On the Best Approximation of Certain Classes of Periodic Functions by Trigonometric Polynomials

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Abstract. We obtain the estimates for the best approximation in the uniform metric of the classes of 2π-periodic functions whose \((\psi, \beta)\)-derivatives have a given majorant \(\omega\) of the modulus of continuity. It is shown that the estimates obtained here are asymptotically exact under some natural conditions on the parameters \(\psi\), \(\omega\) and \(\beta\) defining the classes.

Key Words and Phrases: Best approximation, Modulus of continuity, Asymptotic formula, \((\psi, \beta)\)-derivative, Convolution

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1. Introduction

Let \(L\) be the space of 2π-periodic functions summable over the period with the norm \(\|f\|_1 = \int_{-\pi}^{\pi} |f(t)| \, dt\) and let \(C\) be the space of 2π-periodic continuous functions \(f\) with the norm \(\|f\|_C = \max_t |f(t)|\). Suppose \(f \in L\) and

\[
S[f] = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)
\]

is its Fourier series. Suppose also that \(\psi(k)\) is an arbitrary numerical sequence and \(\beta\) is a fixed real number \((\beta \in \mathbb{R})\). If the series

\[
\sum_{k=1}^{\infty} \frac{1}{\psi(k)} \left( a_k \cos \left( kx + \frac{\beta \pi}{2} \right) + b_k \sin \left( kx + \frac{\beta \pi}{2} \right) \right)
\]

is the Fourier series of a certain function \(\varphi \in L\), then \(\varphi\) is called (see, e.g., [10, 11]) the \((\psi, \beta)\)-derivative of the function \(f\) and is denoted by \(f_\psi^\beta\). The set of continuous functions \(f(x)\) having \((\psi, \beta)\)-derivatives such that \(f_\psi^\beta \in H_\omega\), where

\[
H_\omega = \{ \varphi \in C : |\varphi(t') - \varphi(t'')| \leq \omega(|t' - t''|) \quad \forall t', t'' \in \mathbb{R} \},
\]

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and $\omega(t)$ is a fixed modulus of continuity is usually denoted by $C^\psi_\beta H_\omega$.

For $\psi(k) = k^{-r}$, $r > 0$, the classes $C^\psi_\beta H_\omega$ become the well-know Weyl-Nagy classes $W^\mu_\beta H_\omega$ which, in turn, for $\beta = r$ coincide with the Weyl classes $W^r_\beta H_\omega$ (see, e.g., [11, Chap. 3, Sec. 4, 6]). For natural numbers $\omega$ and $\beta$ the classes $h^{\omega, \beta}$ and $H^{\omega, \beta}$ coincide with the Weyl classes $H^{\omega, \beta}$, which, in turn, for $\beta > 0$ are the functions $\ln t^{\beta}$, where $\beta$ is a fixed modulus of continuity is usually denoted by $h^{\omega, \beta}$.

Let $M$ be the set of all continuous functions $\psi(t)$ convex downwards for $t > 0$ and satisfying the condition $\lim_{t \to \infty} \psi(t) = 0$.

If $\psi \in M'$, where

$$M' := M'(\beta) = \{\psi : \psi \in M \text{ when } \sin \frac{\beta\pi}{2} = 0 \text{ or } \psi \in M \text{ and } \int_1^\infty \frac{\psi(t)}{t} dt < \infty \text{ when } \sin \frac{\beta\pi}{2} \neq 0\},$$

then the classes $C^\psi_\beta H_\omega$ coincide with the classes of functions $f(x)$, which are representable by the convolutions

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi} \int_{-\pi}^{\pi} \psi(x+t) \Psi_\beta(t) dt, \quad x \in \mathbb{R},$$

(see, e.g., [10, p. 31]), where $H^0_\omega = \{\varphi \in H_\omega : \int_{-\pi}^{\pi} \varphi(t) dt = 0\}$, and $\Psi_\beta(t)$ is a summable function, whose Fourier series have the form $\sum_{k=1}^{\infty} \psi(k) \cos(kt + \beta\pi/2)$.

The set $M$ is very inhomogeneous in the rate of convergence of functions $\psi(t)$ to zero as $t \to \infty$. This is why it was suggested in [10, pp. 115, 116] (see also [13, Subsec. 1.3]) to select subsets $M_0$ and $M_C$ from $M$ as follows:

$$M_0 = \{\psi \in M : 0 < \mu(t) \leq K < \infty, \quad \forall t \geq 1\},$$

$$M_C = \{\psi \in M : 0 < K_1 \leq \mu(t) \leq K_2 < \infty, \quad \forall t \geq 1\},$$

where $\mu(t) = \mu(\psi; t) = \frac{t}{\psi(t)}$, $\eta(t) = \eta(\psi; t) = \psi^{-1}(\psi(t)/2)$, $\psi^{-1}(\cdot)$ is the inverse function of $\psi(\cdot)$, and $K_1, K_2$ are positive constants (possibly dependent on $\psi(\cdot)$). The function $\mu(\psi; t)$ is called the modulus of half-decay of the function $\psi(t)$. It is obvious that $M_C \subset M_0$. Typical representatives of the set $M_C$ are the functions $t^{-r}$, $r > 0$, representatives of the set $M_0 \setminus M_C$ are the functions $\ln^{-\alpha}(t+1)$, $\alpha > 0$. Let $M'_0 = M' \cap M_0$. Natural representatives of the set $M'_0$ are the functions $\ln^{-\alpha}(t+1)$, $\alpha > 1$. It is easy to see that if $\beta = 2l$, $l \in \mathbb{Z}$, the set $M'_0$ coincide with $M_0$. Moreover, since for all $\psi \in M_C$

$$\int_0^\infty \frac{\psi(t)}{t} dt \leq K \psi(n), \quad n \in \mathbb{N},$$

(see [11, p. 204]) then $M_C \subset M'_0$. Throughout the paper we denote the positive constants that may be different in different relations by $K, K_i, i = 1, 2$. 
Let us denote the best approximation of the classes $C^\psi H_\omega$ by trigonometric polynomials $t_{n-1}(\cdot)$ of order not more than $n - 1$ by $E_n(C^\psi H_\omega)$, that is

$$E_n(C^\psi H_\omega) = \sup_{f \in C^\psi H_\omega} \inf_{t_{n-1}} \|f(\cdot) - t_{n-1}(\cdot)\|_C.$$  

