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UNIFORM CONVERGENCE OF ORTHOGONAL EXPANSIONS ON THE REAL PROJECTIVE SPACES

Let $\mathbb{P}^d(\mathbb{R})$ be the real projective space, $d\nu$ its invariant normalized measure, Δ its Laplace-Beltrami operator. Let $0 \le \theta_0, \cdots \le \theta_n \le \cdots$ be the eigenvalues and H_{2k}, $k \in \mathbb{N} \cup \{0\}$ be the respective eigenspaces of Δ, $\dim \mathrm{H}_{2k} = d_{2k}$. Let $\{Y_j^{2k}\}_{j=1}^{d_{2k}}$ be an orthonormal basis of H_{2k}. For any $\phi \in L_\infty(\mathbb{P}^d(\mathbb{R}))$ with the formal Fourier expansion

$$\phi \sim c_0 + \sum_{k\in\mathbb{N}} \sum_{j=1}^{d_{2k}} c_{2k,j}(\phi)\, Y_j^{2k}, \quad c_{2k,j}(\phi) = \int_{\mathbb{P}^d(\mathbb{R})} \phi\, \overline{Y_j^{2k}}\, d\nu$$

consider the sequence of Fourier sums

$$S_{2n}(\phi) \;=\; c_0(\phi) + \sum_{k=1}^{n} \sum_{j=1}^{d_{2k}} c_{2k,j}(\phi)\, Y_j^{2k}.$$

Our main result establishes sharp asymptotic for the norm of Fourier projection. Namely, it is shown that

$$\|S_{2n}\|_{L_\infty(\mathbb{P}^d(\mathbb{R})) \to L_\infty(\mathbb{P}^d(\mathbb{R}))} \;=$$

$$= \frac{4n^{(d-1)/2}}{\pi^{3/2}\Gamma(d/2)} \int_0^{\pi/2} (\sin\eta)^{(d-3)/2}\, d\eta \left(1 + \left\{ \begin{array}{ll} O(n^{-1/2}), & d=2 \\ O(n^{-1}), & d \ge 3 \end{array} \right\}\right).$$

In particular, if $d=2$ then

$$\|S_{2n}\|_{L_\infty(\mathbb{P}^2(\mathbb{R})) \to L_\infty(\mathbb{P}^2(\mathbb{R}))} \;=\; n^{1/2}\, 2^{5/2} \left(\Gamma\left(\frac{3}{4}\right)\right)^{-2} + O(1).$$

We give some applications of this result to the problem of uniform convergence of orthogonal developments on $\mathbb{P}^d(\mathbb{R})$.

Introduction. Let $\mathbb{P}^d(\mathbb{R})$ be the real d-dimensional projective space, ν its normalized volume element, Δ its Laplace-Beltrami operator. It is well-known that the eigenvalues θ_m, $m = 2k$, $k \in \mathbb{N} \cup \{0\}$ of Δ are discrete, nonnegative and form an increasing sequence $0 \le \theta_0 \le \theta_2 \le \cdots \le \theta_{2k} \le \cdots$ with $+\infty$ the only accumulation point. Corresponding eigenspaces H_{2k}, $k \in \mathbb{N} \cup \{0\}$, are finite dimensional, $d_{2k} = \dim \mathrm{H}_{2k} < \infty$, orthogonal and $L_2(\mathbb{P}^d(\mathbb{R}), \nu) = \oplus_{2k=0}^{\infty} \mathrm{H}_{2k}$. Let $\{Y_j^{2k}\}_{j=1}^{d_{2k}}$ be an orthonormal basis of H_{2k}. Assume that $\phi \in L_\infty(\mathbb{P}^d(\mathbb{R}))$ with the formal Fourier expansion

$$\phi \sim c_0 + \sum_{k \in \mathbb{N}} \sum_{j=1}^{d_{2k}} c_{2k,j}(\phi) Y_j^{2k}, \;\; c_{2k,j}(\phi) = \int_{\mathbb{P}^d(\mathbb{R})} \phi \;\overline{Y_j^{2k}} d\nu.$$

Consider the sequence of Fourier sums

$$S_{2n}(\phi, x) = c_0 + \sum_{k=1}^{n} \sum_{j=1}^{d_{2k}} c_{2k,j}(\phi) Y_j^{2k}(x).$$

