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APPROXIMATION OF THE CLASSES $C^{\psi}_{\beta} H_{\omega}$ BY GENERALIZED ZYGMUND SUMS

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We obtain asymptotic equalities for the least upper bounds of approximations by Zygmund sums in the uniform metric on the classes of continuous 2π-periodic functions whose (ψ, β)-derivatives belong to the set H_{ω} in the case where the sequences ψ that generate the classes tend to zero not faster than a power function.

Let L be the space of 2π-periodic functions $f(t)$ summable in $(0, 2\pi)$ with the norm

$$\| f \|_L = \int_{-\pi}^{\pi} |f(t)|\, dt,$$

let M be the space of measurable, essentially bounded, 2π-periodic functions $f(t)$ with the norm

$$\| f \|_M = \operatorname*{ess\,sup}_t |f(t)|,$$

and let C be the space of continuous 2π-periodic functions $f(t)$ with the norm

$$\| f \|_C = \max_t |f(t)|.$$

By C^{ψ}_{β} we denote the classes of continuous 2π-periodic functions introduced by Stepanets [1, 2] as follows: Let $f \in C$ and let

$$S[f] = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \tag{1}$$

be its Fourier series. If a sequence $\psi = \psi(k)$, $k \in \mathbb{N}$, of real numbers and a number $\beta \in \mathbb{R}$ are such that the series

$$\sum_{k=1}^{\infty} \frac{1}{\psi(k)} \left(a_k \cos\left(kx + \frac{\beta\pi}{2}\right) + b_k \sin\left(kx + \frac{\beta\pi}{2}\right)\right)$$

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is the Fourier series of a certain function $\varphi \in L$, then $\varphi(\cdot)$ is called the (ψ, β)-derivative of the function $f(\cdot)$ and is denoted by $f_\beta^\psi(\cdot)$. In this case, we say that the function $f(\cdot)$ belongs to the set C_β^ψ. If $f \in C_\beta^\psi$ and

$$\left\| f_\beta^\psi \right\|_M \leq 1,$$

then we say that $f \in C_{\beta,\infty}^\psi$. If $f \in C_\beta^\psi$ and $f_\beta^\psi \in H_\omega$, where

$$H_\omega = \{\varphi \in C\colon |\varphi(t_1) - \varphi(t_2)| \leq \omega(|t_1 - t_2|) \quad \forall t_1, t_2 \in \mathbb{R}\}$$

and $\omega(t)$ is a fixed modulus of continuity, then we write $f \in C_\beta^\psi H_\omega$.

For $\psi(k) = k^{-r}$, $r > 0$, the classes $C_{\beta,\infty}^\psi$ and $C_\beta^\psi H_\omega$ coincide with the known Weyl–Nagy classes W_β^r and $W_\beta^r H_\omega$, respectively (see, e.g., [2, pp. 25–33]).

In what follows, we assume that the sequence $\psi(k)$ that defines the classes $C_\beta^\psi H_\omega$ is the restriction of a certain continuous function $\psi(t)$ of a continuous argument t that belongs to the set

$$\mathfrak{M} = \left\{\psi(t), t \geq 1\colon \psi(t) > 0, \psi(t_1) - 2\psi\left(\frac{t_1 + t_2}{2}\right) + \psi(t_2) \geq 0 \ \ \forall t_1, t_2 \in [1; \infty), \ \lim_{t \to \infty} \psi(t) = 0\right\}$$

to the set $\mathbb{N}$. Following Stepanets (see, e.g., [3, p. 160], we consider the following subsets $\mathfrak{M}_0$, $\mathfrak{M}_C$, and $\mathfrak{M}_\infty^+$ of the set $\mathfrak{M}$:

$$\mathfrak{M}_0 = \{\psi \in \mathfrak{M}\colon 0 < \mu(\psi; t) \leq K < \infty \quad \forall t \geq 1\},$$

$$\mathfrak{M}_C = \{\psi \in \mathfrak{M}\colon 0 < K_1 \leq \mu(\psi; t) \leq K_2 < \infty \quad \forall t \geq 1\},$$

$$\mathfrak{M}_\infty^+ = \left\{\psi \in \mathfrak{M}\colon \mu(\psi; t) \uparrow \infty, t \to \infty\right\},$$

where

$$\mu(\psi; t) = \frac{t}{\eta(\psi; t) - t},$$

$$\eta(\psi; t) = \psi^{-1}\left(\frac{\psi(t)}{2}\right),$$

$\psi^{-1}(\cdot)$ is the function inverse to $\psi(\cdot)$, and the constants K, K_1, and K_2 may, generally speaking, depend on the function ψ. Natural representatives of the set $\mathfrak{M}_C$ are, e.g., the functions t^{-r}, $r > 0$, representatives of the set $\mathfrak{M}_0 \setminus \mathfrak{M}_C$ are the functions $\ln(t + e)^{-\alpha}$, $\alpha > 0$, and representatives of the set $\mathfrak{M}_\infty^+$ are functions of the form $e^{-\alpha t^r}$, $\alpha > 0$, $r > 0$. Let $\mathfrak{M}'$ denote the subset of functions $\psi(\cdot)$ from $\mathfrak{M}$ for which the following condition is satisfied:

$$\int_1^\infty \frac{\psi(t)}{t}\,dt \;<\; \infty.$$

We also set $\mathfrak{M}'_0 = \mathfrak{M}_0 \cap \mathfrak{M}'$.

Let $f(x)$ be a summable 2π-periodic function and let series (1) be its Fourier series. Consider polynomials of the form

$$Z_n^\varphi(f;x) \;=\; \frac{a_0}{2} \;+\; \sum_{k=1}^{n-1}\left(1-\frac{\varphi(k)}{\varphi(n)}\right)(a_k\cos kx + b_k \sin kx), \qquad n\in\mathbb{N}, \tag{2}$$

where $\varphi(k)$ are the values of a certain function $\varphi \in F$ at integer points, and F is the set of all continuous functions $\varphi(u)$ monotonically increasing to infinity on $[1,\infty)$. The polynomials $Z_n^\varphi(f;x)$ were introduced in [4, 5] and are called the generalized Zygmund sums. It is clear that if $\varphi(t) = t^s$, $s>0$, then $Z_n^\varphi(f;x)$ coincide with the classical Zygmund sums $Z_n^s(f;x)$, i.e., with polynomials of the form

$$Z_n^s(f;x) \;=\; \frac{a_0}{2} \;+\; \sum_{k=1}^{n-1}\left(1-\frac{k^s}{n^s}\right)(a_k\cos kx + b_k \sin kx), \qquad n\in\mathbb{N}.$$

For $s = 1$, the Zygmund sums $Z_n^s(f;x)$ turn into the known Fejér sums $\sigma_n(f;x)$ of order $n-1$ for the function $f(x)$.

