \sin kx), \quad n \in \mathbb{N}.$$ 

For $s = 1$, the Zygmund sums $Z_n^1(f; x)$ turn into the known Fejér sums $\sigma_n(f; x)$ of order $n - 1$ for the function $f(x)$.

Based on the known results of Nikol’skii [6, p. 261] (see also [3, pp. 18, 20]) concerning necessary and sufficient conditions for the regularity of linear summation methods for Fourier series, one can easily establish the following statement for the polynomials $Z^\varphi_n(f; x)$:

**Proposition 1.** Suppose that a function $\varphi(u) \geq 0$, $u \in [0, \infty)$, is such that $\varphi(0) = 0$, $\varphi \in F$, and, for any $n = 2, 3, \ldots$, $\varphi(u)$ is convex upward or downward for $u \in [0, n]$. Then the condition

$$\frac{1}{\varphi(n)} \sum_{k=1}^{n-1} \frac{\varphi(n) - \varphi(k)}{n-k} \leq K \quad (3)$$

is necessary and sufficient for the uniform convergence of the polynomials $Z^\varphi_n(f; x)$ to the function $f(x)$ in the entire space $C$.

By using Theorem 2.1 from [3, p. 92], which contains sufficient conditions and saturation orders for general linear summation methods for Fourier series, one can easily verify that the method $Z^\varphi_n$ generated by a positive function $\varphi$ is saturated in the space $C$ with saturation order $\frac{1}{\varphi(n)}$. This means that, for generalized Zygmund sums, the relation

$$\left\| f(\cdot) - Z^\varphi_n(f; \cdot) \right\|_C = \frac{\varphi(1)}{\varphi(n)}, \quad n \to \infty,$$
implies that \( f(x) \equiv \text{const} \) and there exists at least one nonconstant function \( f(x) \) for which

\[
\left\| f(\cdot) - Z_n^\varphi(f; \cdot) \right\|_C = O(1) \frac{\varphi(n)}{\varphi(n)}, \quad n \to \infty.
\]

The aim of the present work is to establish asymptotic equalities for the quantities

\[
\mathcal{E}\left( C^W_\beta H_\omega; Z_n^\varphi \right)_C = \sup_{f \in C^W_\beta H_\omega} \left\| f(\cdot) - Z_n^\varphi(f; \cdot) \right\|_C, \quad n \to \infty,
\]

under certain natural restrictions imposed on the functions \( \varphi(\cdot), \psi(\cdot), \omega(\cdot) \), and the parameter \( \beta \). If these equalities are obtained, then one says [3, 7] that the Kolmogorov–Nikol’skii problem is solved for the method \( Z_n^\varphi \) on the class \( C^W_\beta H_\omega \) in the metric of the space \( C \). For various linear summation methods for Fourier series on various functional classes, this problem was solved in numerous works (see, e.g., [3, 8–19]). For more details on the history of the problem, see the bibliography in [3, 7, 11, 12, 14].

The most complete results related to finding asymptotic equalities for the quantities

\[
\mathcal{E}\left( \mathcal{Y}; Z_n^\varphi \right)_C = \sup_{f \in \mathcal{Y}} \left\| f(\cdot) - Z_n^\varphi(f; \cdot) \right\|_C, \quad n \to \infty,
\]

were obtained by Telyakovskii [16] in the case where \( \mathcal{Y} = W^r_\beta, \quad r > 0, \beta \in \mathbb{R} \), and by Bushev [20] in the case where \( \mathcal{Y} = C^W_\beta \omega, \beta \in \mathbb{R}, \psi \in \mathcal{W}' \). In [4, 5, 21–24], the approximative properties of the generalized Zygmund sums \( Z_n^\varphi(f; x) \) were studied on the classes \( C^W_\beta \omega, \psi \) for different \( \psi(\cdot) \). In [5], it was shown, in particular, that if \( \psi \in \mathcal{W}_C \cup \mathcal{W}^+_\infty, \beta = 0, \) \( \sum\varphi(t)\psi(t) = 1, \ t \geq 1, \) then the following estimate holds for any \( f \in C^W_\beta \omega : \)

\[
\left\| f(\cdot) - Z_n^\varphi(f; \cdot) \right\|_C = O(1) \psi(n) \ln(1 + \min \{ \mu(\psi; n), n \}), \quad n > 1.
\]