(4)

As is shown in [10, p. 330] if $\omega(t)$ is an arbitrary modulus of continuity and $\psi \in \mathcal{M}$, $\beta \in \mathbb{R}$ or $\psi \in \mathcal{M}'_0$, $\beta = 0$, then the following estimates hold for the quantity $E_n(C^\psi H_\omega)$:

$$K_1 \psi(n)\omega(1/n) \leq E_n(C^\psi H_\omega) \leq K_2 \psi(n)\omega(1/n).$$  

(5)

When $\psi(k) = k^{-r}$, $r > 0$, $\beta \in \mathbb{R}$, the orders of decrease of quantity (4) have been known earlier [3] (see also [15, p. 508]). It should be noted that unlike order estimates, exact values for the quantity $E_n(C^\psi H_\omega)$ have been found for $\psi(k) = k^{-r}$, $r \in \mathbb{N}$, $\beta = r$ and for the convex upwards modulus of continuity by Korneichuk [5] (see also [6, p. 319], [2, p. 344]). The similar problem on the class of real-valued functions defined on the entire real axis and having the $r$-th continuous derivatives $f^{(r)}$ such that $\omega(f^{(r)}; t) \leq \omega(t)$, $t \in [0, \infty)$, is solved in the paper of Ganzburg [4].

The aim of the present work is to study the rate of decrease of quantity (4) when $\psi \in \mathcal{M}_0$ and $\beta \in \mathbb{R}$.

2. Main Results

The following statements are true.

**Theorem 1.** Let $\psi \in \mathcal{M}_0$, $\beta \in \mathbb{R}$ and let $\omega(t)$ be an arbitrary modulus of continuity. Then, as $n \to \infty$,

$$E_n(C^\psi H_\omega) = \frac{\theta_n(\omega)}{\pi} \left| \sin \frac{\beta \pi}{2} \right| \int_0^{1/n} \psi\left(\frac{1}{t}\right) \frac{\omega(t)}{t} dt + O(1)\psi(n)\omega(1/n),$$  

(6)

where $\theta_n(\omega) \in [2/3, 1]$ and $O(1)$ is a quantity uniformly bounded in $n$ and $\beta$. If $\omega(t)$ is a convex upwards modulus of continuity, then $\theta_n(\omega) = 1$.

We give an example of functions $\psi$ and $\omega$ for which (6) is an asymptotic formula.

**Example 1.** Let $\psi(t) = \ln^{-\gamma}(t + 1)$, $\gamma > 1$, $\beta \neq 2l$, $l \in \mathbb{Z}$ and

$$\omega(t) = \begin{cases} 0, & t = 0, \\ \ln^{-\alpha}\left(\frac{1}{t} + 1\right), & t > 0, \quad 0 < \alpha \leq 1. \end{cases}$$

Then by virtue of (6) the following asymptotic formula holds as $n \to \infty$:

$$E_n(C^\psi H_\omega) = \ln^{-\gamma+\alpha}(n + 1)\left(\frac{1}{\pi(\gamma + \alpha - 1)}\right) \sin \frac{\beta \pi}{2} \ln n + O(1),$$

where $O(1)$ is a quantity uniformly bounded in $n$ and $\beta$. 

Note that if
\[
\lim_{n \to \infty} \frac{|\psi'(n)|}{\psi(n)} = 0, \quad \psi'(n) := \psi'(n+), \tag{7}
\]
and
\[
\lim_{n \to \infty} \frac{\omega'(1/n)}{\omega(1/n)n} = 0, \quad \omega'(1/n) := \omega'(1/n+), \tag{8}
\]
then equalities
\[
\lim_{n \to \infty} \frac{\psi(n) \omega(1/n)}{\int_0^{1/n} \frac{\psi(t) \omega(t)}{t} \, dt} = \lim_{n \to \infty} \frac{|\psi'(n)|}{\psi(n)} + \lim_{n \to \infty} \frac{\omega'(1/n)}{\omega(1/n)n} = 0,
\]
are valid.

Therefore from Theorem 1 we obtain

**Corollary 1.** Assume that \( \psi \in \mathcal{M}_0' \), \( \beta \neq 2l \), \( l \in \mathbb{Z} \), \( \omega(t) \) is a convex upwards modulus of continuity and conditions (7) and (8) are fulfilled. Then the following asymptotic formula holds as \( n \to \infty \):
\[
E_n(C_\beta^\psi H_\omega) = \frac{1}{\pi} \left| \sin \frac{\beta \pi}{2} \right| \int_0^{1/n} \frac{\psi(t) \omega(t)}{t} \, dt + O(1) \psi(n) \omega(1/n),
\]
where \( O(1) \) is a quantity uniformly bounded in \( n \) and \( \beta \).

The functions \( \psi \) and \( \omega \) from Example 1 can serve as an example of the functions which satisfy conditions (7) and (8), respectively.