Based on the known results of Nikol'skii [6, p. 261] (see also [3, pp. 18, 20]) concerning necessary and sufficient conditions for the regularity of linear summation methods for Fourier series, one can easily establish the following statement for the polynomials $Z_n^\varphi(f;x)$:

Proposition 1. *Suppose that a function* $\varphi(u)\ge 0$, $u\in[0,\infty)$, *is such that* $\varphi(0) = 0$, $\varphi\in F$, *and, for any* $n = 2, 3, \ldots$, $\varphi(u)$ *is convex upward or downward for* $u\in[0,n]$. *Then the condition*

$$\frac{1}{\varphi(n)}\sum_{k=1}^{n-1}\frac{\varphi(n)-\varphi(k)}{n-k} \;\le\; K \tag{3}$$

is necessary and sufficient for the uniform convergence of the polynomials $Z_n^\varphi(f;x)$ *to the function* $f(x)$ *in the entire space* C.

By using Theorem 2.1 from [3, p. 92], which contains sufficient conditions and saturation orders for general linear summation methods for Fourier series, one can easily verify that the method Z_n^φ generated by a positive function φ is saturated in the space C with saturation order $\dfrac{1}{\varphi(n)}$. This means that, for generalized Zygmund sums, the relation

$$\left\| f(\cdot) - Z_n^\varphi(f;\cdot)\right\|_C \;=\; \frac{o(1)}{\varphi(n)}, \qquad n\to\infty,$$

implies that $f(x) \equiv \text{const}$ and there exists at least one nonconstant function $f(x)$ for which

$$\left\| f(\cdot) - Z_n^{\varphi}(f;\cdot) \right\|_C = \frac{O(1)}{\varphi(n)}, \qquad n \to \infty.$$

The aim of the present work is to establish asymptotic equalities for the quantities

$$\mathcal{E}\left(C_\beta^\psi H_\omega; Z_n^\varphi\right)_C = \sup_{f \in C_\beta^\psi H_\omega} \left\| f(\cdot) - Z_n^{\varphi}(f;\cdot) \right\|_C, \qquad n \to \infty, \tag{4}$$

under certain natural restrictions imposed on the functions $\varphi(\cdot)$, $\psi(\cdot)$, $\omega(\cdot)$, and the parameter β. If these equalities are obtained, then one says [3, 7] that the Kolmogorov–Nikol'skii problem is solved for the method Z_n^φ on the class $C_\beta^\psi H_\omega$ in the metric of the space C. For various linear summation methods for Fourier series on various functional classes, this problem was solved in numerous works (see, e.g., [3, 8 –19]). For more details on the history of the problem, see the bibliography in [3, 7, 11, 12, 14].

The most complete results related to finding asymptotic equalities for the quantities

$$\mathcal{E}\left(\mathfrak{N}; Z_n^s\right)_C = \sup_{f \in \mathfrak{N}} \left\| f(\cdot) - Z_n^{s}(f;\cdot) \right\|_C, \qquad n \to \infty,$$

were obtained by Telyakovskii [16] in the case where $\mathfrak{N} = W_\beta^r$, $r > 0$, $\beta \in \mathbb{R}$, and by Bushev [20] in the case where $\mathfrak{N} = C_{\beta,\infty}^\psi$, $\beta \in \mathbb{R}$, $\psi \in \mathfrak{M}'$. In [4, 5, 21–24], the approximative properties of the generalized Zygmund sums $Z_n^\varphi(f;x)$ were studied on the classes $C_{\beta,\infty}^\psi$ for different $\psi(\cdot)$. In [5], it was shown, in particular, that if $\psi \in \mathfrak{M}_C \cup \mathfrak{M}_\infty^+$, $\beta = 0$, and $\varphi(t)\psi(t) = 1$, $t \ge 1$, then the following estimate holds for any $f \in C_{\beta,\infty}^\psi$:

$$\left\| f(\cdot) - Z_n^{\varphi}(f;\cdot) \right\|_C = O(1)\,\psi(n)\ln\left(1 + \min\{\mu(\psi;n), n\}\right), \qquad n > 1.$$

In [23, 25], asymptotic equalities were obtained for

$$\mathcal{E}\left(\mathfrak{N}; Z_n^\varphi\right)_C = \sup_{f \in \mathfrak{N}} \left\| f(\cdot) - Z_n^{\varphi}(f;\cdot) \right\|_C, \tag{5}$$

in the case where $\mathfrak{N} = C_\beta^\psi H_\omega$, $\psi \in \mathfrak{M}_\infty^+$, $\varphi(t)\,\psi(t) = 1$, $t \ge 1$, $\beta \in \mathbb{R}$, under certain additional restrictions on the functions $\omega(t)$ and $\mu(\psi;t)$. In particular, it was shown in [23, p. 81] that if $\beta = 2l$, $l \in \mathbb{Z}$, and

$$\omega(1/n)\ln\left(\min\{\mu(\psi;n), n\}\right) = o(1), \qquad n \to \infty,$$

then the following asymptotic equality holds as $n \to \infty$:

$$\mathcal{E}\left(C_\beta^\psi H_\omega; Z_n^\varphi\right)_C = \frac{2\psi(n)}{\pi}\int_0^{\pi/2} \omega(2t)\,dt + O(1)\,\psi(n)\,\omega(1/n)\ln\left(\min\{\mu(\psi;n), n\}\right). \tag{6}$$