In [23, 25], asymptotic equalities were obtained for

\[
\mathcal{E}\left( \mathcal{Y}; Z_n^\varphi \right)_C = \sup_{f \in \mathcal{Y}} \left\| f(\cdot) - Z_n^\varphi(f; \cdot) \right\|_C,
\]

in the case where \( \mathcal{Y} = C^W_\beta H_\omega, \psi \in \mathcal{W}^+_\infty, \sum\varphi(t)\psi(t) = 1, \ t \geq 1, \beta \in \mathbb{R}, \) under certain additional restrictions on the functions \( \omega(t) \) and \( \mu(\psi; t) \). In particular, it was shown in [23, p. 81] that if \( \beta = 2l, \ l \in \mathbb{Z}, \) and

\[
\omega(1/n) \ln(\min \{ \mu(\psi; n), n \}) = o(1), \quad n \to \infty,
\]

then the following asymptotic equality holds as \( n \to \infty : \)

\[
\mathcal{E}\left( C^W_\beta H_\omega; Z_n^\varphi \right)_C = \frac{2\psi(n)}{\pi} \int_0^{\pi/2} \omega(2t) dt + O(1) \psi(n) \omega(1/n) \ln(\min \{ \mu(\psi; n), n \}).
\]
It follows from the results of [24], in particular, that if \( \psi \in \mathcal{M}'_0 \), \( \beta \in \mathbb{R} \), and the function \( \varphi(u) \psi(u) \) is nondecreasing and convex upward for \( u \geq 1 \), then the following equality holds as \( n \to \infty \):

\[
\mathcal{E}\left( C^{\psi}_\beta; Z_n^\psi \right)_C = \frac{2}{\pi} \sin \frac{\beta \pi}{2} \left( \frac{1}{\varphi(n)} \int_1^n \varphi(u) \psi(u) \frac{du}{u} + \int_n^{\infty} \frac{\psi(u)}{u} du \right) + O(1) \psi(n).
\]

In the present work, we study the asymptotic behavior of \( \mathcal{E}\left( C^{\psi}_\beta; Z_n^\psi \right)_C \) for \( \psi \in \mathcal{M}'_0 \), \( \beta \in \mathbb{R} \), and \( \varphi(u) \psi(u) = 1 \) for all \( u \geq 1 \). Then the following relation holds as \( n \to \infty \):

\[
\mathcal{E}\left( C^{\psi}_\beta; Z_n^\psi \right)_C = \frac{\theta_\omega}{\pi} \sin \frac{\beta \pi}{2} \left( \varphi(n) \int_1^{1/n} \frac{\omega(t)}{t} dt + \int_1^{\omega(2t)} \frac{\psi(u)}{n \sin u} du \right) + O(1) \psi(n),
\]

where \( \theta_\omega \in [2/3, 1] \) and \( \theta_\omega = 1 \) if \( \omega(t) \) is a convex modulus of continuity.

If, in addition,

\[
\int_0^{1/2} \frac{\omega(t)}{t} dt \leq K,
\]

then the following estimate holds as \( n \to \infty \):

\[
\mathcal{E}\left( C^{\psi}_\beta; Z_n^\psi \right)_C = O(1) \psi(n).
\]

In relations (7) and (9), \( O(1) \) is a value uniformly bounded in \( n \) and \( \beta \).

It follows from [3, pp. 214, 216] that, in the case where \( \psi(t) = t^{-r}, \ r > 0 \), the second term on the right-hand side of (7) does not exceed the remainder in order. In this case, equality (7) was obtained in [15, p. 42]. In the case where \( \psi \in \mathcal{M}'_0 \), a statement analogous to Theorem 1 was proved in [25, p. 185].

Comparing equality (3.10) in [3, p. 216] with equality (10) in [26, p. 662], we obtain the asymptotic relation

\[
\int_0^{1/n} \omega(2t) \int_n^{\infty} \psi(u) \sin u du dt = \int_0^{1/n} \psi(1/t) \omega(t) dt + O(1) \psi(n) \omega(1/n) + O(1) n(\psi(n) - \psi(n + 1)) \omega(1/n), \quad \psi \in \mathcal{M}'_0.
\]
Taking into account that, for an arbitrary function \( \psi \in \mathcal{W}'_0 \), one has

\[
n(n(n - \psi(n + 1)) = O(1)n(n),
\]

\[
\int_0^{1/n} \omega(2t) \int \psi(u) \sin tu \; du \; dt = \int_0^{1/n} \psi(1/t) \omega(t) \; dt + O(1)n(n)\omega(1/n), \quad \psi \in \mathcal{W}'_0,
\]

which readily follows from relation (12.10) in [3, p. 161], we can rewrite equality (7) in the form

\[
\mathcal{E}(C^\psi_{\beta} H_\omega; Z^\psi_n) = \frac{\theta_\omega}{\pi} \int_0^{1/n} \int \psi(1/t) \omega(t) \; dt + O(1)n(n)\omega(1/n).
\]

Note that, e.g., for the function \( \psi(t) = \ln(t + 1)^{-\alpha} \), \( \alpha > 1 \), and for a majorant \( \omega(t) \) that coincides with the function \( \ln(1/t)^{-\gamma} \), \( 0 < \gamma < 1 \), in the interval \( (0, 1/e) \), the first and the second term on the right-hand side of (7') are the leading terms, and, therefore, in this case, Theorem 1 contains a solution of the Kolmogorov–Nikol’skii problem for the method \( Z^\psi_n \) on the classes \( C^\psi_{\beta} H_\omega \). At the same time, one can easily give an example of a majorant \( \omega(t) \) for which this theorem allows one to obtain only an equality exact in order for the quantity (5) in the case where \( \mathcal{W} = C^\psi_{\beta} H_\omega \) (by choosing, in particular, \( \omega(t) = t^\alpha \), \( 0 < \alpha \leq 1 \)). In the case where \( \beta \in \mathbb{Z} \), one can obtain sharper estimates for \( \mathcal{E}(C^\psi_{\beta} H_\omega; Z^\psi_n) \), which are presented in the following theorems:

**Theorem 2.** Let \( \psi \in \mathcal{W}'_0 \), \( \beta = 2l + 1 \), \( l \in \mathbb{Z} \), and \( \varphi(u)\psi(u) = 1 \) for all \( u \geq 1 \). If

\[
\int_0^{\delta} \frac{\omega(t)}{t} \; dt = O(1)\omega(\delta),
\]

then the following asymptotic equality holds as \( n \to \infty \):

\[
\mathcal{E}(C^\psi_{\beta} H_\omega; Z^\psi_n) = \frac{\theta_\omega}{\pi} \psi(n) \int_0^{\pi/2} \frac{\omega(2t)}{\sin t} \; dt + O(1)n(n)\omega(1/n),
\]

where \( \theta_\omega \in [2/3, 1] \), \( \theta_\omega = 1 \) if \( \omega(t) \) is a convex modulus of continuity, and \( O(1) \) is uniformly bounded in \( n \) and \( \beta \).

For \( \psi(k) = k^{-1} \), \( \beta = 1 \), and \( \omega(t) = t^\alpha \), \( 0 < \alpha \leq 1 \), equality (12) was proved by Nikol’skii [9, p. 26]; for an arbitrary convex modulus of continuity, it was proved by Stepanets (see Theorem 5 in [27]). For \( \psi(k) = k^{-r} \), \( r = 1, 3, \ldots \), \( \beta = r \), and \( \omega(t) = t^\alpha \), \( 0 < \alpha \leq 1 \), equality (12) was proved by Nagy [10, p. 48].

**Theorem 3.** Let \( \psi \in \mathcal{W}'_0 \), \( \beta = 2l \), \( l \in \mathbb{Z} \), and \( \varphi(u)\psi(u) = 1 \) for all \( u \geq 1 \). Then the following asymptotic equality holds as \( n \to \infty \):
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$$\mathcal{E}\left( C^y_{\beta} H_\omega ; Z_n^\varphi \right)_C = \frac{2\psi(n)}{\pi} \int_0^{\pi/2} \omega(2t) dt + O(1) \psi(n) \omega(1/n),$$

(12)

where $O(1)$ is uniformly bounded in $n$ and $\beta$.

For $\psi(k) = k^{-2}$ and $\beta = 2$, equality (13) was proved by Stepanets [28, p. 352].

Taking (6) and (13) into account, we conclude that if $\psi \in \mathcal{M}_0 \cup \mathcal{M}_\infty$, $\beta = 2l$, $l \in \mathbb{Z}$, and $\varphi(u) \psi(u) = 1$, $u \geq 1$, then

$$\mathcal{E}\left( C^y_{\beta} H_\omega ; Z_n^\varphi \right)_C = \frac{2\psi(n)}{\pi} \int_0^{\pi/2} \omega(2t) dt + O(1) \psi(n) \omega(1/n) \left( 1 + \ln^+ \left( \min \{ \mu(\psi; n), n \} \right) \right) \quad \text{as } n \to \infty,$$

where $\ln^+(t) = \ln(t)$ for $t > 1$ and $\ln^+(t) = 0$ for $t \leq 1$.

Prior to the proof of Theorems 1–3, note that, since the classes $C^y_{\beta} H_\omega$ are invariant under the shift of an argument (see, e.g., [1, pp. 121, 122]), the following equality is true:

$$\mathcal{E}\left( C^y_{\beta} H_\omega ; Z_n^\varphi \right)_C = \sup_{f \in C^y_{\beta} H_\omega} |\rho_n(f; 0)|,$$

(14)

where

$$\rho_n(f; 0) = \rho_n(f; 0 ; Z_n^\varphi) \overset{\text{df}}{=} f(0) - Z_n^\varphi(f; 0).$$

For the estimation of $\rho_n(f; 0)$, we need the following statement:

**Lemma 1.** Let $\psi \in \mathcal{M}_0^\prime$, $\beta \in \mathbb{R}$ or $\psi \in \mathcal{M}_0$, $\beta = 2l$, $l \in \mathbb{Z}$. If $\varphi(u) \psi(u) = 1$ for $u \geq 1$, then the following equality holds for any $f \in C^y_{\beta} H_\omega$ and $n \in \mathbb{N}$:

$$\rho_n(f; 0) = -\frac{1}{\pi} \sin \frac{\beta \pi}{2} \left[ \psi(n) n \int_{|r| \geq 1} \delta\left( \frac{r}{n} \right) \frac{\sin \frac{t}{r}}{t} dt + \delta\left( \frac{r}{n} \right) \int_{|r| \leq 1} \psi(nu) \sin \nu \frac{u}{r} du \right]$$

$$+ \frac{n}{\pi} \cos \frac{\beta \pi}{2} \psi(n) \int_{-\infty}^{\infty} \delta\left( \frac{r}{n} \right) \cos \frac{t}{r} - 1 dt + O(1) \psi(n) \omega(1/n),$$

(15)

where $\delta(\cdot) \overset{\text{df}}{=} f^y_{\beta}(\cdot) - f^y_{\beta}(0)$ and $O(1)$ is uniformly bounded in $n$ and $\beta$.