Relation (6) implies that if \( \psi \in \mathcal{M}_0' \) and
\[
\left| \sin \frac{\beta \pi}{2} \right| \int_0^{1/n} \frac{\omega(t)}{t} \, dt = O(1) \omega(1/n), \quad \beta \in \mathbb{R}, \tag{9}
\]
or \( \psi \in \mathcal{M}_C \) (see (3)), then
\[
E_n(C_\beta^\psi H_\omega) = O(1) \psi(n) \omega(1/n).
\]
Taking into account that function \( \psi(t) \) is monotonically decreasing for \( t \geq 1 \) and using the estimate
\[
E_n(C_\beta^\psi H_\omega) \geq K \psi(n) \omega(1/n), \quad \forall \psi \in \mathcal{M}_0', \tag{10}
\]
(see [10, pp. 329, 330]), by virtue of relation (6) we arrive at the following statement:

**Corollary 2.** Let \( \beta \in \mathbb{R} \) and let \( \omega(t) \) be an arbitrary modulus of continuity. If \( \psi \in \mathcal{M}_C \) or \( \psi \in \mathcal{M}_0' \) and \( \omega(t) \) satisfies condition (9), then
\[
K_1 \psi(n) \omega(1/n) \leq E_n(C_\beta^\psi H_\omega) \leq K_2 \psi(n) \omega(1/n), \tag{11}
\]
where \( K_1 \) and \( K_2 \) are positive constants.

Thus, estimates (5) obtained by Stepanets [10, p. 330] (see also [11, Chap. 5, Sec. 22; Chap. 7, Sec. 4]) for the arbitrary modulus of continuity \( \omega(t) \) and for \( \psi \in \mathcal{M}_C, \beta \in \mathbb{R} \) or for \( \psi \in \mathcal{M}_0', \beta = 0 \), hold also in the case when \( \psi \in \mathcal{M}_0', \beta \neq 0 \) and \( \omega(t) \) satisfies condition (9). For example, the function \( \omega(t) = t^\alpha, \ 0 < \alpha \leq 1 \), satisfies (9).
3. Proof of Theorem 1

Suppose that all conditions of the theorem are satisfied. Let us carry out the proof in two stages.

1. We shall find an upper estimate for \( E_n(C^\psi_\beta H_\omega) \).

We set

\[
U_n^\psi(f; x) = \frac{a_0}{2} + \sum_{k=1}^{n-1} \left( 1 - \frac{\psi(n) k^2}{\psi(k) n^2} \right) (a_k \cos kx + b_k \sin kx), \quad n \in \mathbb{N},
\]

where \( a_k \) and \( b_k \) are the Fourier coefficients of a function \( f \in C^\psi_\beta H_\omega \). Show that for the quantity

\[
\mathcal{E}_n(C^\psi_\beta H_\omega) = \sup_{f \in C^\psi_\beta H_\omega} \| f(\cdot) - U_{n-1}^\psi(f; \cdot) \|_C,
\]

the inequality

\[
\mathcal{E}_n(C^\psi_\beta H_\omega) \leq \frac{1}{\pi} \left| \sin \frac{\beta \pi}{2} \right| \left( \int_0^{1/n} \psi \left( \frac{1}{t} \right) \frac{\omega(t)}{t} dt + O(1) \psi(n) \omega(1/n) \right),
\]

(13)

is true. Since

\[
E_n(C^\psi_\beta H_\omega) \leq \mathcal{E}_n(C^\psi_\beta H_\omega),
\]

(14)

then the required upper estimate for \( E_n(C^\psi_\beta H_\omega) \) follows from (13).

For further reasoning, we need the one statement, which follows from the results of book [10, p. 65]. We will give a few notations before formulating it. Let \( f \) be a summable function, whose Fourier series have the form (1). Further, let \( \lambda_n = \{\lambda_1(u), \lambda_2(u), \ldots, \lambda_n(u)\} \) be a collection of continuous functions on \([0, 1]\) such that \( \lambda(k/n) = \lambda_k^{(n)}, k = 0, n, n \in \mathbb{N} \), where \( \lambda_k^{(n)} \) are elements of the triangular matrix \( \Lambda = \|\lambda_k^{(n)}\|, k = 1, n, \lambda_0^{(n)} = 1 \), that determine a polynomial of the form

\[
U_n(f; x; \Lambda) = \frac{a_0}{2} + \sum_{k=1}^{n} \lambda_k^{(n)} (a_k \cos kx + b_k \sin kx), \quad n \in \mathbb{N}.
\]

(15)

The following statement is true:

**Lemma A** [10, p. 65]. Suppose that \( f \in C^\psi_\beta H_\omega \) and \( \tau_n(u) \) is the continuous function defined by relation

\[
\tau_n(u) = \tau_n(u; \lambda; \psi) = \begin{cases} (1 - \lambda_n(u)) \psi(1 nu), & 0 \leq u \leq \frac{1}{n}, \\ (1 - \lambda_n(u)) \psi(nu), & \frac{1}{n} \leq u \leq 1, \\ \psi(nu), & u \geq 1, \end{cases}
\]

(16)

and such that its Fourier transform

\[
\hat{\tau}_n(t) := \hat{\tau}_n(t; \beta) = \frac{1}{\pi} \int_0^{\infty} \tau_n(u) \cos \left( ut + \frac{\beta \pi}{2} \right) du, \quad \beta \in \mathbb{R},
\]
is summable on the whole real line, i.e. \( \int_{-\infty}^{\infty} |\hat{\tau}_n(t)| \, dt < \infty \). Then at any point \( x \) the following equality holds:

\[
f(x) - U_n(f; x; \Lambda) = \int_{-\infty}^{\infty} f_\beta^\psi \left( x + \frac{t}{n} \right) \hat{\tau}_n(t) \, dt, \quad n \in \mathbb{N}. \tag{17}
\]

Using Lemma A, let us show that

\[
f(x) - U_\psi^\psi(n) - 1(f; x) = \int_{-\infty}^{\infty} f_\beta^\psi \left( x + \frac{t}{n} \right) \hat{\tau}_n(t) \, dt, \quad \forall f \in C^\psi_H(\omega), \quad n \in \mathbb{N}, \tag{18}
\]

where \( \hat{\tau}_n(t) \) is the Fourier transform of the function

\[
\tau_n(u) = \tau_n(u; \psi) = \begin{cases} 
\psi(n)u^2, & 0 \leq u \leq 1, \\
\psi(nu), & u \geq 1.
\end{cases} \tag{19}
\]