It follows from the results of [24], in particular, that if $\psi \in \mathfrak{M}'_0$, $\beta \in \mathbb{R}$, and the function $\varphi(u)\,\psi(u)$ is nondecreasing and convex upward for $u \geq 1$, then the following equality holds as $n \to \infty$:

$$\mathcal{E}\Big(C^{\psi}_{\beta,\infty};\, Z^{\varphi}_{n}\Big)_{C} \;=\; \frac{2}{\pi}\left|\sin\frac{\beta\pi}{2}\right|\left(\frac{1}{\varphi(n)}\int_{1}^{n}\frac{\varphi(u)\psi(u)}{u}\,du + \int_{n}^{\infty}\frac{\psi(u)}{u}\,du\right) \;+\; O(1)\,\psi(n).$$

In the present work, we study the asymptotic behavior of $\mathcal{E}\Big(C^{\psi}_{\beta} H_{\omega};\, Z^{\varphi}_{n}\Big)_{C}$ for $\varphi(t)\,\psi(t) = 1$, $t \geq 1$, in the case where either $\psi \in \mathfrak{M}_0$ and $\beta = 0$ or $\psi \in \mathfrak{M}'_0$ and $\beta \in \mathbb{R}$. The results obtained here complement the aforementioned investigations of [23, 25] on the classes $C^{\psi}_{\beta} H_{\omega}$, and, moreover, as corollaries, numerous new statements are established for the classical Zygmund sums $Z^{s}_{n}(f; x)$.

The following theorem is true:

Theorem 1. *Let* $\psi \in \mathfrak{M}'_0$, $\beta \in \mathbb{R}$, *and* $\varphi(u)\psi(u) = 1$ *for all* $u \geq 1$. *Then the following relation holds as* $n \to \infty$:

$$\mathcal{E}\Big(C^{\psi}_{\beta} H_{\omega};\, Z^{\varphi}_{n}\Big)_{C} \;=\; \frac{\theta_{\omega}}{\pi}\left|\sin\frac{\beta\pi}{2}\right|\left(\psi(n)\int_{1/n}^{1}\frac{\omega(2t)}{t}\,dt \;+\; \int_{0}^{1/n}\omega(2t)\int_{n}^{\infty}\psi(u)\sin ut\,du\,dt\right) \;+\; O(1)\,\psi(n), \tag{7}$$

where $\theta_{\omega} \in [2/3, 1]$ *and* $\theta_{\omega} = 1$ *if* $\omega(t)$ *is a convex modulus of continuity.*

If, in addition,

$$\int_{0}^{1}\frac{\omega(t)}{t}\,dt \;\leq\; K, \tag{8}$$

then the following estimate holds as $n \to \infty$:

$$\mathcal{E}\Big(C^{\psi}_{\beta} H_{\omega};\, Z^{\varphi}_{n}\Big)_{C} \;=\; O(1)\,\psi(n). \tag{9}$$

In relations (7) and (9), $O(1)$ *is a value uniformly bounded in* n *and* β.

It follows from [3, pp. 214, 216] that, in the case where $\psi(t) = t^{-r}$, $r > 0$, the second term on the right-hand side of (7) does not exceed the remainder in order. In this case, equality (7) was obtained in [15, p. 42]. In the case where $\psi \in \mathfrak{M}^{+}_{\infty}$, a statement analogous to Theorem 1 was proved in [25, p. 185].

Comparing equality (3.10) in [3, p. 216] with equality (10) in [26, p. 662], we obtain the asymptotic relation

$$\int_{0}^{1/n}\omega(2t)\int_{n}^{\infty}\psi(u)\sin ut\,du\,dt$$

$$= \int_{0}^{1/n}\psi\!\left(\frac{1}{t}\right)\frac{\omega(t)}{t}\,dt \;+\; O(1)\,\psi(n)\,\omega(1/n) \;+\; O(1)\,n\big(\psi(n) - \psi(n+1)\big)\,\omega(1/n), \qquad \psi \in \mathfrak{M}'_0.$$

Taking into account that, for an arbitrary function $\psi \in \mathfrak{M}'_0$, one has

$$n(\psi(n) - \psi(n+1)) = O(1)\psi(n),$$

$$\int_0^{1/n} \omega(2t) \int_n^{\infty} \psi(u) \sin ut\, du dt = \int_0^{1/n} \psi\left(\frac{1}{t}\right) \frac{\omega(t)}{t} dt + O(1)\psi(n)\omega(1/n), \qquad \psi \in \mathfrak{M}'_0, \tag{10}$$

which readily follows from relation (12.10) in [3, p. 161], we can rewrite equality (7) in the form

$$\mathcal{E}\left(C^{\psi}_{\beta} H_{\omega}; Z^{\varphi}_n\right)_C = \frac{\theta_{\omega}}{\pi}\left|\sin\frac{\beta\pi}{2}\right|\left(\psi(n)\int_{1/n}^{1} \frac{\omega(2t)}{t} dt + \int_0^{1/n} \psi\left(\frac{1}{t}\right)\frac{\omega(t)}{t} dt\right) + O(1)\psi(n). \tag{7'}$$

Note that, e.g., for the function $\psi(t) = \ln(t+1)^{-\alpha}$, $\alpha > 1$, and for a majorant $\omega(t)$ that coincides with the function $\ln(1/t)^{-\gamma}$, $0 < \gamma < 1$, in the interval $(0, 1/e]$, the first and the second term on the right-hand side of (7') are the leading terms, and, therefore, in this case, Theorem 1 contains a solution of the Kolmogorov–Nikol'skii problem for the method Z^{φ}_n on the classes $C^{\psi}_{\beta} H_{\omega}$. At the same time, one can easily give an example of a majorant $\omega(t)$ for which this theorem allows one to obtain only an equality exact in order for the quantity (5) in the case where $\mathfrak{N} = C^{\psi}_{\beta} H_{\omega}$ (by choosing, in particular, $\omega(t) = t^{\alpha}$, $0 < \alpha \le 1$). In the case where $\beta \in \mathbb{Z}$, one can obtain sharper estimates for $\mathcal{E}\left(C^{\psi}_{\beta} H_{\omega}; Z^{\varphi}_n\right)_C$, which are presented in the following theorems:

Theorem 2. *Let* $\psi \in \mathfrak{M}'_0$, $\beta = 2l + 1$, $l \in \mathbb{Z}$, *and* $\varphi(u)\psi(u) = 1$ *for all* $u \ge 1$. *If*

$$\int_0^{\delta} \frac{\omega(t)}{t} dt = O(1)\omega(\delta), \tag{11}$$

then the following asymptotic equality holds as $n \to \infty$:

$$\mathcal{E}\left(C^{\psi}_{\beta} H_{\omega}; Z^{\varphi}_n\right)_C = \frac{\theta_{\omega}\,\psi(n)}{\pi}\int_0^{\pi/2} \frac{\omega(2t)}{\sin t} dt + O(1)\psi(n)\omega(1/n), \tag{12}$$

where $\theta_{\omega} \in [2/3, 1]$, $\theta_{\omega} = 1$ *if* $\omega(t)$ *is a convex modulus of continuity, and* $O(1)$ *is uniformly bounded in* n *and* β.