We shall be study an asymptotic behavior of the norms of Fourier projections $S_{2n} : L_\infty(\mathbb{P}^d(\mathbb{R})) \to L_\infty(\mathbb{P}^d(\mathbb{R}))$, as $n \to \infty$. Observe that this problem is closely connected with the problem of uniform convergence of Fourier series on $\mathbb{P}^d(\mathbb{R})$. Indeed, let

$$E_{2n}(\phi) = \inf \left\{ \|\phi - t_{2n}\|_{L_\infty(\mathbb{P}^d(\mathbb{R}))} \mid t_{2n} \in \mathcal{T}_{2n} \right\}$$

be the best approximation of a function $\phi \in L_\infty(\mathbb{P}^d(\mathbb{R}))$ by the subspace $\mathcal{T}_{2n}$ of polynomials of order $\leq 2n$, $\mathcal{T}_{2n} = \oplus_{k=0}^{n} \mathrm{H}_{2k}$. Then, by the Lebesgue inequality we get

$$\|\phi - S_{2n}(\phi, x)\|_{L_\infty(\mathbb{P}^d(\mathbb{R}))} \leq \left(1 + \|S_{2n}\|_{L_\infty(\mathbb{P}^d(\mathbb{R})) \to L_\infty(\mathbb{P}^d(\mathbb{R}))}\right) E_{2n}(\phi),$$

where

$$\|S_{2n}\|_{L_\infty(\mathbb{P}^d(\mathbb{R})) \to L_\infty(\mathbb{P}^d(\mathbb{R}))} = \sup\{\|S_{2n}(\phi)\|_{L_\infty(\mathbb{P}^d(\mathbb{R}))} | \phi \in L_\infty(\mathbb{P}^d(\mathbb{R}))\}.$$

It means that $S_{2n}(\phi, x)$ converges uniformly to f if

$$E_{2n}(\phi) = \overline{\overline{o}} \left(\|S_{2n}\|_{L_\infty(\mathbb{P}^d(\mathbb{R})) \to L_\infty(\mathbb{P}^d(\mathbb{R}))}\right)^{-1}, \;\; n \to \infty.$$

It is well-known that in the case of the circle, $\mathbb{S}^1$, we have

$$\|S_n\|_{L_\infty(\mathbb{S}^1) \to L_\infty(\mathbb{S}^1)} = \frac{1}{\pi} \int_{-\pi}^{\pi} |D_n(t)| dt \; = \; \frac{4}{\pi^2} \ln n + O(1).$$

Remark that in the case of $\mathbb{S}^d$, the unit Euclidean sphere in $\mathbb{R}^{d+1}$, sharp asymptotics of the norms of Fourier projections have been found in Gronwall [5] if $d = 2$ and in Kushpel [9] in the case $d \geq 3$. Namely, it was shown that

$$\|S_n\|_{L_\infty(\mathbb{S}^d)\to L_\infty(\mathbb{S}^d)} = \mathcal{K}(\mathbb{S}^d)n^{(d-1)/2} + \left\{ \begin{array}{ll} O(n^{d/2-1}), & d=2 \\ O(n^{(d-3)/2}), & d\geq 3 \end{array} \right\},$$

and

$$\mathcal{K}(\mathbb{S}^d) = \frac{4}{\pi^{3/2}\Gamma(d/2)} \int_0^{\pi/2} (\sin\ \eta)^{(d-3)/2}\ (\cos\ \eta)^{(d-1)/2} d\eta,$$

The cases of complex and quternionic projective spaces, $\mathbb{P}^d(\mathbb{C})$ and $\mathbb{P}^d(\mathbb{H})$ respectively and the Cayley elliptic plane $\mathbb{P}^{16}(\mathrm{Cay})$ have been considered in Kushpel [10],

$$\|S_n\|_{L_\infty(\mathbb{M}^d)\to L_\infty(\mathbb{M}^d)} = \mathcal{K}(\mathbb{R}^d)n^{(d-1)/2} + \left\{ \begin{array}{ll} O(n^{d/2-1}), & d=2 \\ O(n^{(d-3)/2}), & d\geq 3 \end{array} \right\},$$

and

$$\mathcal{K}(\mathbb{M}^d) = \frac{4}{\pi^{3/2}\Gamma(d/2)} \int_0^{\pi/2} (\sin\ \eta)^{(d-3)/2}\ (\cos\ \eta)^{\chi(\mathbb{M}^d)} d\eta,$$

where

$$\chi(\mathbb{M}^d) = \left\{ \begin{array}{ll} 1/2, & \mathbb{M}^d = \mathbb{P}^d(\mathbb{C}),\ d = 4, 6, 8, \cdots, \\ 2, & \mathbb{M}^d = \mathbb{P}^d(\mathbb{H}),\ d = 8, 12, 16, \cdots, \\ 7/2, & \mathbb{M}^d = \mathbb{P}^{16}(\mathrm{Cay}). \end{array} \right.$$

The main result of this article is the following statement.