Since polynomial (12) can be represented in the form

\[
U_\psi^\psi(n-1)(f; x) = \frac{a_0}{2} + \sum_{k=1}^{n} \lambda^\psi(k/n)(a_k \cos kx + b_k \sin kx),
\]

where \( \lambda^\psi(k/n) \) are the values of continuous function

\[
\lambda^\psi(u) = \lambda^\psi_n(u) = \begin{cases} 
1 - \frac{\psi(n)}{\psi(1)} u, & 0 \leq u \leq \frac{1}{n}, \\
1 - \frac{\psi(n)}{\psi(nu)} u^2, & \frac{1}{n} \leq u \leq 1,
\end{cases} \tag{20}
\]

at the points \( u = k/n \) and

\[
\tau_n(u) = \tau_n(u; \psi) = \begin{cases} 
(1 - \lambda^\psi(u))\psi(1)nu, & 0 \leq u \leq \frac{1}{n}, \\
(1 - \lambda^\psi(u))\psi(nu), & \frac{1}{n} \leq u \leq 1, \\
\psi(nu), & u \geq 1,
\end{cases}
\]

then it follows from Lemma A that for proving (18) it is sufficient to establish the inequality

\[
\int_{-\infty}^{\infty} |\hat{\tau}_n(t)| \, dt < \infty. \tag{21}
\]

With this aim we put

\[
\mu_n(u) = \begin{cases} 
\psi(n)(u^2 - u), & 0 \leq u \leq 1, \\
0, & u \geq 1,
\end{cases} \quad \nu_n(u) = \tau_n(u) - \mu_n(u).
\]

Integrating twice by parts, we get

\[
\hat{\mu}_n(t) := \hat{\mu}_n(t; \beta) = \frac{1}{\pi} \int_{0}^{\infty} \mu_n(u) \cos \left( ut + \frac{\beta \pi}{2} \right) \, du = \frac{O(1)}{t^2}, \quad t > 0,
\]
which yields
\[ \int_{|t| \geq 1} |\hat{\mu}_n(t)| \, dt < \infty. \] (22)

It is obvious that
\[ \int_{|t| \leq 1} |\hat{\mu}_n(t)| \, dt < \infty. \] (23)

Taking (22), (23) together and using the estimates
\[ \int_{-\infty}^{\infty} |\hat{\nu}_n(t)| \, dt < \infty \quad \forall \psi \in \mathfrak{M}_0', \]
(see, e.g., [11, p. 174]) and
\[ |\hat{\tau}_n(t)| \leq |\hat{\mu}_n(t)| + |\hat{\nu}_n(t)|, \]
we obtain (21).

Furthermore, since the function \( \tau_n(u) \) satisfies all conditions of Lemma 3 from [14] according to which
\[ \tau_n(u) = \int_{-\infty}^{\infty} \cos \left( ut + \frac{\beta \pi}{2} \right) \hat{\tau}_n(t) \, dt, \quad u \geq 0, \]
we have
\[ \int_{-\infty}^{\infty} \hat{\tau}_n(t) \, dt = \frac{\tau_n(0)}{\cos \frac{\beta \pi}{2}} = 0, \quad \beta \neq 2l - 1, \quad l \in \mathbb{Z}. \]
If \( \beta = 2l - 1, \ l \in \mathbb{Z} \), the equality \( \int_{-\infty}^{\infty} \hat{\tau}_n(t) \, dt = 0 \) is obvious, because \( \hat{\tau}_n(t) \) is odd. Hence, starting from (18) we can write
\[ f(x) - U_{n-1}^{\psi}(f;x) = \int_{-\infty}^{\infty} \left( f_{\beta}(x + \frac{t}{n}) - f_{\beta}^{\psi}(x) \right) \hat{\tau}_n(t) \, dt \quad \forall f \in C_{\beta}^{\psi} H_{\omega}, \ n \in \mathbb{N}. \] (24)

Since \( f_{\beta}^{\psi} \in H_{\omega}^0 \) and, as it is not hard to see, for every \( \varphi \in H_{\omega}^0 \) function \( \varphi_1(u) = \varphi(u + h), \ h \in \mathbb{R} \), also belongs to \( H_{\omega}^0 \), then using the notation
\[ \delta(t, \varphi) = \varphi(t) - \varphi(0), \]
it follows from (24) that
\[ E_n(C_{\beta}^{\psi} H_{\omega}) \leq \sup_{\varphi \in H_{\omega}^0} \int_{-\infty}^{\infty} \left( \varphi \left( \frac{t}{n} \right) - \varphi(0) \right) \hat{\tau}_n(t) \, dt \leq \sup_{\varphi \in H_{\omega}^0} \int_{-\infty}^{\infty} \delta \left( \frac{t}{n}, \varphi \right) \hat{\tau}_n(t) \, dt. \] (25)

Now we shall simplify the integral in the right-hand side of (25) without loss of its principal value. The following relations are true:
\[ \int_{-\infty}^{\infty} \delta \left( \frac{t}{n}, \varphi \right) \hat{\tau}_n(t) \, dt = \]
\[ \int_{-\infty}^{\infty} \delta\left(\frac{t}{n}, \varphi\right) \int_{0}^{\infty} \tau_{n}(u) \cos ut du dt - \sin \frac{\beta \pi}{2} \int_{-\infty}^{\infty} \delta\left(\frac{t}{n}, \varphi\right) \int_{0}^{\infty} \tau_{n}(u) \sin ut du dt = \]

\[ = \cos \frac{\beta \pi}{2} \pi \int_{-\infty}^{\infty} \delta\left(\frac{t}{n}, \varphi\right) \int_{0}^{\infty} \tau_{n}(u) \cos ut du dt - \sin \frac{\beta \pi}{2} \frac{1}{\pi} \int_{0}^{\infty} \delta\left(\frac{t}{n}, \varphi\right) \int_{0}^{\infty} \tau_{n}(u) \sin ut du dt \]

\[ + \int_{|t| \leq 1} \delta\left(\frac{t}{n}, \varphi\right) \int_{0}^{1} \tau_{n}(u) \sin ut du dt + \int_{|t| \leq 1} \delta\left(\frac{t}{n}, \varphi\right) \int_{1}^{\infty} \psi(nu) \sin ut du dt \]. (26)