**Proof.** We set
\[ \tau_n(u) = \begin{cases} 
\psi(n)nu, & 0 \leq u \leq \frac{1}{n}, \\
\psi(n), & \frac{1}{n} \leq u \leq 1, \\
\psi(nu), & u \geq 1, 
\end{cases} \quad \nu_n(u) = \begin{cases} 
\psi(nu), & 0 \leq u \leq 1, \\
\psi(n), & u \geq 1, 
\end{cases} \]

and \( \mu_n(u) = \tau_n(u) - \nu_n(u) \) for \( u \geq 0 \). Assume that \( \psi \) satisfies the conditions of Lemma 1. Then, by virtue of Lemma 3.1 in [3, p. 186], the transformation

\[ \hat{\nu}_n(t) = \frac{1}{\pi} \int_0^\infty \nu_n(u) \cos \left( ut + \frac{\beta \pi}{2} \right) du \]

(understood as an improper integral) is a function summable on the entire axis, i.e.,

\[ \int_{-\infty}^{\infty} \left| \hat{\nu}_n(t) \right| dt < \infty. \tag{16} \]

Since the function \( \mu_n(u) \) is absolutely continuous in \([0, 1]\), \( \mu_n(1) = 0 \), and, as can easily be verified, the integrals

\[ \int_0^1 \frac{1}{u(1-u)} |d\mu_n'(u)|, \quad \int_0^1 \frac{\left| \mu_n(u) \right|}{u} du, \quad \int_0^1 \frac{\left| \mu_n(u) \right|}{1-u} du \]

are convergent, by virtue of the theorem in [16, p. 70]) we have

\[ \int_{-\infty}^{\infty} \left| \int_0^1 \mu_n(u) \cos \left( ut + \frac{\beta \pi}{2} \right) du \right| dt < \infty. \tag{17} \]

Taking (16) and (17) into account, we conclude that the function

\[ \hat{\tau}_n(t) = \frac{1}{\pi} \int_0^\infty \tau_n(u) \cos \left( ut + \frac{\beta \pi}{2} \right) du \]

is summable on the entire axis. Since

\[ \tau_n(u) = \begin{cases} 
(1 - \lambda_n(1/n)) \psi(1)nu, & 0 \leq u \leq \frac{1}{n}, \\
(1 - \lambda_n(u)) \psi(nu), & \frac{1}{n} \leq u \leq 1, \quad n \in \mathbb{N}, 
\end{cases} \]
where

\[ \lambda_n(u) = \begin{cases} 
1 - \frac{\varphi(1)}{\varphi(n)} nu, & 0 \leq u \leq \frac{1}{n}, \\
1 - \frac{\varphi(nu)}{\varphi(n)}, & \frac{1}{n} \leq u \leq 1, \; \varphi \in F,
\end{cases} \]

by virtue of Theorem 3.2 in [2, p. 56] the following equality holds for every function \( f \in C^y_\beta H_{\omega} \):

\[ \rho_n(f; 0) = \int_{-\infty}^{\infty} f^\psi_{\beta, \omega}(t) \hat{\tau}_n(t) dt. \] (18)

Taking into account that \( \tau_n(0) = 0 \) and using Lemma 3 from [16, p. 71], according to which

\[ \int_{-\infty}^{\infty} \hat{\tau}_n(t) dt = 0, \]

we rewrite equality (18) in the form

\[ \rho_n(f; 0) = \int_{-\infty}^{\infty} \left( f_{\beta, \omega}(t) - f_{\beta, \omega}(0) \right) \hat{\tau}_n(t) dt = \int_{-\infty}^{\infty} \delta\left( t \right) \hat{\tau}_n(t) dt \]

\[ = \frac{\cos \frac{\beta \pi}{2}}{\pi} \int_{-\infty}^{\infty} \delta\left( \frac{t}{n} \right) \tau_n(u) \cos u t du - \frac{\sin \frac{\beta \pi}{2}}{\pi} \int_{-\infty}^{\infty} \delta\left( \frac{t}{n} \right) \tau_n(u) \sin u t du. \] (19)

Integrating by parts and taking into account that \( \tau_n(\infty) = 0, \) we get

\[ \int_{0}^{\infty} \tau_n(u) \cos u t du = \psi(n) n \frac{\cos \frac{t}{n} - 1}{t^2} - \frac{n}{t} \int_{1}^{\infty} \psi'(nu) \sin u t du, \quad \psi'(t) \overset{df}{=} \psi(t + 0) \] (20)

and

\[ \int_{0}^{\infty} \tau_n(u) \sin u t du = \psi(n) \frac{n \sin \frac{t}{n}}{t^2} + \frac{n}{t} \int_{1}^{\infty} \psi'(nu) \cos u t du. \] (21)