For $\psi(k) = k^{-1}$, $\beta = 1$, and $\omega(t) = t^{\alpha}$, $0 < \alpha \le 1$, equality (12) was proved by Nikol'skii [9, p. 26]; for an arbitrary convex modulus of continuity, it was proved by Stepanets (see Theorem 5 in [27]). For $\psi(k) = k^{-r}$, $r = 1, 3, \ldots$, $\beta = r$, and $\omega(t) = t^{\alpha}$, $0 < \alpha \le 1$, equality (12) was proved by Nagy [10, p. 48].

Theorem 3. *Let* $\psi \in \mathfrak{M}_0$, $\beta = 2l$, $l \in \mathbb{Z}$, *and* $\varphi(u)\psi(u) = 1$ *for all* $u \ge 1$. *Then the following asymptotic equality holds as* $n \to \infty$:

$$\mathcal{E}\Big(C^{\psi}_{\beta}H_{\omega}; Z^{\varphi}_{n}\Big)_{C} = \frac{2\psi(n)}{\pi}\int\limits_{0}^{\pi/2}\omega(2t)\,dt + O(1)\psi(n)\omega(1/n), \tag{12}$$

where $O(1)$ *is uniformly bounded in* n *and* β.

For $\psi(k) = k^{-2}$ and $\beta = 2$, equality (13) was proved by Stepanets [28, p. 352].

Taking (6) and (13) into account, we conclude that if $\psi \in \mathfrak{M}_0 \cup \mathfrak{M}^{+}_{\infty}$, $\beta = 2l$, $l \in \mathbb{Z}$, and $\varphi(u)\psi(u) = 1$, $u \geq 1$, then

$$\mathcal{E}\Big(C^{\psi}_{\beta}H_{\omega}; Z^{\varphi}_{n}\Big)_{C} = \frac{2\psi(n)}{\pi}\int\limits_{0}^{\pi/2}\omega(2t)\,dt + O(1)\psi(n)\omega(1/n)\Big(1 + \ln^{+}(\min\{\mu(\psi; n), n\})\Big) \quad \text{as} \quad n \to \infty,$$

where $\ln^{+}(t) = \ln(t)$ for $t > 1$ and $\ln^{+}(t) = 0$ for $t \leq 1$.

Prior to the proof of Theorems 1–3, note that, since the classes $C^{\psi}_{\beta}H_{\omega}$ are invariant under the shift of an argument (see, e.g., [1, pp. 121, 122]), the following equality is true:

$$\mathcal{E}\Big(C^{\psi}_{\beta}H_{\omega}; Z^{\varphi}_{n}\Big)_{C} = \sup_{f \in C^{\psi}_{\beta}H_{\omega}} |\rho_n(f; 0)|, \tag{14}$$

where

$$\rho_n(f; 0) = \rho_n\big(f; 0; Z^{\varphi}_{n}\big) \overset{\text{df}}{=} f(0) - Z^{\varphi}_{n}(f; 0).$$

For the estimation of $\rho_n(f; 0)$, we need the following statement:

Lemma 1. *Let* $\psi \in \mathfrak{M}'_0$, $\beta \in \mathbb{R}$ *or* $\psi \in \mathfrak{M}_0$, $\beta = 2l$, $l \in \mathbb{Z}$. *If* $\varphi(u)\psi(u) = 1$ *for* $u \geq 1$, *then the following equality holds for any* $f \in C^{\psi}_{\beta}H_{\omega}$ *and* $n \in \mathbb{N}$:

$$\rho_n(f; 0) = -\frac{1}{\pi}\sin\frac{\beta\pi}{2}\left(\psi(n)n\int\limits_{|t|\geq 1}\delta\Big(\frac{t}{n}\Big)\frac{\sin\frac{t}{n}}{t^2}\,dt + \int\limits_{|t|\leq 1}\delta\Big(\frac{t}{n}\Big)\int\limits_{1}^{\infty}\psi(nu)\sin ut\,du\,dt\right)$$

$$+ \frac{n}{\pi}\cos\frac{\beta\pi}{2}\,\psi(n)\int\limits_{-\infty}^{\infty}\delta\Big(\frac{t}{n}\Big)\frac{\cos\frac{t}{n} - 1}{t^2}\,dt + O(1)\psi(n)\omega(1/n), \tag{15}$$

where $\delta(\cdot) \overset{\text{df}}{=} f^{\psi}_{\beta}(\cdot) - f^{\psi}_{\beta}(0)$ *and* $O(1)$ *is uniformly bounded in* n *and* β.

Proof. We set

$$\tau_n(u) = \begin{cases} \psi(n)nu, & 0 \le u \le \dfrac{1}{n}, \\ \psi(n), & \dfrac{1}{n} \le u \le 1, \\ \psi(nu), & u \ge 1, \end{cases} \qquad \nu_n(u) = \begin{cases} \psi(n)u, & 0 \le u \le 1, \\ \psi(nu), & u \ge 1, \end{cases}$$

and $\mu_n(u) = \tau_n(u) - \nu_n(u)$ for $u \ge 0$. Assume that ψ satisfies the conditions of Lemma 1. Then, by virtue of Lemma 3.1 in [3, p. 186], the transformation

$$\hat{\nu}_n(t) = \frac{1}{\pi}\int_0^\infty \nu_n(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du$$