Integrating by parts, taking into account the equality \( \tau_{n}(0) = \tau_{n}(\infty) = 0 \) and assuming that \( \psi'(u) := \psi'(u+) \), we have

\[ \int_{0}^{\infty} \tau_{n}(u) \cos ut du = -\frac{1}{t} \int_{0}^{\infty} \tau'_{n}(u) \sin ut du = \]

\[ = -\frac{2 \psi(n)}{t} \int_{0}^{1} u \sin ut du - \frac{n}{t} \int_{1}^{\infty} \psi'(nu) \sin ut du, \] (27)

and similarly

\[ \int_{0}^{\infty} \tau_{n}(u) \sin ut du = \frac{2 \psi(n)}{t} \int_{0}^{1} u \cos ut du + \frac{n}{t} \int_{1}^{\infty} \psi'(nu) \cos ut du, \] (28)

Combining (26)–(28), we obtain

\[ \int_{-\infty}^{\infty} \delta\left(\frac{t}{n}, \varphi\right) \tilde{\tau}_{n}(t) dt = \]

\[ = -\sin \frac{\beta \pi}{2} \int_{|t| \leq 1} \delta\left(\frac{t}{n}, \varphi\right) \int_{1}^{\infty} \psi(nu) \sin ut du dt + r_{n}(\psi, \varphi, \beta), \quad \varphi \in H_{0}^{1}, \ n \in \mathbb{N}, \] (29)

where

\[ r_{n}(\psi, \varphi, \beta) = \frac{\cos \frac{\beta \pi}{2}}{\pi} \left( -2 \psi(n) \int_{-\infty}^{\infty} \delta\left(\frac{t}{n}, \varphi\right) \frac{1}{t} \int_{0}^{1} u \sin ut du dt - \right. \]

\[ -n \int_{-\infty}^{\infty} \delta\left(\frac{t}{n}, \varphi\right) \frac{1}{t} \int_{1}^{\infty} \psi'(nu) \sin ut du dt \bigg) - \]

\[ -\sin \frac{\beta \pi}{2} \left( 2 \psi(n) \int_{|t| \geq 1} \delta\left(\frac{t}{n}, \varphi\right) \frac{1}{t} \int_{0}^{1} u \cos ut du dt + \right. \]

\[ + n \int_{|t| \geq 1} \delta\left(\frac{t}{n}, \varphi\right) \frac{1}{t} \int_{1}^{\infty} \psi'(nu) \cos ut du dt + \]

\[ + \int_{|t| \leq 1} \delta\left(\frac{t}{n}, \varphi\right) \int_{0}^{1} \tau_{n}(u) \sin ut du dt \bigg) = \frac{\cos \frac{\beta \pi}{2}}{\pi} \sum_{i=1}^{2} J_{i,n} - \frac{\sin \frac{\beta \pi}{2}}{\pi} \sum_{i=3}^{5} J_{i,n}. \] (30)
Let us show that
\[ r_n(\psi, \varphi, \beta) = O(1)\psi(n)\omega(1/n). \] (31)

Since for \( t \in [-1,1] \) the quantity
\[ \frac{1}{t} \int_0^1 u \sin ut \, du, \]
is bounded by a constant, then using the inequality \(|\delta(t, \varphi)| \leq \omega(|t|)\), we get
\[ J_{1,n} = -2\psi(n) \int_{|t| \geq 1} \delta(t, \varphi) \frac{1}{t} \int_0^1 u \sin ut \, du \, dt + O(1)\psi(n)\omega(1/n). \] (32)

To estimate the integral in (32) we establish the following auxiliary statements.

**Lemma 1.** On every interval \((x_k^{(i)} , x_{k+1}^{(i)})\), \( x_k^{(i)} = (2k - 1 + i)\pi/2a, \ i = 0, 1, k \in \mathbb{N}, \ a > 0\), the function
\[ \int_x^\infty \frac{1}{t} \int_0^a u^s \sin \left( ut + \frac{i\pi}{2} \right) \, du \, dt, \quad x > 0, \ s \geq 1, \]
has at least one zero.

**Proof.** We will give a proof of the lemma only for the case \( i = 0 \), because the proof in case \( i = 1 \) is similar. On the basis of the estimate \(|\int_x^\infty \sin t \, dt| \leq \frac{2}{x}, \ x > 0\) (see, e.g., \([1, \text{p. 5}], [9, \text{p. 343}]\)) it is simple to see that the integral
\[ \int_x^\infty \frac{u^s \sin ut}{t} \, dt = u^s \int_0^\infty \frac{\sin t}{t} \, dt, \]
converges uniformly with respect to \( u \in [0, a], \ a > 0\). Therefore, changing the order of integration, we obtain
\[ S(x) := \int_x^\infty \frac{1}{t} \int_0^a u^s \sin ut \, du \, dt = \int_0^a u^s \int_x^\infty \frac{\sin ut}{t} \, dt \, du. \]

Making the change of variables and integrating by parts, we have
\[ S(x) = \int_0^a u^s \int_{ux}^\infty \frac{\sin t}{t} \, dt \, du = \frac{1}{s+1} \left( a^{s+1} \int_{ax}^\infty \frac{\sin t}{t} \, dt + \int_0^a u^s \sin ux \, du \right) = \]
\[ = \frac{1}{s+1} \left( a^{s+1} \int_{ax}^\infty \frac{\sin t}{t} \, dt - a^s \cos ax \frac{x}{a} \right) + \frac{s}{x} \int_0^a u^{s-1} \cos ux \, du. \]