Relations (19)–(21) yield
\[ \rho_n(f; 0) = \frac{n \cos \frac{\beta \pi}{2}}{\pi} \left( \psi(n) \int_{-\infty}^{\infty} \delta(t) \frac{\cos \frac{t}{n^2} - 1}{t^2} dt - \int_{-\infty}^{\infty} \delta(t) \frac{1}{t} \int_{-1}^{1} \psi'(nu) \sin nt \, du \, dt \right) \]

\[ - \frac{\sin \frac{\beta \pi}{2}}{\pi} \left( \psi(n) \int_{|t| \leq 1} \delta(t) \frac{\sin t}{t^2} dt + \int_{|t| \geq 1} \delta(t) \frac{1}{t} \int_{-1}^{1} \psi'(nu) \cos nt \, du \, dt \right) \]

\[ + \int_{|t| \leq 1} \delta \left( \frac{t}{n} \right) \int_{0}^{1} \tau_n(u) \sin nt \, du \, dt + \int_{|t| \leq 1} \delta \left( \frac{t}{n} \right) \int_{1}^{\infty} \psi(\nu u) \sin nt \, du \, dt \right). \] (22)

Since \( \tau_n(u) \leq \psi(n) \) for \( u \in [0, 1] \) and \( |\delta(t)| \leq \omega(|t|) \), we have

\[ \int_{|t| \leq 1} \delta \left( \frac{t}{n} \right) \int_{0}^{1} \tau_n(u) \sin nt \, du \, dt = O(1) \psi(n) \omega(1/n). \] (23)

It follows from [3, pp. 223, 226] (see also [29, p. 285]) that the following estimates are true:

\[ n \int_{-\infty}^{\infty} \delta \left( \frac{t}{n} \right) \int_{-1}^{1} \psi'(nu) \sin nt \, du \, dt = O(1) \psi(n) \omega(1/n), \quad \psi \in \mathcal{W}_0, \] (24)

and

\[ n \int_{|t| \geq 1} \delta \left( \frac{t}{n} \right) \int_{1}^{\infty} \psi'(nu) \cos nt \, du \, dt = O(1) \psi(n) \omega(1/n), \quad \psi \in \mathcal{W}'_0. \] (25)

Combining (22)–(25), we obtain equality (15).

The lemma is proved.

**Proof of Theorem 1.** We begin with equalities (14) and (15). Performing the change of variables in the first and the third integral on the right-hand side of (15) and using the relation (see, e.g., [30, p. 1084])

\[ \int_{-\infty}^{\infty} y(t) \frac{1 - \cos t}{t^2} \, dt = \frac{1}{2} \int_{-\pi}^{\pi} y(t) \, dt \quad \forall y \in L, \] (26)

after elementary transformations we obtain

\[ \rho_n(f; 0) = - \frac{\sin \frac{\beta \pi}{2}}{\pi} \left( \psi(n) \int_{|t| \geq 1/n} \delta(t) \frac{\sin t}{t^2} dt + \int_{|t| \leq 1} \delta \left( \frac{t}{n} \right) \int_{1}^{\infty} \psi(\nu u) \sin nt \, du \, dt \right) + O(1) \psi(n). \] (27)
Further, we simplify the right-hand side of (27) without losing its principal value. It is easy to see that

\[ \int_{|t| \geq 1/n} \delta(t) \frac{\sin t}{t} \, dt = \int_{1/n \leq |t| \leq 1} \delta(t) \frac{\sin t}{t^2} \, dt + O(1) \]

\[ = \int_{1/n \leq |t| \leq 1} \delta(t) \frac{t}{t} \, dt + \int_{1/n \leq |t| \leq 1} \delta(t) \frac{\sin t - t}{t^2} \, dt + O(1). \quad (28) \]

Since the function \( \frac{\sin t - t}{t^2} \) is bounded on the segment \([-1, 1]\), combining relations (27) and (28) we get

\[ \rho_n(f; 0) = -\frac{\sin \beta \pi}{\pi} \left( \psi(n) \int_{1/n \leq |t| \leq 1} \delta(t) \, dt + \int_{|t| \leq 1} \delta(t) \int_{1}^{\infty} \psi(nu) \sin u t \, du \, dt \right) + O(1) \psi(n). \quad (29) \]

By virtue of Lemma 3.1.6 in [7, p. 143], the function

\[ \int_{1}^{\infty} \psi(nu) \sin u t \, du \]

is positive for \( t \in (0, 1] \), and, since it is odd, we get

\[ \left| \int_{|t| \leq 1} \delta\left(\frac{t}{n}\right) \int_{1}^{\infty} \psi(nu) \sin u t \, du \right| = \left| \int_{0}^{1/n} \left( \delta\left(\frac{t}{n}\right) - \delta\left(-\frac{t}{n}\right) \right) \int_{1}^{\infty} \psi(nu) \sin u t \, du \right| \]

\[ \leq \int_{0}^{1/n} \omega(2t) \int_{1}^{\infty} \psi(u) \sin u t \, du. \quad (30) \]