Theorem 1.

$$\|S_{2n}\|_{L_\infty(\mathbb{P}^d(\mathbb{R}))\to L_\infty(\mathbb{P}^d(\mathbb{R}))} = \mathcal{K}(\mathbb{P}^d(\mathbb{R}))n^{(d-1)/2} \times \left(1 + \left\{ \begin{array}{ll} O(n^{-1/2}), & d=2 \\ O(n^{-1}), & d\geq 3 \end{array} \right\}\right),$$

where

$$\mathcal{K}(\mathbb{P}^d(\mathbb{R})) = \frac{4}{\pi^{3/2}\ \Gamma(d/2)} \int_0^{\pi/2} (\sin\eta)^{(d-3)/2} d\eta.$$

Elements of Harmonic Analysis. The real projective spaces $\mathbb{P}^d(\mathbb{R})$ can be obtained by identifying of the antipodal points on $\mathbb{S}^d$. This quotient space of the sphere is homeomorphic with the collection of all lines passing through the origin in $\mathbb{R}^d$. Also, $\mathbb{P}^d(\mathbb{R})$ can be defined as the cosets of the orthogonal group $\mathbf{O}(d+1)$, i.e.,

$$\mathbb{P}^d(\mathbb{R}) = \frac{\mathbf{O}(d+1)}{\mathbf{O}(1)\times\mathbf{O}(d)}.$$

Let

$$\pi:\ \mathbf{O}(d+1) \to \frac{\mathbf{O}(d+1)}{\mathbf{O}(1)\times\mathbf{O}(d)}$$

be the natural mapping and $\mathbf{e}$ be the identity of $\mathbf{O}(d+1)$. The point $\mathbf{o} = \pi(\mathbf{e})$ which is invariant under all motions of $\mathbf{O}(1)\times\mathbf{O}(d)$ is called the pole (or the north pole) of $\mathbb{P}^d(\mathbb{R})$. On $\mathbb{P}^d(\mathbb{R})$ there is an invariant Riemannian metric $d(\cdot,\cdot)$, an invariant Haar measure $d\nu$ and an invariant second order differential operator, the Laplace-Beltrami operator Δ. A function $Z(\cdot) : \mathbb{P}^d(\mathbb{R}) \to \mathbb{R}$ is called zonal if $Z(h^{-1}\cdot) = Z(\cdot)$ for any $h \in \mathbf{O}(1)\times\mathbf{O}(d)$.

For more details see, e.g., Cartan [3], Gangolli [4], and Helgason [6], [7].

A function on $\mathbb{P}^d(\mathbb{R})$ is invariant under the left action of $\mathbf{O}(1)\times\mathbf{O}(d)$ on $\mathbb{P}^d(\mathbb{R})$ if and only if it depends only the distance of its argument from $\mathbf{o}$. Since the distance of any point of $\mathbb{P}^d(\mathbb{R})$ from $\mathbf{o}$ is at most $\pi/2$, it follows that a spherical function Z on $\mathbb{P}^d(\mathbb{R})$ can be identified with a function $\tilde{Z}$ on $[0,\pi/2]$. Let θ be the distance of a point from $\mathbf{o}$. We may choose a geodesic polar coordinate system $(\theta,\mathbf{u})$ where $\mathbf{u}$ is an angular parameter. In this coordinate system the radial part Δ_θ of the Laplace-Beltrami operator Δ has the expression

$$\Delta_\theta = \frac{1}{A(\theta)}\ \frac{d}{d\theta}\left(A(\theta)\frac{d}{d\theta}\right),$$

where $A(\theta)$ is the area of the sphere of radius θ in $\mathbb{P}^d(\mathbb{R})$. It is interesting to remark that an explicit form the function $A(\theta)$ can be computed using methods of Lie algebras (see Helgason [7, p.251], [6, p.168] for the details). It can be shown that

$$A(\theta) = \omega_d(\sin\theta)^{d-1},$$

where ω_d is the area of the unit sphere in $\mathbb{R}^d$. Now we can write the operator Δ_θ (up to some numerical constant) in the form

$$\Delta_\theta = \frac{1}{(\sin\theta)^{d-1}}\ \frac{d}{d\theta}(\sin\theta)^{d-1}\ \frac{d}{d\theta}.$$

Using a simple change of variables $t = \cos\theta$, this operator takes the form (up to a positive multiple),

$$\Delta_t = (1-t^2)^{-(d-2)/2}\ \frac{d}{dt}(1-t^2)^{d/2}\ \frac{d}{dt}. \tag{1}$$

We will need the following statement Szegö [11, p.60]:

Proposition 1. *The Jacobi polynomials* $y = P_k^{(\alpha,\beta)}$ *satisfy the following linear homogeneous differential equation of the second order:*

$$(1-t^2)y^{''} + (\beta - \alpha - (\alpha+\beta+2)t)y^{'} + k(k+\alpha+\beta+1)y = 0,$$

or

$$\frac{d}{dt}((1-t)^{\alpha+1}(1-t)^{\beta+1}y^{'}) + k(k+\alpha+\beta+1)(1-t)^{\alpha}(1+t)^{\beta}y = 0.$$

It follows from the above proposition that the eigenfunctions of the operator Δ_t which has been defined in (1) are well-known Jacobi polynomials $P_k^{(\alpha,\beta)}(t)$ and the corresponding eigenvalues are $\theta_k = -k(k+\alpha+\beta+1)$, where $\alpha = \beta = (d-2)/2$. In this way zonal functions on $\mathbb{P}^d(\mathbb{R})$ can be easily identified since the elementary zonal functions are eigenfunctions of the Laplace-Beltrami operator. Note, that on the real projective spaces, $\mathbb{P}^d(\mathbb{R})$, the only polynomials of even degree appear because, due to the identification of antipodal points on $\mathbb{S}^d$, only the even order polynomials $P_{2k}^{(\alpha,\alpha)}$, $k \in \mathbb{N}$ can be lifted to be functions on $\mathbb{P}^d(\mathbb{R})$. Let Z_{2k}, $k \in \mathbb{N}$, with $Z_0 \equiv 1$ be a zonal function corresponding to the eigenvalue $\theta_{2k} = -2k(2k+d-1)$ and $\tilde{Z}_{2k}$ be the corresponding functions induced on $[0,\pi/2]$ by Z_{2k}. Then, Koornwinder [8],

$$\tilde{Z}_{2k}(\theta) = C_{2k}(\mathbb{P}^d(\mathbb{R})P_{2k}^{((d-2)/2,(d-2)/2)}(\cos\theta). \tag{2}$$

Remark that for any $k \in \mathbb{N}$ the polynomial $P_k^{((d-2)/2,(d-2)/2)}$ is just a multiple of the Gegenbauer polynomial $P_k^{(d-1)/2}$. A detailed treatment of the Jacobi polynomials can be found in Szegö [11]. In particular, the Jacobi polynomials $P_k^{(\alpha,\beta)}(t)$, $\alpha > -1$, $\beta > -1$ are orthogonal with respect to $\omega^{\alpha,\beta}(t) = c^{-1}(1-t)^\alpha(1+t)^\beta$ on $(-1,1)$. The above constant c can be found using the normalization condition $\int_{\mathbb{P}^d(\mathbb{R})} d\nu = 1$ for the invariant measure $d\nu$ on $\mathbb{P}^d(\mathbb{R})$ and a well-known formula for the Euler integral of the first kind

$$\mathrm{B}(p\,,q) = \int_0^1 \xi^{p-1}(1-\xi)^{q-1}d\xi = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},\ \ p>0,\ q>0. \tag{3}$$

Applying (3) and a simple change of variables we get

$$1 = \int_{\mathbb{P}^d(\mathbb{R})} d\nu = \int_0^1 \omega^{(d-2)/2,(d-2)/2}(t)dt = c^{-1}\int_0^1 (1-t^2)^{(d-2)/2}dt,$$

so that,

$$c = \int_0^1 (1-t^2)^{(d-2)/2}dt = 2^{d-2}\frac{(\Gamma(d/2))^2}{\Gamma(d)}. \tag{4}$$

We normalize the Jacobi polynomials as follows:

$$P_k^{(\alpha,\beta)}(1) = \frac{\Gamma(k+\alpha+1)}{\Gamma(\alpha+1)\Gamma(k+1)}.$$

This way of normalization is coming from the definition of Jacoby polynomials using the generating function Szegö [11, p.69]. In particular,

$$P_{2k}^{((d-2)/2,(d-2)/2)}(1) = \frac{\Gamma(2k+d/2)}{\Gamma(d/2)\Gamma(2k+1)}.$$

Let $L_p(\mathbb{P}^d(\mathbb{R}))$ be the set of functions of finite norm given by

$$\parallel \varphi \parallel_p = \parallel \varphi \parallel_{L_p(\mathbb{P}^d(\mathbb{R}))} = \begin{cases} (\int_{\mathbb{P}^d(\mathbb{R})} |\varphi(x)|^p d\nu(x))^{1/p}, & 1 \le p < \infty, \\ \text{ess sup}\{|\varphi(x)| \mid x \in \mathbb{P}^d(\mathbb{R})\}, & p = \infty. \end{cases}$$