Hence, taking into account the equation
\[ \int_{ax}^\infty \frac{\sin t}{t} \, dt = \frac{\cos ax}{ax} - \int_{ax}^\infty \frac{\cos t}{t^2} \, dt, \]
we get
\[ S(x) = \frac{1}{s+1} \left( -a^{s+1} \int_{ax}^{\infty} \cos \frac{t}{t^2} dt + \frac{s}{x^{s+1}} \int_{0}^{ax} u^{s-1} \cos u \, du \right). \] (33)

On every interval \((t_j, t_{j+1}), t_j = (2j + 1)\pi/2, j = 0, 1, \ldots\), the function \(\int_{x}^{\infty} \frac{\cos t}{t^2} dt\) vanishes with a change of sign at some point \(\tilde{x}_j\). Since
\[ \int_{\pi/2}^{\infty} \cos \frac{t}{t^2} dt = -\int_{\pi/2}^{\infty} \sin \frac{t}{t} dt < 0, \]
then for any \(k \in \mathbb{N}\)
\[ \text{sign} \int_{(2k-1)\pi/2}^{\infty} \frac{\cos t}{t^2} dt = (-1)^{k}. \] (34)

Further, we have
\[ \int_{0}^{(2k-1)\pi/2} u^{s-1} \cos u \, du = \alpha_0 + \sum_{j=1}^{k-1} \alpha_j, \]
where
\[ \alpha_0 = \int_{0}^{\pi/2} u^{s-1} \cos u \, du, \quad \alpha_j = \int_{(2j-1)\pi/2}^{(2j+1)\pi/2} u^{s-1} \cos u \, du. \]
If \(k = 1\), then
\[ \text{sign} \int_{0}^{(2k-1)\pi/2} u^{s-1} \cos u \, du = \text{sign} \alpha_0 = 1. \] (35)
Let \(k = 2, 3, \ldots\) Since the function \(u^{s-1}\) does not decrease \((s \geq 1)\) for \(u \geq 0\), we can write
\[ |\alpha_0| < |\alpha_j| \leq |\alpha_{j+1}|, \quad j \geq 1, \]
and respectively
\[ \text{sign} \int_{0}^{(2k-1)\pi/2} u^{s-1} \cos u \, du = \text{sign} \int_{(2k-3)\pi/2}^{(2k-1)\pi/2} u^{s-1} \cos u \, du = (-1)^{k+1}, \quad k = 2, 3, \ldots. \] (36)

Taking account of (33)–(36), we have
\[ \text{sign} S\left(\frac{2k-1}{2a} \pi\right) = (-1)^{k+1}, \quad k \in \mathbb{N}, \quad a > 0. \] (37)

The function \(S(x)\) is continuous for any \(x > 0\). Therefore, it follows from (37) that on every interval \((x_k, x_{k+1})\), where \(x_k = (2k - 1)\pi/2a, k \in \mathbb{N}, a > 0\), the function \(S(x)\) has the required zero. Lemma 1 is proved. \(\blacksquare\)

Lemma 2. Let \(\varphi \in H_{\omega}, 1 \leq a \leq n, \ n \in \mathbb{N} \text{ and } s \geq 1\). Then for \(i = 0, 1\), the following estimate holds:
\[ \int_{|t| \geq 1} \left( \varphi\left(\frac{t}{n}\right) - \varphi(0) \right) \frac{1}{t} \int_{0}^{a/n} u^s \sin\left(ut + \frac{i\pi}{2}\right) du \, dt = O(1)\omega(1/n), \] (38)
where \(O(1)\) is a quantity uniformly bounded in \(n, \varphi, a\) and \(s\).
Proof. Making the change of variables, we get

$$
\int_{|t| \geq 1} \left( \varphi \left( \frac{t}{n} \right) - \varphi(0) \right) \frac{1}{t} \int_0^{a/n} u^s \sin \left( ut + \frac{i\pi}{2} \right) du \, dt =
$$

$$
= \frac{1}{n^{s+1}} \int_{|t| \geq 1/n} (\varphi(t) - \varphi(0)) \frac{1}{t} \int_0^{a} u^s \sin \left( ut + \frac{i\pi}{2} \right) du \, dt, \quad i = 0, 1. \quad (39)
$$

Let us denote by $t_k^{(i)}$ the zero of function

$$
\int_{x_k}^{x_{k+1}} u^s \sin \left( ut + \frac{i\pi}{2} \right) du, \quad i = 0, 1,
$$
on interval $(x_k, x_{k+1})$, $x_k = \frac{2k-1+i}{2a}$, which exists according to Lemma 1. Using the notation $\delta(t) = \varphi(t) - \varphi(0)$, we have

$$
\left| \int_{1/n}^{\infty} \delta(t) \frac{1}{t} \int_0^{a} u^s \sin \left( ut + \frac{i\pi}{2} \right) du \, dt \right| = \left| \int_{1/n}^{t_k^{(i)}} \delta(t) \frac{1}{t} \int_0^{a} u^s \sin \left( ut + \frac{i\pi}{2} \right) du \, dt + \right.
$$

$$
\left. + \sum_{k=1}^{\infty} \int_{t_k^{(i)}}^{t_{k+1}^{(i)}} (\delta(t) - \delta(t_k^{(i)})) \frac{1}{t} \int_0^{a} u^s \sin \left( ut + \frac{i\pi}{2} \right) du \, dt \right| \leq
$$

$$
\leq \omega(t_k^{(i)}) \int_{1/n}^{t_k^{(i)}} \left| \frac{1}{t} \int_0^{a} u^s \sin \left( ut + \frac{i\pi}{2} \right) du \right| dt + \omega(\Delta_i) \int_{t_k^{(i)}}^{\infty} \left| \frac{1}{t} \int_0^{a} u^s \sin \left( ut + \frac{i\pi}{2} \right) du \right| dt, \quad (40)
$$

where $\Delta_i = \sup_k (t_{k+1}^{(i)} - t_k^{(i)})$. Since $t_1^{(i)} < \frac{2\pi}{a}$ and $\Delta_i < \frac{2\pi}{a}$, it follows from (40) that

$$
\left| \int_{1/n}^{\infty} \delta(t) \frac{1}{t} \int_0^{a} u^s \sin \left( ut + \frac{i\pi}{2} \right) du \, dt \right| < \omega \left( \frac{2\pi}{a} \right) \int_{1/n}^{\infty} \left| \frac{1}{t} \int_0^{a} u^s \sin \left( ut + \frac{i\pi}{2} \right) du \right| dt. \quad (41)
$$