Taking into account the estimate

\[ \left| \int_{1/n \leq |t| \leq 1} \delta(t) \frac{t}{t} \, dt \right| = \left| \int_{1/n}^{1} \delta(t) - \delta(-t) \frac{t}{t} \, dt \right| \leq \int_{1/n}^{1} \frac{\omega(2t)}{t} \, dt \]

and relations (14), (29), and (30), we obtain

\[ \mathcal{E}\left(C_{\beta}^{\psi} H_{n}; Z_{n}^{\psi}\right) \leq \left| \frac{\mathbb{E}}{2} \left( \psi(n) \frac{\omega(2t)}{t} \int_{1/n}^{1} \frac{\omega(2t)}{t} \, dt + \frac{1}{n} \int_{0}^{1/n} \omega(2t) \int_{n}^{\infty} \psi(u) \sin u t \, du \right) \right| + O(1) \psi(n). \quad (31) \]

Let
and let \( \omega(t) \) be a convex modulus of continuity. In this case (see, e.g., [15, pp. 28, 29]), the function \( \varphi^*(t) \) belongs to the class \( H_{\omega} \). It is clear that \( \varphi^* \in H_{\omega}^0 \), where \( H_{\omega}^0 = \{ \varphi: \varphi \in H_{\omega}, \varphi \perp \mathbf{1} \} \). Then, according to Sec. 7.2 of [2, pp. 109, 110], the class \( C_{\beta}^\psi H_{\omega} \), \( \psi \in M_0^\prime \), contains a function \( g^*(\cdot) \) whose \((\psi, \beta)\)-derivative \( g_{\beta}^\psi(t) \) satisfies the equation

\[
g_{\beta}^\psi(t) = \varphi^*(t). \tag{32}
\]

For the function \( g^*(t) \), according to (29), we have

\[
\left| \rho_n(g^*; 0) \right| = \left| \sin \frac{\beta \pi}{2} \left( \frac{\psi(n)}{\pi} \int_0^{1/n} \frac{\omega(2t)}{t} dt + \frac{1}{\pi} \int_0^{1/n} \omega(2t) \int_0^\infty \psi(u) \sin ut \, du \, dt \right) + O(1) \psi(n). \right.
\]

This implies that we can take the equality sign in (31). Thus, equality (7) is proved in the case where \( \omega(t) \) is a convex modulus of continuity.

If \( \omega(t) \) is an arbitrary modulus of continuity, then the function \( \varphi^*(t) \) need not belong to the class \( H_{\omega} \). However, as is shown in [15, p. 11], the function

\[
\varphi_*(t) = \frac{2\varphi^*(t)}{3}
\]

already belongs to the class \( H_{\omega} \). This means that the class \( C_{\beta}^\psi H_{\omega} \) contains a function \( g_*(t) \) whose \((\psi, \beta)\)-derivative \( g_{\beta}^\psi(t) \) satisfies the equality

\[
g_{\beta}^\psi(t) = \varphi_*(t). \tag{33}
\]

For the function \( g_*(t) \), according to formula (29), we have

\[
\left| \rho_n(g_*; 0) \right| = \left| \sin \frac{\beta \pi}{2} \left( \frac{\psi(n)}{\pi} \int_0^{1/n} \frac{\omega(2t)}{t} dt + \frac{1}{\pi} \int_0^{1/n} \omega(2t) \int_0^\infty \psi(u) \sin ut \, du \, dt \right) + O(1) \psi(n). \right.
\]

This implies that equality (7) is true in the case of an arbitrary modulus of continuity \( \omega(t) \).
Assume, in addition, that the majorant $\omega(t)$ satisfies condition (8). It was shown in [3, p. 191] that

$$\left| \int_{n}^{\infty} \psi(u) \sin ut \, du \right| < \int_{n}^{\infty} \psi(u) \, du \quad \forall \psi \in \mathcal{M}_0.$$ 

Hence,

$$\int_{0}^{1/n} \omega(2t) \int_{n}^{\infty} \psi(u) \sin ut \, du \, dt = O(1) \int_{0}^{1/n} \omega(2t) \int_{n}^{\infty} \psi(u) \, du \, dt = O(1) \psi(n) \int_{0}^{1/n} \omega(t) \frac{t}{t^2} \, dt. \quad (34)$$

Then, using relations (8), (31), and (34), we obtain (9).

Theorem 1 is proved.