(understood as an improper integral) is a function summable on the entire axis, i.e.,

$$\int_{-\infty}^{\infty} |\hat{\nu}_n(t)|\,dt < \infty. \tag{16}$$

Since the function $\mu_n(u)$ is absolutely continuous in $[0, 1]$, $\mu_n(1) = 0$, and, as can easily be verified, the integrals

$$\int_0^1 u(1-u)\,|d\mu_n'(u)|, \qquad \int_0^1 \frac{|\mu_n(u)|}{u}du, \qquad \int_0^1 \frac{|\mu_n(u)|}{1-u}du$$

are convergent, by virtue of the theorem in [16, p. 70]) we have

$$\int_{-\infty}^{\infty}\left|\int_0^1 \mu_n(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du\right|dt < \infty. \tag{17}$$

Taking (16) and (17) into account, we conclude that the function

$$\hat{\tau}_n(t) = \frac{1}{\pi}\int_0^\infty \tau_n(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du$$

is summable on the entire axis. Since

$$\tau_n(u) = \begin{cases} \big(1 - \lambda_n(1/n)\big)\,\psi(1)nu, & 0 \le u \le \dfrac{1}{n}, \\ \big(1 - \lambda_n(u)\big)\,\psi(nu), & \dfrac{1}{n} \le u \le 1, \quad n \in \mathbb{N}, \end{cases}$$

where

$$\lambda_n(u) = \begin{cases} 1 - \dfrac{\varphi(1)}{\varphi(n)} nu, & 0 \le u \le \dfrac{1}{n}, \\ 1 - \dfrac{\varphi(nu)}{\varphi(n)}, & \dfrac{1}{n} \le u \le 1, \quad \varphi \in F, \end{cases}$$

by virtue of Theorem 3.2 in [2, p. 56] the following equality holds for every function $f \in C^{\psi}_{\beta}H_{\omega}$:

$$\rho_n(f;0) = \int_{-\infty}^{\infty} f^{\psi}_{\beta}\left(\frac{t}{n}\right)\hat{\tau}_n(t)\,dt. \tag{18}$$

Taking into account that $\tau_n(0) = 0$ and using Lemma 3 from [16, p. 71], according to which

$$\int_{-\infty}^{\infty} \hat{\tau}_n(t)dt = 0,$$

we rewrite equality (18) in the form

$$\rho_n(f;0) = \int_{-\infty}^{\infty} \left(f^{\psi}_{\beta}\left(\frac{t}{n}\right) - f^{\psi}_{\beta}(0)\right)\hat{\tau}_n(t)\,dt = \int_{-\infty}^{\infty} \delta\left(\frac{t}{n}\right)\hat{\tau}_n(t)\,dt$$

$$= \frac{\cos\frac{\beta\pi}{2}}{\pi}\int_{-\infty}^{\infty}\delta\left(\frac{t}{n}\right)\int_{0}^{\infty}\tau_n(u)\cos ut\,du\,dt - \frac{\sin\frac{\beta\pi}{2}}{\pi}\int_{-\infty}^{\infty}\delta\left(\frac{t}{n}\right)\int_{0}^{\infty}\tau_n(u)\sin ut\,du\,dt. \tag{19}$$

Integrating by parts and taking into account that $\tau_n(\infty) = 0$, we get

$$\int_{0}^{\infty}\tau_n(u)\cos ut\,du = \psi(n)n\,\frac{\cos\frac{t}{n} - 1}{t^2} - \frac{n}{t}\int_{1}^{\infty}\psi'(nu)\sin ut\,du, \qquad \psi'(t) \stackrel{\mathrm{df}}{=} \psi'(t+0) \tag{20}$$

and

$$\int_{0}^{\infty}\tau_n(u)\sin ut\,du = \psi(n)\,\frac{n\sin\frac{t}{n}}{t^2} + \frac{n}{t}\int_{1}^{\infty}\psi'(nu)\cos ut\,du. \tag{21}$$

Relations (19)–(21) yield

$$\rho_n(f;0) = \frac{n\cos\frac{\beta\pi}{2}}{\pi}\left(\psi(n)\int\limits_{-\infty}^{\infty}\delta\left(\frac{t}{n}\right)\frac{\cos\frac{t}{n}-1}{t^2}dt - \int\limits_{-\infty}^{\infty}\delta\left(\frac{t}{n}\right)\frac{1}{t}\int\limits_{1}^{\infty}\psi'(nu)\sin ut\,du\,dt\right)$$

$$-\frac{\sin\frac{\beta\pi}{2}}{\pi}\left(\psi(n)n\int\limits_{|t|\ge 1}\delta\left(\frac{t}{n}\right)\frac{\sin\frac{t}{n}}{t^2}dt + n\int\limits_{|t|\ge 1}\delta\left(\frac{t}{n}\right)\frac{1}{t}\int\limits_{1}^{\infty}\psi'(nu)\cos ut\,du\,dt\right.$$

$$\left.+\int\limits_{|t|\le 1}\delta\left(\frac{t}{n}\right)\int\limits_{0}^{1}\tau_n(u)\sin ut\,du\,dt + \int\limits_{|t|\le 1}\delta\left(\frac{t}{n}\right)\int\limits_{1}^{\infty}\psi(nu)\sin ut\,du\,dt\right). \tag{22}$$

Since $\tau_n(u)\le\psi(n)$ for $u\in[0,1]$ and $|\delta(t)|\le\omega(|t|)$, we have

$$\int\limits_{|t|\le 1}\delta\left(\frac{t}{n}\right)\int\limits_{0}^{1}\tau_n(u)\sin ut\,du\,dt = O(1)\psi(n)\omega(1/n). \tag{23}$$

It follows from [3, pp. 223, 226] (see also [29, p. 285]) that the following estimates are true:

$$n\int\limits_{-\infty}^{\infty}\delta\left(\frac{t}{n}\right)\frac{1}{t}\int\limits_{1}^{\infty}\psi'(nu)\sin ut\,du\,dt = O(1)\psi(n)\omega(1/n), \qquad \psi\in\mathfrak{M}_0, \tag{24}$$

and

$$n\int\limits_{|t|\ge 1}\delta\left(\frac{t}{n}\right)\frac{1}{t}\int\limits_{1}^{\infty}\psi'(nu)\cos ut\,du\,dt = O(1)\psi(n)\omega(1/n), \qquad \psi\in\mathfrak{M}'_0. \tag{25}$$

Combining (22)–(25), we obtain equality (15).

The lemma is proved.