After integrating by parts it is easy to see, that

$$
\left| \int_0^{a} u^s \sin \left( ut + \frac{i\pi}{2} \right) du \right| \leq \frac{2a^s}{t}, \quad t > 0, \quad i = 0, 1. \quad (42)
$$

From (41) and (42) follows the inequality

$$
\left| \int_{1/n}^{\infty} \delta(t) \frac{1}{t} \int_0^{a} u^s \sin \left( ut + \frac{i\pi}{2} \right) du \, dt \right| < 2a^s \omega \left( \frac{2\pi}{a} \right) \int_{1/n}^{\infty} \frac{dt}{t^2} = 2a^s \omega \left( \frac{2\pi}{a} \right) n \leq
$$

$$
\leq 2a^s \left( \frac{2\pi n}{a} + 1 \right) \omega \left( \frac{1}{n} \right) n < 8a^{s-1} \pi n^2 \omega \left( \frac{1}{n} \right) \leq 8\pi n^{s+1} \omega \left( \frac{1}{n} \right), \quad i = 0, 1. \quad (43)
$$

The estimate

$$
\int_{-\infty}^{-1/n} \delta(t) \frac{1}{t} \int_0^{a} u^s \sin \left( ut + \frac{i\pi}{2} \right) du \, dt = O(1)n^{s+1} \omega(1/n), \quad i = 0, 1, \quad (44)
$$
A. Serdyuk, I. Ovsii

is similarly proved. Comparing relations (43), (44) and (39), we obtain (38). Lemma 2 is proved. ▶

Applying Lemma 2 to the integral in (32) and, at the same time, to $J_{3,n}$, we have

$$J_{1,n} = O(1)\psi(n)\omega(1/n),$$  
(45)  
$$J_{3,n} = O(1)\psi(n)\omega(1/n).$$  
(46)

In the monograph [11, pp. 212, 216, see relations (4.26′) and (4.42), (4.45), (4.46)] it is shown, that

$$J_{2,n} = O(1)\psi(n)\omega(1/n), \quad \forall \psi \in M_0,$$  
(47)  
and

$$J_{4,n} = O(1)\psi(n)\omega(1/n) \quad \forall \psi \in M'_0, \quad \beta \neq 2l, \quad l \in \mathbb{Z}.$$  
(48)

Since $|\tau_n(u)| \leq \psi(n), \quad u \in [0,1]$, it is clear that

$$J_{5,n} = O(1)\psi(n)\omega(1/n).$$  
(49)

Comparing (30), (45)–(49), we arrive at (31). Then from (29) for any function $\varphi \in H_0^\omega$ and $n \in \mathbb{N}$, we obtain

$$\int_{-\infty}^{\infty} \delta\left(\frac{t-n}{n}, \varphi\right) \tilde{\tau}_n(t) \, dt = -\frac{\sin \frac{\beta \pi}{2}}{\pi} \int_{|t|<1} \delta\left(\frac{t-n}{n}, \varphi\right) \int_{1}^{\infty} \psi(nu) \sin ut \, du \, dt + O(1)\psi(n)\omega(1/n) =$$

$$= -\frac{\sin \frac{\beta \pi}{2}}{\pi} \int_{0}^{1} \left( \delta\left(\frac{t}{n}, \varphi\right) - \delta\left(-\frac{t}{n}, \varphi\right) \right) \int_{1}^{\infty} \psi(nu) \sin ut \, du \, dt +$$

$$+ O(1)\psi(n)\omega(1/n), \quad \psi \in M_0, \quad \beta \in \mathbb{R}.$$  
(50)

Since

$$\int_{1}^{\infty} \psi(nu) \sin ut \, du > 0, \quad t \in (0,1], \quad \psi \in M'_0, \quad \beta \neq 2l, \quad l \in \mathbb{Z},$$  
(51)

(see, e.g., [12, p. 143]) and

$$\int_{0}^{1} \omega\left(\frac{2t}{n}\right) \int_{1}^{\infty} \psi(nu) \sin ut \, du \, dt =$$

$$= \int_{0}^{1/n} \psi\left(\frac{1}{t} \frac{\omega(t)}{t}\right) \, dt + O(1)\psi(n)\omega(1/n), \quad \psi \in M'_0, \quad \beta \neq 2l, \quad l \in \mathbb{Z},$$  
(52)

(see [8, p. 528]), from (25) and (50) we obtain (13). Putting together inequalities (13) and (14) we find a required estimate for quantity (4):

$$E_n(C^\psi_{\beta} H_\omega) \leq \frac{1}{\pi} \left| \sin \frac{\beta \pi}{2} \right| \int_{0}^{1/n} \psi\left(\frac{1}{t} \frac{\omega(t)}{t}\right) \, dt + O(1)\psi(n)\omega(1/n), \quad \psi \in M_0, \quad \beta \in \mathbb{R}.$$  
(53)

2. Let us find a lower bound for $E_n(C^\psi_{\beta} H_\omega)$. 


Let \( \varphi_n(t) \) be an odd \( 2\pi/n \)-periodic function defined on \([0, \pi/n]\) by the equalities

\[
\varphi_n(t) = \begin{cases} 
\frac{c_\omega}{2} \omega(2t), & t \in [0, \pi/2n], \\
\frac{c_\omega}{2} \omega(\frac{2\pi}{n} - 2t), & t \in [\pi/2n, \pi/n], 
\end{cases}
\]

where \( c_\omega = 1 \) if \( \omega(t) \) is a convex upwards modulus of continuity and \( c_\omega = 2/3 \) otherwise. As shown in [10, pp. 83–85] if \( \omega(t) \) is an arbitrary modulus of continuity, then

\[ |\varphi_n(t') - \varphi_n(t'')| \leq \omega(|t' - t''|), \quad t', t'' \in [-\pi/2n, \pi/2n]. \]

This implies that

\[ |\varphi_n(t') - \varphi_n(t'')| \leq \omega(|t' - t''|), \quad t', t'' \in \mathbb{R}, \]

and, hence, \( \varphi_n \in H_\omega \). We denote by \( f^*(\cdot) \) the function from the set \( C_\beta^\psi H_\omega, \psi \in \mathfrak{M}' \), whose \((\psi, \beta)\)-derivative \( f^*_\beta(t) \) coincides with the function \( \varphi_n(t) \) on a period. By relations (2), such a function \( f^*(\cdot) \) exists.