Further, let $U_p = \{\varphi \mid \varphi \in L_p(\mathbb{P}^d(\mathbb{R})),\ \ \parallel \varphi \parallel_p \le 1\}$ be the unit ball of the space $L_p(\mathbb{P}^d(\mathbb{R}))$. The Hilbert space $L_2(\mathbb{P}^d(\mathbb{R}))$ with usual scalar product

$$\langle f, g\rangle = \int_{\mathbb{P}^d(\mathbb{R})} f(x)\overline{g(x)}d\nu(x)$$

has the decomposition

$$L_2(\mathbb{P}^d(\mathbb{R})) = \bigoplus_{k=0}^{\infty} \mathrm{H}_k,$$

where H_{2k} is the eigenspace of the Laplace-Beltrami operator corresponding to the eigenvalue $\theta_{2k} = -2k(2k+\alpha+\beta+1)$. Let $\{Y_j^{2k}\}_{j=1}^{d_{2k}}$ be an orthonormal basis of H_{2k}. The following addition formula is known, Koornwinder [8],

$$\sum_{j=1}^{d_{2k}} Y_j^{2k}(x)\overline{Y_j^{2k}(y)} = \tilde{Z}_{2k}(\cos\theta), \tag{5}$$

where $\theta = d(x,y)$ or comparing (5) with (2) we get

$$\sum_{j=1}^{d_{2k}} Y_j^{2k}(x)\overline{Y_j^{k}(y)} = \tilde{Z}_k(\cos\theta) = C_{2k}(\mathbb{P}^d(\mathbb{R}))P_{2k}^{(\alpha,\beta)}(\cos\theta). \tag{6}$$

See Helgason [7], [6], Cartan [3], Koornwinder [8], and Gangolli [4] for more information concerning the harmonic analysis on homogeneous spaces.

Using multiplier operators we can introduce a wide range of smooth functions on $\mathbb{P}^d(\mathbb{R})$. Let $\phi \in L_p(\mathbb{P}^d(\mathbb{R}))$, $1 \le p \le \infty$, with the formal Fourier expansion

$$\phi \sim \sum_{k=0}^{\infty}\sum_{j=1}^{d_{2k}} c_{2k,j}(\phi)Y_j^{2k}, \;\; c_{2k,j}(\phi) = \int_{\mathbb{P}^d(\mathbb{R})} \phi\overline{Y_j^{2k}}d\nu.$$

Let $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}\cup\{0\}}$ be a sequence of real (complex) numbers. If for any $\phi \in L_p(\mathbb{P}^d(\mathbb{R}))$ there is a function $f := \Lambda\phi \in L_q(\mathbb{P}^d(\mathbb{R}))$ such that

$$f \sim \sum_{k=0}^{\infty} \lambda_k \sum_{j=1}^{d_{2k}} c_{2k,j}(\phi)Y_j^{2k},$$

then we shall say that the multiplier operator Λ is of (p,q)-type with norm $\|\Lambda\|_{p,q} := \sup_{\varphi\in U_p}\|\Lambda\varphi\|_q$. We shall say that the function f is in

$\Lambda U_p \oplus \mathbb{R}$ if

$$\Lambda\phi = f \sim C \ + \sum_{k=1}^{\infty} \lambda_k \sum_{j=1}^{d_{2k}} c_{2k,j}(\phi) Y_j^{2k},$$

where $C \in \mathbb{R}$ and $\phi \in U_p$. In particular, the γ-th fractional integral ($\gamma > 0$) of a function $\phi \in L_1(\mathbb{P}^d(\mathbb{R}))$ is defined by the sequence $\lambda_k = (2k(2k+d-1))^{-\gamma/2}$. Sobolev's classes $W_p^\gamma(\mathbb{P}^d(\mathbb{R}))$ on $\mathbb{P}^d(\mathbb{R})$ are defined as sets of functions with formal Fourier expansions

$$C + \sum_{k=1}^{\infty} (2k(2k+d-1))^{-\gamma/2} \sum_{j=1}^{d_{2k}} c_{2k,j}(\phi) Y_j^{2k},$$

where $C \in \mathbb{R}$ and $\|\phi\|_p \leq 1$. Let Z be a zonal integrable function on $\mathbb{P}^d(\mathbb{R})$. For any integrable function g we can define convolution h on $\mathbb{P}^d(\mathbb{R})$ as the following

$$h(\cdot) = (Z * g)(\cdot) = \int_{\mathbb{P}^d(\mathbb{R})} Z(\cos(d(\cdot, x)) g(x) d\nu(x).$$

For the convolution on $\mathbb{P}^d(\mathbb{R})$ we have Young's inequality

$$\|(z * g)\|_q \leq \|z\|_p \|g\|_r,$$

where $1/q = 1/p + 1/r - 1$ and $1 \leq p, q, r \leq \infty$. It is possible to show that for any $\gamma > 0$ the function

$$G_\gamma = G_{\gamma,\eta} \sim \sum_{k=1}^{\infty} (2k(2k+d-1))^{-\gamma/2} Z_{2k}^{\eta}$$

with pole η is integrable on $\mathbb{P}^d(\mathbb{R})$ and for any function $g \in W_p^\gamma(\mathbb{P}^d(\mathbb{R}))$ we have an integral representation

$$g = C + G_\gamma * \phi,$$

where $C \in \mathbb{R}$ and $\phi \in U_p$.