**Proof of Theorem 2.** Without loss of generality, we can consider only the case $\beta = 1$. Performing the change of variables in the first integral on the right-hand side of (15) and using the equality (see, e.g., [30, p. 1084])

$$\int_{-\infty}^{\infty} y(t) \frac{t - \sin t}{t^2} \, dt = 0 \quad \forall y \in L,$$

we get

$$\rho_n(f; 0) = -\frac{\psi(n)}{\pi} \left( \int_{-\infty}^{\infty} \left( \int_{-1/n}^{1/n} \delta(t) \frac{\sin t}{t^2} \, dt \right) - \frac{1}{\pi} \int_{|t| \leq 1} \delta\left( \frac{t}{n} \right) \int_{1}^{\infty} \psi(nu) \sin ut \, du \, dt + O(1) \psi(n) \omega(1/n) \right)$$

$$= -\frac{\psi(n)}{\pi} \left( \int_{-\infty}^{\infty} \delta(t) \frac{\sin t}{t^2} \, dt + \int_{-\infty}^{\infty} \delta(t) \frac{\sin t - t}{t^2} \, dt + \int_{0}^{1/n} \delta(t) \frac{\sin t}{t^2} \, dt \right)$$

$$- \frac{1}{\pi} \int_{|t| \leq 1} \delta\left( \frac{t}{n} \right) \int_{1}^{\infty} \psi(nu) \sin ut \, du \, dt + O(1) \psi(n) \omega(1/n)$$

$$= -\frac{\psi(n)}{\pi} \left( \int_{-\infty}^{\infty} \delta(t) \frac{\sin t}{t^2} \, dt - \frac{1}{\pi} \int_{|t| \leq 1} \delta\left( \frac{t}{n} \right) \int_{1}^{\infty} \psi(nu) \sin ut \, du \, dt \right)$$

$$+ O(1) \psi(n) \left( \int_{0}^{1/n} \omega(2t) \frac{\sin t}{t^2} \, dt + O(1/n) \right)$$

$$= -\frac{\psi(n)}{\pi} \left( \int_{-\infty}^{\infty} \delta(t) \frac{\sin t}{t^2} \, dt - \frac{1}{\pi} \int_{|t| \leq 1} \delta\left( \frac{t}{n} \right) \int_{1}^{\infty} \psi(nu) \sin ut \, du \, dt + O(1) \psi(n) \left( \int_{0}^{1/n} \omega(t) \frac{t}{t^2} \, dt + O(1/n) \right) \right). \quad (35)$$
According to relation (1.33) in [2, p. 43], the following equality is true:

\[
\int_{-\infty}^{\infty} \frac{y(t)}{t} \, dt = \frac{1}{2} \int_{-\pi}^{\pi} y(t) \cot \frac{t}{2} \, dt \quad \forall y \in L.
\]

Using this equality and relations (11), (30), and (35), we obtain

\[
\rho_n(f; 0) = -\frac{\psi(n)}{2\pi} \int_{-\pi}^{\pi} \delta(t) \cot \frac{t}{2} \, dt + O(1) \psi(n) \omega(1/n).
\]  

(36)

Further, since

\[
\int_{-\pi}^{\pi} \delta(t) \cot \frac{t}{2} \, dt = \left( \int_{-\pi/2}^{\pi/2} + \int_{\pi/2}^{3\pi/2} \right) \delta(t) \cot \frac{t}{2} \, dt
\]

\[
= \int_{0}^{\pi/2} \left( \delta(t) - \delta(-t) \right) \cot \frac{t}{2} \, dt - \int_{0}^{\pi/2} \left( \delta(\pi + t) - \delta(\pi - t) \right) \tan \frac{t}{2} \, dt,
\]

we have

\[
\left| \int_{-\pi}^{\pi} \delta(t) \cot \frac{t}{2} \, dt \right| \leq \int_{0}^{\pi/2} \omega(2t) \cot \frac{t}{2} \, dt + \int_{0}^{\pi/2} \omega(2t) \tan \frac{t}{2} \, dt = 2 \int_{0}^{\pi/2} \frac{\omega(2t)}{\sin t} \, dt.
\]  

(37)

Combining relations (14), (36), and (37), we get

\[
E\left( C_p^w H_\omega; Z_n^w \right) \leq \frac{\psi(n)}{\pi} \int_{0}^{\pi/2} \frac{\omega(2t)}{\sin t} \, dt + O(1) \psi(n) \omega(1/n).
\]  

(38)

Using equality (36), one can easily verify that, for the function \( \tilde{g}(t) \) that coincides with the function \( g^*(t) \) considered above in the case where \( \omega(t) \) is a convex majorant and with the function \( g^+(t) \) otherwise, the following relation is true:

\[
|\rho_n(\tilde{g}; 0)| = \alpha(\omega) \frac{\psi(n)}{\pi} \int_{0}^{\pi/2} \frac{\omega(2t)}{\sin t} \, dt + O(1) \psi(n) \omega(1/n),
\]  

(39)

where \( \alpha(\omega) = 1 \) if \( \omega(t) \) is a convex modulus of continuity and \( \alpha(\omega) = 2/3 \) otherwise. Combining relations (38) and (39), we obtain (12).