In virtue of formula (3.4) from the book [10, Chap. 2, Subsec. 3.1] the following equality holds for any \( f \in C_\beta^\psi H_\omega, \psi \in \mathfrak{M}' \):

\[
f(x) - U_{n-1}(f; x; \Lambda) = \frac{1}{\pi} \int_{-\pi}^\pi f(x + t) \left( \sum_{k=1}^\infty \psi(k) \cos \left( kt + \frac{\beta \pi}{2} \right) - \sum_{k=1}^{n-1} \lambda_k^{(n)} \psi(k) \cos \left( kt + \frac{\beta \pi}{2} \right) \right) dt, \quad x \in \mathbb{R}, \quad n \in \mathbb{N},
\]

where \( U_{n-1}(f; x; \Lambda) \) is a trigonometric polynomial of the form (15), such that \( \lambda_k^{(n)} = 0 \). Since function \( \varphi_n(t) \) is odd \( 2\pi/n \)-periodic, the equalities

\[ \int_{-\pi}^\pi \varphi_n(t) \sin kt \, dt = 0, \quad k = 1, 2, \ldots, n - 1, \quad n \geq 2, \]

(see, e.g., [6, p. 159]) and

\[ \varphi_n \left( \frac{i\pi}{n} + t \right) = (-1)^i \varphi_n(t), \quad i \in \mathbb{Z}, \]

hold. Then, using relation (54) for \( f^*(\cdot) \), we obtain

\[
f^* \left( \frac{i\pi}{n} \right) - U_{n-1}(f^*; \frac{i\pi}{n}; \Lambda) =
\]

\[= \frac{(-1)^i}{\pi} \int_{-\pi}^\pi \varphi_n(t) \left( \sum_{k=1}^\infty \psi(k) \cos \left( kt + \frac{\beta \pi}{2} \right) - \sum_{k=1}^{n-1} \lambda_k^{(n)} \psi(k) \cos \left( kt + \frac{\beta \pi}{2} \right) \right) dt =
\]

\[= \frac{(-1)^i}{\pi} \int_{-\pi}^\pi \varphi_n(t) \sum_{k=1}^\infty \psi(k) \cos \left( kt + \frac{\beta \pi}{2} \right) dt =
\]
\[ E_n(f^*) \geq \inf_{t_{n-1}} ||f^*(\cdot) - t_{n-1}(\cdot)||_C, \quad n \in \mathbb{N}. \]  \hfill (57)

From (56) and (57) it follows, in particular, that

\[ E_n(f^*) \geq |f^*(0) - U_{n-1}(f^*; 0; \Lambda)|, \quad n = 2, 3, \ldots. \hfill (58) \]

Inequality (58) is satisfied for triangular matrix \( \Lambda = ||\lambda_k(n)||, k = 1, n \), such that \( \lambda_n^{(n)} = 0 \). Let’s define its remaining elements in the following way:

\[ \lambda_k^{(n)} = \lambda^{\psi}(k/n), \quad k = 1, n-1, \quad n \in \mathbb{N}, \]

where \( \lambda^{\psi}(\cdot) \) is defined by (20). Since in this case

\[ U_{n-1}(f^*; 0; \Lambda) = U_{n-1}^\psi(f^*; 0), \]

then from (58) we obtain, taking the inequality \( E_n(C_{\beta}^\psi H_{\omega}) \geq E_n(f^*) \) into account,

\[ E_n(C_{\beta}^\psi H_{\omega}) \geq |f^*(0) - U_{n-1}^\psi(f^*; 0)|, \quad n = 2, 3, \ldots, \psi \in \mathcal{M}'. \hfill (59) \]

By virtue of (24) and (50)

\[ f^*(0) - U_{n-1}^\psi(f^*; 0) = \int_{-\infty}^{\infty} \left( f^\psi_{\beta}(\frac{t}{n}) - f^\psi_{\beta}(0) \right) \hat{\tau}_n(t) dt = \]

\[ = -\frac{\sin \frac{\beta \pi}{2}}{\pi} \int_0^1 \left( \varphi_n(t/n) - \varphi_n(-t/n) \right) \int_0^{\infty} \psi(\nu u) \sin ut du dt + O(1)\psi(n)\omega(1/n) = \]

\[ = -\omega \frac{\sin \frac{\beta \pi}{2}}{\pi} \int_0^1 \omega(\frac{2t}{n}) \int_0^{\infty} \psi(\nu u) \sin ut du dt + O(1)\psi(n)\omega(1/n), \quad \psi \in \mathcal{M}'_0. \hfill (60) \]

Combining (51), (52), (59) and (60), we arrive at the desired estimate

\[ E_n(C_{\beta}^\psi H_{\omega}) \geq \frac{c_\omega}{\pi} \left| \sin \frac{\beta \pi}{2} \right| \int_0^{1/n} \psi\left(\frac{1}{t}\right) \frac{\omega(t)}{t} dt + O(1)\psi(n)\omega(1/n), \quad \psi \in \mathcal{M}'_0, \beta \neq 2l, l \in \mathbb{Z}. \hfill (61) \]

From (53) and (61) we obtain formula (6). Theorem 1 is proved.
References


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