The Orthogonal Projection. In this section we prove Theorem 1 which has been noticed in the Introduction.

Proof. We will need an explicit representation for the constant $C_{2k}(\mathbb{P}^d(\mathbb{R}))$ defined in (6). Putting $y = x$ in (6) and then integrating both sides with respect to $d\nu(x)$ we get

$$d_{2k} = \dim \mathrm{H}_{2k} = \sum_{j=1}^{d_{2k}} \int_{\mathbb{P}^d(\mathbb{R})} |Y_j^{2k}(x)|^2 d\nu(x)$$

$$= C_{2k}(\mathbb{P}^d(\mathbb{R})) P_{2k}^{((d-2)/2,(d-2)/2)}(1). \tag{7}$$

Taking the square of both sides of (6) and then integrating with respect to $d\nu(x)$ we find

$$\sum_{j=1}^{d_{2k}} |Y_j^{2k}(y)|^2 = C_{2k}^2(\mathbb{P}^d(\mathbb{R})) \int_{\mathbb{P}^d(\mathbb{R})} \left(P_{2k}^{((d-2)/2,(d-2)/2)}(\cos d(x,y)) \right)^2 d\nu(x). \tag{8}$$

Since $d\nu$ is shift invariant then

$$\int_{\mathbb{P}^d(\mathbb{R})} \left(P_{2k}^{((d-2)/2,(d-2)/2)}(\cos(d(x,y))) \right)^2 d\nu(x) = c^{-1} \left\| P_{2k}^{((d-2)/2,(d-2)/2)} \right\|_2^2,$$

where the constant c is defined by (4) and (see Szegö [11], p.68)

$$\left\| P_{2k}^{((d-2)/2,(d-2)/2)} \right\|_2^2 = \int_0^1 \left(P_{2k}^{(d-2)/2,(d-2)/2)}(t) \right)^2 (1-t^2)^{(d-2)/2} dt =$$

$$= \frac{2^{d-2}}{4k+d-1} \; \frac{(\Gamma(2k+d/2))^2}{\Gamma(2k+1)\Gamma(2k+d-1)}.$$

So that, (8) can be written in the form

$$\sum_{j=1}^{d_{2k}} |Y_j^{2k}(y)|^2 = c^{-1} C_{2k}^2(\mathbb{P}^d(\mathbb{R})) \left\| P_{2k}^{((d-2)/2,(d-2)/2)} \right\|_2^2.$$

Integrating the last line with respect to $d\nu(y)$ we obtain

$$d_{2k} = c^{-1} C_{2k}^2(\mathbb{P}^d(\mathbb{R})) \left\| P_{2k}^{(\alpha,\beta)} \right\|_2^2.$$

It is sufficient to compare this with (7) to obtain

$$C_{2k}(\mathbb{P}^d(\mathbb{R})) = \frac{cP_{2k}^{((d-2)/2,(d-2)/2)}(1)}{\left\|P_{2k}^{((d-2)/2,(d-2)/2)}\right\|_2^2}. \tag{9}$$

We get now an integral representation for the Fourier sums $S_{2n}(\phi, x)$ of a function $\phi \in L_\infty(\mathbb{P}^d(\mathbb{R}))$,

$$S_{2n}(\phi, x) = c_0(\phi) + \sum_{k=1}^{n}\sum_{j=1}^{d_{2k}} c_{2k,j}(\phi) Y_j^{2k}(x) =$$

$$= \int_{\mathbb{P}^d(\mathbb{R})} \phi(y)\overline{Y_1^0(y)} d\nu(y) + \sum_{k=1}^{n}\sum_{j=1}^{d_{2k}} \left(\int_{\mathbb{P}^d(\mathbb{R})} \phi(y)\overline{Y_j^{2k}(y)} d\nu(y)\right) Y_j^{2k}(x) =$$

$$= \int_{\mathbb{P}^d(\mathbb{R})} \sum_{k=0}^{n} \left(\sum_{j=1}^{d_{2k}} \overline{Y_j^{2k}(y)} Y_j^{2k}(x)\right) \phi(y) d\nu(y) =$$