Theorem 2 is proved.
**Proof of Theorem 3.** Without loss of generality, we can consider only the case $\beta = 0$. Performing the change of variables in the third integral on the right-hand side of (15) and taking equality (26) into account, we get

$$|ho_n(f; 0)| = \frac{1}{\pi} \psi(n) \left| \int_{-\pi}^{\pi} \delta(t) \frac{\cos t - 1}{t^2} dt \right| + O(1) \psi(n) \omega(1/n)$$

$$= \frac{\psi(n)}{2\pi} \left| \int_{-\pi}^{\pi} \left( f_\beta^\psi(t) - f_\beta^\psi(0) \right) dt \right| + O(1) \psi(n) \omega(1/n). \quad (40)$$

Relation (40) yields

$$|ho_n(f; 0)| = \frac{\psi(n)}{2\pi} \left| \int_{0}^{\pi/2} \left( f_\beta^\psi(t) - f_\beta^\psi(0) \right) dt + \int_{0}^{\pi} \left( f_\beta^\psi(-t) - f_\beta^\psi(0) \right) dt \right| + O(1) \psi(n) \omega(1/n)$$

$$\leq \frac{2}{\pi} \psi(n) \int_{0}^{\pi/2} \omega(2t) dt + O(1) \psi(n) \omega(1/n). \quad (41)$$

Using (14) and (41), we obtain

$$E \left( C_\beta^\psi H_\omega : Z_\psi^\beta \right)_C \leq \frac{2}{\pi} \psi(n) \int_{0}^{\pi/2} \omega(2t) dt + O(1) \psi(n) \omega(1/n). \quad (42)$$

We set

$$\varphi_0(t) = \begin{cases} \frac{2}{\pi} \int_{0}^{\pi/2} \omega(2\tau)d\tau - \omega(t), & 0 \leq t \leq \pi, \\ \varphi_0(-t), & -\pi \leq t \leq 0. \end{cases} \quad \varphi_0(t + 2\pi) = \varphi_0(t).$$

It is clear that $\varphi_0 \in H_\omega^0$, and, therefore (see Sec. 7.2 of [2, pp. 109, 110]), the class $C_\beta^\psi H_\omega$, $\psi \in W_1^0$, contains a function $g_0(\cdot)$ whose $(\psi, \beta)$-derivative coincides with the function $\varphi_0(t)$ on a period. For the function $g_0(t)$, relation (40) yields

$$|ho_n(g_0; 0)| = \frac{2}{\pi} \psi(n) \int_{0}^{\pi/2} \omega(2t) dt + O(1) \psi(n) \omega(1/n).$$

This implies that the we can take the equality sign in (42).

Theorem 3 is proved.
Setting $\psi(t) = t^{-r}$, $r > 0$, in Theorems 2 and 3 and taking into account that $C^W_\beta H_\omega = W^r_\beta H_\omega$ in this case, we obtain the following statement:

**Corollary 1.** Suppose that $r > 0$, $s = r$, $\beta \in \mathbb{Z}$, and condition (11) is satisfied for $\beta = 2l + 1$, $l \in \mathbb{Z}$. Then the following asymptotic equalities hold as $n \to \infty$:

$$
E\left( W^r_\beta H_\omega; Z^s_n \right)_C = \begin{cases} 
\frac{2}{\pi r} \int_0^{\pi/2} \omega(2t) \, dt + O(1) n^{-r} \omega(1/n), & \beta = 2l, \\
\frac{\Theta_\omega}{\pi n^r} \int_0^{\pi/2} \frac{\omega(2t)}{\sin t} \, dt + O(1) n^{-r} \omega(1/n), & \beta = 2l + 1,
\end{cases} \tag{43}
$$

where $\Theta_\omega$ and $O(1)$ have the same meaning as in Theorem 2.

Note that, for $\omega(t) = t$, one has $W^r_{r-1} H_\omega = W^r$, $r = 2, 3, \ldots$, where $W^r$ is the class of $2\pi$-periodic functions such that their $(r - 1)$th derivatives are absolutely continuous and $|f^{(r)}| \leq 1$ almost everywhere. Therefore, taking into account the relation

$$
\int_0^{\pi/2} \frac{t}{\sin t} \, dt = 2G,
$$

where $G$ is the Catalan constant (see, e.g., [31, p. 431]), and using Corollary 1, we arrive at the following statement:

**Corollary 2.** Let $s = r - 1$ and $r = 2, 3, \ldots$. Then the following asymptotic equalities hold as $n \to \infty$:

$$
E\left( W^r; Z^s_n \right)_C = \begin{cases} 
\frac{4G}{\pi n^s} + O(1) n^{-r}, & r = 2, 4, \ldots, \\
\frac{\pi}{2n^s} + O(1) n^{-r}, & r = 3, 5, \ldots,
\end{cases} \tag{44}
$$

where $G$ is the Catalan constant and $O(1)$ is uniformly bounded in $n$.

It is easy to see that, since (see, e.g., [31, p. 21])

$$
G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)^2},
$$

the constants in the leading terms of (44) coincide with the so-called Favard–Akhiezer–Krein constants $\tilde{K}_j$ and $K_j$ for $j = 1$ (see, e.g., [11, pp. 89, 329]):
The asymptotic equalities (44) were proved by Nagy [10, p. 47].

It is easy to see that, by virtue of (2), the condition \( \varphi(t) \psi(t) = 1, \quad t \geq 1, \) in Theorems (1)–(3) can be replaced by the condition \( \varphi(t) \psi(t) = \text{const}, \) \( t \geq 1. \)

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