Proof of Theorem 1. We begin with equalities (14) and (15). Performing the change of variables in the first and the third integral on the right-hand side of (15) and using the relation (see, e.g., [30, p. 1084])

$$\int\limits_{-\infty}^{\infty}y(t)\frac{1-\cos t}{t^2}dt = \frac{1}{2}\int\limits_{-\pi}^{\pi}y(t)\,dt \qquad \forall\, y\in L, \tag{26}$$

after elementary transformations we obtain

$$\rho_n(f;0) = -\frac{\sin\frac{\beta\pi}{2}}{\pi}\left(\psi(n)\int\limits_{|t|\ge 1/n}\delta(t)\frac{\sin t}{t^2}dt + \int\limits_{|t|\le 1}\delta\left(\frac{t}{n}\right)\int\limits_{1}^{\infty}\psi(nu)\sin ut\,du\,dt\right) + O(1)\psi(n). \tag{27}$$

Further, we simplify the right-hand side of (27) without losing its principal value. It is easy to see that

$$\int\limits_{|t|\ge 1/n} \delta(t)\frac{\sin t}{t^2}dt = \int\limits_{1/n\le|t|\le 1} \delta(t)\frac{\sin t}{t^2}dt + O(1)$$

$$= \int\limits_{1/n\le|t|\le 1} \frac{\delta(t)}{t}dt + \int\limits_{1/n\le|t|\le 1} \delta(t)\frac{\sin t - t}{t^2}dt + O(1). \tag{28}$$

Since the function $\dfrac{\sin t - t}{t^2}$ is bounded on the segment $[-1, 1]$, combining relations (27) and (28) we get

$$\rho_n(f;0) = -\frac{\sin\frac{\beta\pi}{2}}{\pi}\left(\psi(n)\int\limits_{1/n\le|t|\le 1}\frac{\delta(t)}{t}dt + \int\limits_{|t|\le 1}\delta\left(\frac{t}{n}\right)\int\limits_1^\infty \psi(nu)\sin ut\,du\,dt\right) + O(1)\psi(n). \tag{29}$$

By virtue of Lemma 3.1.6 in [7, p. 143], the function

$$\int\limits_1^\infty \psi(nu)\sin ut\,du$$

is positive for $t\in(0,1]$, and, since it is odd, we get

$$\left|\int\limits_{|t|\le 1}\delta\left(\frac{t}{n}\right)\int\limits_1^\infty\psi(nu)\sin ut\,du\,dt\right| = \left|\int\limits_0^1\left(\delta\left(\frac{t}{n}\right)-\delta\left(-\frac{t}{n}\right)\right)\int\limits_1^\infty\psi(nu)\sin ut\,du\,dt\right|$$

$$\le \int\limits_0^{1/n}\omega(2t)\int\limits_n^\infty\psi(u)\sin ut\,du\,dt. \tag{30}$$

Taking into account the estimate

$$\left|\int\limits_{1/n\le|t|\le 1}\frac{\delta(t)}{t}dt\right| = \left|\int\limits_{1/n}^1\frac{\delta(t)-\delta(-t)}{t}dt\right| \le \int\limits_{1/n}^1\frac{\omega(2t)}{t}dt$$

and relations (14), (29), and (30), we obtain

$$\mathcal{E}\left(C_\beta^\psi H_\omega; Z_n^\varphi\right)_C \le \left|\sin\frac{\beta\pi}{2}\right|\left(\frac{\psi(n)}{\pi}\int\limits_{1/n}^1\frac{\omega(2t)}{t}dt + \frac{1}{\pi}\int\limits_0^{1/n}\omega(2t)\int\limits_n^\infty\psi(u)\sin ut\,du\,dt\right) + O(1)\psi(n). \tag{31}$$

Let

$$\varphi^*(t) = \begin{cases} \frac{1}{2}\,\omega(2t), & 0 \le t \le \frac{\pi}{2}, \\ \frac{1}{2}\,\omega(2\pi - 2t), & \frac{\pi}{2} \le t \le \pi, \\ -\varphi^*(-t), & -\pi \le t \le 0, \end{cases} \qquad \varphi^*(t + 2\pi) = \varphi^*(t)$$

and let $\omega(t)$ be a convex modulus of continuity. In this case (see, e.g., [15, pp. 28, 29]), the function $\varphi^*(t)$ belongs to the class H_ω. It is clear that $\varphi^* \in H_\omega^0$, where $H_\omega^0 = \{\varphi : \varphi \in H_\omega,\ \varphi \perp 1\}$. Then, according to Sec. 7.2 of [2, pp. 109, 110], the class $C_\beta^\psi H_\omega$, $\psi \in \mathfrak{M}'_0$, contains a function $g^*(\cdot)$ whose (ψ, β)-derivative $g^{*\psi}_{\beta}(t)$ satisfies the equation

$$g^{*\psi}_{\beta}(t) = \varphi^*(t). \tag{32}$$

For the function $g^*(t)$, according to (29), we have

$$\left|\rho_n(g^*; 0)\right| = \left|\sin\frac{\beta\pi}{2}\right|\left(\frac{\psi(n)}{\pi}\int\limits_{1/n}^{1}\frac{\omega(2t)}{t}\,dt + \frac{1}{\pi}\int\limits_{0}^{1/n}\omega(2t)\int\limits_{n}^{\infty}\psi(u)\sin ut\,du\,dt\right) + O(1)\psi(n).$$

This implies that we can take the equality sign in (31). Thus, equality (7) is proved in the case where $\omega(t)$ is a convex modulus of continuity.

If $\omega(t)$ is an arbitrary modulus of continuity, then the function $\varphi^*(t)$ need not belong to the class H_ω. However, as is shown in [15, p. 11], the function

$$\varphi_*(t) = \frac{2\varphi^*(t)}{3}$$

already belongs to the class H_ω. This means that the class $C_\beta^\psi H_\omega$ contains a function $g_*(t)$ whose (ψ, β)-derivative $g_{*\beta}^{\psi}(t)$ satisfies the equality

$$g_{*\beta}^{\psi}(t) = \varphi_*(t). \tag{33}$$

For the function $g_*(t)$, according to formula (29), we have

$$\left|\rho_n(g_*; 0)\right| = \frac{2}{3}\left|\sin\frac{\beta\pi}{2}\right|\left(\frac{\psi(n)}{\pi}\int\limits_{1/n}^{1}\frac{\omega(2t)}{t}\,dt + \frac{1}{\pi}\int\limits_{0}^{1/n}\omega(2t)\int\limits_{n}^{\infty}\psi(u)\sin ut\,du\,dt\right) + O(1)\psi(n).$$

This implies that equality (7) is true in the case of an arbitrary modulus of continuity $\omega(t)$.