$$= \int_{\mathbb{P}^d(\mathbb{R})} \sum_{k=0}^{n} Z_{2k}^x(y)\phi(y) d\nu(y) = \int_{\mathbb{P}^d(\mathbb{R})} K_{2n}(x, y)\phi(y) d\nu(y), \tag{10}$$

where

$$K_{2n}(x, y) = \sum_{k=0}^{n} Z_{2k}^x(y). \tag{11}$$

By (2) and (9) we have

$$K_{2n}(x, y) = c \sum_{k=0}^{n} \frac{P_{2k}^{((d-2)/2,(d-2)/2)}(1)}{\left\|P_{2k}^{((d-2)/2,(d-2)/2)}\right\|_2^2} P_{2k}^{((d-2)/2,(d-2)/2)}(\cos d(x, y)).$$

Let us denote

$$G_n^{(\alpha,\beta)}(\gamma, \delta) = \sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(\gamma) P_k^{(\alpha,\beta)}(\delta)}{\left\|P_k^{(\alpha,\beta)}\right\|_{2,*}^2},$$

where

$$\left\| P_k^{(\alpha,\beta)} \right\|_{2,*}^2 = \int_{-1}^{1} \left(P_k^{(\alpha,\beta)}(t) \right)^2 (1-t)^\alpha (1+t)^\beta dt$$

Then by Szegö [11], p.71,

$$G_n^{(\alpha,\beta)}(\gamma, 1) = \sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(\gamma) P_k^{(\alpha,\beta)}(1)}{\left\| P_k^{(\alpha,\beta)} \right\|_{2,*}^2} =$$

$$= 2^{-\alpha-\beta-1} \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)} P_n^{(\alpha+1,\beta)}(\gamma). \tag{12}$$

Remark that, Szegö [11],

$$P_k^{(\alpha,\beta)}(\gamma) = (-1)^k P_k^{(\beta,\alpha)}(-\gamma) \tag{13}$$

for any $\gamma \in \mathbb{R}$ and $k \in \mathbb{N}$. By the definitions of the norms $\|\cdot\|_2$ and $\|\cdot\|_{2,*}$

$$\left\| P_{2k}^{((d-2)/2,(d-2)/2)} \right\|_{2,*}^2 = 2 \left\| P_{2k}^{((d-2)/2,(d-2)/2)} \right\|_2^2, \tag{14}$$

for any $k \in \mathbb{N}$ since $P_{2k}^{((d-1)/2,(d-1)/2)}$ is an even function. Comparing (12) - (14) we get an explicit representation for the kernel function (11) in the integral representation (10), i.e.,

$$K_{2n}(x,y) = c 2^{-\alpha-\beta-1} 2 \frac{\Gamma(2n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(2n+\beta+1)} \times$$

$$\times \frac{P_{2n}^{(\alpha+1,\beta)}(\cos d(x,y)) + P_{2n}^{(\beta,\alpha+1)}(\cos d(x,y))}{2} =$$

$$= c 2^{-d+1} \frac{\Gamma(2n+d)}{\Gamma(d/2)\Gamma(2n+d/2)} \times$$

$$\times \left(P_{2n}^{(d/2,(d-2)/2)}(\cos d(x,y)) + P_{2n}^{((d-2)/2,d/2)}(\cos d(x,y)) \right) \tag{15}$$

since $\alpha = \beta = (d-2)/2$. It is known, Szegö [11], p.196, that for $0 < \eta < \pi$,

$$P_n^{(\alpha,\beta)}(\cos\eta) = n^{-1/2}\ \kappa^{(\alpha,\beta)}(\eta) \cos(N\eta + \gamma) + O(n^{-3/2}), \tag{16}$$

where

$$\kappa^{(\alpha,\beta)}(\eta) = \pi^{-1/2}\left(\sin\frac{\eta}{2}\right)^{-\alpha-1/2}\left(\cos\frac{\eta}{2}\right)^{-\beta-1/2},$$

$$N = n + \frac{\alpha+\beta+1}{2} = n + \frac{d-1}{2}, \quad \text{and} \quad \gamma = -\frac{\alpha+1/2}{2}\pi.$$