Assume, in addition, that the majorant $\omega(t)$ satisfies condition (8). It was shown in [3, p. 191] that

$$\left|\int_n^{\infty}\psi(u)\sin ut\,du\right| < \int_n^{n+2\pi/t}\psi(u)\,du \quad \forall\,\psi\in\mathfrak{M}'_0.$$

Hence,

$$\int_0^{1/n}\omega(2t)\int_n^{\infty}\psi(u)\sin ut\,du\,dt = O(1)\int_0^{1/n}\omega(2t)\int_n^{n+2\pi/t}\psi(u)\,du\,dt = O(1)\,\psi(n)\int_0^{1/n}\frac{\omega(t)}{t}\,dt. \tag{34}$$

Then, using relations (8), (31), and (34), we obtain (9).

Theorem 1 is proved.

Proof of Theorem 2. Without loss of generality, we can consider only the case $\beta = 1$. Performing the change of variables in the first integral on the right-hand side of (15) and using the equality (see, e.g., [30, p. 1084])

$$\int_{-\infty}^{\infty} y(t)\,\frac{t-\sin t}{t^2}\,dt = 0 \quad \forall\,y\in L,$$

we get

$$\begin{aligned}
\rho_n(f;0) &= -\frac{\psi(n)}{\pi}\left(\int_{-\infty}^{\infty} - \int_{-1/n}^{1/n}\right)\delta(t)\,\frac{\sin t}{t^2}\,dt - \frac{1}{\pi}\int_{|t|\le 1}\delta\!\left(\frac{t}{n}\right)\int_1^{\infty}\psi(nu)\sin ut\,du\,dt + O(1)\,\psi(n)\,\omega(1/n)\\
&= -\frac{\psi(n)}{\pi}\left(\int_{-\infty}^{\infty}\frac{\delta(t)}{t}\,dt + \int_{-\infty}^{\infty}\delta(t)\,\frac{\sin t - t}{t^2}\,dt - \int_0^{1/n}\big(\delta(t)-\delta(-t)\big)\frac{\sin t}{t^2}\,dt\right)\\
&\quad - \frac{1}{\pi}\int_{|t|\le 1}\delta\!\left(\frac{t}{n}\right)\int_1^{\infty}\psi(nu)\sin ut\,du\,dt + O(1)\,\psi(n)\,\omega(1/n)\\
&= -\frac{\psi(n)}{\pi}\int_{-\infty}^{\infty}\frac{\delta(t)}{t}\,dt - \frac{1}{\pi}\int_{|t|\le 1}\delta\!\left(\frac{t}{n}\right)\int_1^{\infty}\psi(nu)\sin ut\,du\,dt\\
&\quad + O(1)\,\psi(n)\left(\int_0^{1/n}\omega(2t)\,\frac{\sin t}{t^2}\,dt + \omega(1/n)\right)\\
&= -\frac{\psi(n)}{\pi}\int_{-\infty}^{\infty}\frac{\delta(t)}{t}\,dt - \frac{1}{\pi}\int_{|t|\le 1}\delta\!\left(\frac{t}{n}\right)\int_1^{\infty}\psi(nu)\sin ut\,du\,dt + O(1)\,\psi(n)\left(\int_0^{1/n}\frac{\omega(t)}{t}\,dt + \omega(1/n)\right).
\end{aligned} \tag{35}$$

According to relation (1.33) in [2, p. 43], the following equality is true:

$$\int_{-\infty}^{\infty} \frac{y(t)}{t}\,dt = \frac{1}{2}\int_{-\pi}^{\pi} y(t)\cot\frac{t}{2}\,dt \qquad \forall\, y \in L.$$

Using this equality and relations (11), (30), and (35), we obtain

$$\rho_n(f;0) = -\frac{\psi(n)}{2\pi}\int_{-\pi}^{\pi}\delta(t)\cot\frac{t}{2}\,dt + O(1)\,\psi(n)\,\omega(1/n). \tag{36}$$

Further, since

$$\int_{-\pi}^{\pi}\delta(t)\cot\frac{t}{2}\,dt = \left(\int_{-\pi/2}^{\pi/2} + \int_{\pi/2}^{3\pi/2}\right)\delta(t)\cot\frac{t}{2}\,dt$$

$$= \int_{0}^{\pi/2}\big(\delta(t) - \delta(-t)\big)\cot\frac{t}{2}\,dt - \int_{0}^{\pi/2}\big(\delta(\pi+t) - \delta(\pi-t)\big)\tan\frac{t}{2}\,dt,$$

we have

$$\left|\int_{-\pi}^{\pi}\delta(t)\cot\frac{t}{2}\,dt\right| \le \int_{0}^{\pi/2}\omega(2t)\cot\frac{t}{2}\,dt + \int_{0}^{\pi/2}\omega(2t)\tan\frac{t}{2}\,dt = 2\int_{0}^{\pi/2}\frac{\omega(2t)}{\sin t}\,dt. \tag{37}$$

Combining relations (14), (36), and (37), we get

$$\mathcal{E}\Big(C^{\psi}_{\beta}H_{\omega};Z^{\varphi}_{n}\Big)_{C} \le \frac{\psi(n)}{\pi}\int_{0}^{\pi/2}\frac{\omega(2t)}{\sin t}\,dt + O(1)\,\psi(n)\,\omega(1/n). \tag{38}$$

Using equality (36), one can easily verify that, for the function $\tilde{g}(t)$ that coincides with the function $g^{*}(t)$ considered above in the case where $\omega(t)$ is a convex majorant and with the function $g_{*}(t)$ otherwise, the following relation is true:

$$|\rho_n(\tilde{g};0)| = \alpha(\omega)\,\frac{\psi(n)}{\pi}\int_{0}^{\pi/2}\frac{\omega(2t)}{\sin t}\,dt + O(1)\,\psi(n)\,\omega(1/n), \tag{39}$$

where $\alpha(\omega) = 1$ if $\omega(t)$ is a convex modulus of continuity and $\alpha(\omega) = 2/3$ otherwise. Combining relations (38) and (39), we obtain (12).