Let $\eta = d(x,y)$ and $\mathbf{o}$ be the north pole of $\mathbb{P}^d(\mathbb{R})$, then from (15), (16) and since K_{2n} is a zonal function and $d\nu$ is shift invariant we get

$$\|S_{2n}\|_{L_\infty(\mathbb{P}^d(\mathbb{R}))\to L_\infty(\mathbb{P}^d(\mathbb{R}))} = \sup_{\|\phi\|_{L_\infty(\mathbb{P}^d(\mathbb{R}))}\le 1} \|S_{2n}(\phi,x)\|_{L_\infty(\mathbb{P}^d(\mathbb{R}))} =$$

$$= \sup\left\{\int_{\mathbb{P}^d(\mathbb{R})} |K_{2n}(x,y)| d\nu(y) : \ x \in \mathbb{P}^d(\mathbb{R})\right\} =$$

$$= \int_{\mathbb{P}^d(\mathbb{R})} |K_{2n}(\mathbf{o},y)| d\nu(y) = \frac{c2^{-d+1}\Gamma(2n+d)}{\Gamma(d/2)\Gamma(2n+d/2)} \times$$

$$\times \int_{\mathbb{P}^d(\mathbb{R})} \left| P_{2n}^{(d/2,(d-2)/2)}(\cos(d(\mathbf{o},y)) + P_{2n}^{((d-2)/2,d/2)}(\cos(d(\mathbf{o},y))\right| d\nu(y) =$$

$$= \frac{2^{-d+1}\Gamma(2n+d)}{\Gamma(d/2)\Gamma(2n+d/2)}\, I_n,$$

where

$$I_n := \int_0^1 \left| P_{2n}^{(d/2,(d-2)/2)}(t) + P_{2n}^{((d-2)/2,d/2)}(t)\right| (1-t^2)^{(d-2)/2} dt =$$

$$= \int_0^{\pi/2} \left| P_{2n}^{(d/2,(d-2)/2)}(\cos\eta) + P_{2n}^{((d-2)/2,d/2)}(\cos\eta)\right| (\sin\eta)^{d-1} dt =$$

$$= \frac{2^{d/2+1/2}}{\pi^{1/2}(2n)^{1/2}} \int_0^{\pi/2} (\sin\eta)^{(d-3)/2} \left|\cos\left(\left(2n+\frac{d-1}{2}\right)\eta - \frac{(d+1)\pi}{4}\right)\right| d\eta +$$

$$+O(n^{-3/2}).$$

Applying a simple Tylor series arguments and an elementary estimates of the derivative of the function $(\sin\eta)^{(d-3)/2}$ we get

$$I_n = \frac{2^{d/2+1}}{\pi^{3/2} n^{1/2}} \int_0^{\pi/2} (\sin\eta)^{(d-3)/2} d\eta + \left\{ \begin{array}{ll} O(n^{-1/2}), & d=2, \\ O(n^{-1}), & d \ge 3 \end{array} \right\}.$$

***Remark* 1.** Let $\mathbb{M}^d = \mathbb{S}^d, \mathbb{P}^d(\mathbb{R}), \mathbb{P}^d(\mathbb{C}), \mathbb{P}^d(\mathbb{H}), \mathbb{P}^{16}(\mathrm{Cay})$. It is known [1], [2] that for any $\gamma > 0$,

$$E_n(W_\infty^\gamma(\mathbb{M}^d)) := \sup\{E_n(f) |\ f \in W_\infty^\gamma(\mathbb{M}^d)\} \asymp n^{-\gamma},\ \ n \to \infty.$$

From the Theorem 1 and the Lebesgue inequality it follows that the Fourier series of a function $f \in W_\infty^\gamma(\mathbb{M}^d)$ converges uniformly if $\gamma > (d-1)/2$. In general, let $\Delta^0\lambda_k = \lambda_k$, $\Delta^1\lambda_k = \lambda_k - \lambda_{k+1}$, $\Delta^{s+1}\lambda_k = \Delta^s\lambda_k - \Delta^s\lambda_{k+1}$, $k, s \in \mathbb{N}$ and

$$N := \begin{cases} (d+1)/2, & d = 3, 5, \cdots, \\ (d+2)/2, & d = 2, 4, \cdots \end{cases}$$

Let $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}\cup\{0\}}$ be a multiplier operator, $\Lambda : L_\infty(\mathbb{M}^d) \to L_\infty(\mathbb{M}^d)$ and $\Lambda U_\infty(\mathbb{M}^d)$ be the respective set of smooth functions, then from the Theorem 2, [2], p.317, [1] it follows that the Fourier series of a function $f \in \Lambda U_\infty(\mathbb{M}^d)$ converges uniformly if

$$\lim_{n\to\infty} n^{(d-1)/2} \sum_{k=n+1}^{\infty} |\Delta^{N+1}\lambda_k|\ k^N = 0,$$

since $E_n(\Lambda U_\infty(\mathbb{M}^d) \ll \sum_{k=n+1}^{\infty} |\Delta^{N+1}\lambda_k|\ k^N$ as $n \to \infty$.

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