Theorem 2 is proved.

Proof of Theorem 3. Without loss of generality, we can consider only the case $\beta = 0$. Performing the change of variables in the third integral on the right-hand side of (15) and taking equality (26) into account, we get

$$|\rho_n(f;0)| = \frac{1}{\pi}\psi(n)\left|\int_{-\infty}^{\infty}\delta(t)\frac{\cos t - 1}{t^2}dt\right| + O(1)\psi(n)\omega(1/n)$$

$$= \frac{\psi(n)}{2\pi}\left|\int_{-\pi}^{\pi}\left(f_\beta^\psi(t) - f_\beta^\psi(0)\right)dt\right| + O(1)\psi(n)\omega(1/n). \tag{40}$$

Relation (40) yields

$$|\rho_n(f;0)| = \frac{\psi(n)}{2\pi}\left|\int_0^{\pi}\left(f_\beta^\psi(t) - f_\beta^\psi(0)\right)dt + \int_0^{\pi}\left(f_\beta^\psi(-t) - f_\beta^\psi(0)\right)dt\right| + O(1)\psi(n)\omega(1/n)$$

$$\le \frac{2}{\pi}\psi(n)\int_0^{\pi/2}\omega(2t)\,dt + O(1)\psi(n)\omega(1/n). \tag{41}$$

Using (14) and (41), we obtain

$$\mathcal{E}\left(C_\beta^\psi H_\omega; Z_n^\varphi\right)_C \le \frac{2}{\pi}\psi(n)\int_0^{\pi/2}\omega(2t)\,dt + O(1)\psi(n)\omega(1/n). \tag{42}$$

We set

$$\varphi_0(t) = \begin{cases} \dfrac{2}{\pi}\displaystyle\int_0^{\pi/2}\omega(2\tau)d\tau - \omega(t), & 0 \le t \le \pi, \\ \varphi_0(-t), & -\pi \le t \le 0, \end{cases} \qquad \varphi_0(t + 2\pi) = \varphi_0(t).$$

It is clear that $\varphi_0 \in H_\omega^0$, and, therefore (see Sec. 7.2 of [2, pp. 109, 110]), the class $C_\beta^\psi H_\omega$, $\psi \in \mathfrak{M}_0$, contains a function $g_0(\cdot)$ whose (ψ, β)-derivative coincides with the function $\varphi_0(t)$ on a period. For the function $g_0(t)$, relation (40) yields

$$|\rho_n(g_0;0)| = \frac{2}{\pi}\psi(n)\int_0^{\pi/2}\omega(2t)\,dt + O(1)\psi(n)\omega(1/n).$$

This implies that the we can take the equality sign in (42).

Theorem 3 is proved.

Setting $\psi(t) = t^{-r}$, $r > 0$, in Theorems 2 and 3 and taking into account that $C^{\psi}_{\beta} H_{\omega} = W^{r}_{\beta} H_{\omega}$ in this case, we obtain the following statement:

Corollary 1. *Suppose that* $r > 0$, $s = r$, $\beta \in \mathbb{Z}$, *and condition (11) is satisfied for* $\beta = 2l + 1$, $l \in \mathbb{Z}$. *Then the following asymptotic equalities hold as* $n \to \infty$:

$$\mathcal{E}\left(W^{r}_{\beta} H_{\omega}; Z^{s}_{n}\right)_{C} = \begin{cases} \dfrac{2}{\pi n^{r}} \displaystyle\int_{0}^{\pi/2} \omega(2t)\, dt + O(1)\, n^{-r}\, \omega(1/n), & \beta = 2l, \\ \dfrac{\theta_{\omega}}{\pi n^{r}} \displaystyle\int_{0}^{\pi/2} \dfrac{\omega(2t)}{\sin t}\, dt + O(1)\, n^{-r}\, \omega(1/n), & \beta = 2l + 1, \end{cases} \qquad l \in \mathbb{Z}, \tag{43}$$

where θ_{ω} *and* $O(1)$ *have the same meaning as in Theorem 2.*

Note that, for $\omega(t) = t$, one has $W^{r-1}_{r-1} H_{\omega} = W^{r}$, $r = 2, 3, \ldots$, where W^{r} is the class of 2π-periodic functions such that their $(r-1)$th derivatives are absolutely continuous and $\left|f^{(r)}\right| \le 1$ almost everywhere. Therefore, taking into account the relation

$$\int_{0}^{\pi/2} \frac{t}{\sin t}\, dt = 2G,$$

where G is the Catalan constant (see, e.g., [31, p. 431]), and using Corollary 1, we arrive at the following statement:

Corollary 2. *Let* $s = r - 1$ *and* $r = 2, 3, \ldots$. *Then the following asymptotic equalities hold as* $n \to \infty$:

$$\mathcal{E}\left(W^{r}; Z^{s}_{n}\right)_{C} = \begin{cases} \dfrac{4G}{\pi n^{s}} + O(1) n^{-r}, & r = 2, 4, \ldots, \\ \dfrac{\pi}{2n^{s}} + O(1) n^{-r}, & r = 3, 5, \ldots, \end{cases} \tag{44}$$

where G *is the Catalan constant and* $O(1)$ *is uniformly bounded in* n.

It is easy to see that, since (see, e.g., [31, p. 21])

$$G = \sum_{k=0}^{\infty} \frac{(-1)^{k}}{(2k+1)^{2}},$$

the constants in the leading terms of (44) coincide with the so-called Favard–Akhiezer– Krein constants $\tilde{K}_{j}$ and K_{j} for $j = 1$ (see, e.g., [11, pp. 89, 329]):

$$K_j = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k(j+1)}}{(2k+1)^{j+1}}, \qquad \tilde{K}_j = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{kj}}{(2k+1)^{j+1}}, \qquad j = 0, 1, 2, \ldots .$$

The asymptotic equalities (44) were proved by Nagy [10, p. 47].

It is easy to see that, by virtue of (2), the condition $\varphi(t)\psi(t) = 1$, $t \geq 1$, in Theorems (1)–(3) can be replaced by the condition $\varphi(t)\psi(t) = \text{const}$, $t \geq 1$.

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